Nondango is NP-Complete

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Abstract

Nondango is a pencil puzzle consisting of a rectangular grid partitioned into regions, with some cells containing a white circle. The player has to color some circles black such that every region contains exactly one black circle, and there are no three consecutive circles (horizontally, vertically, or diagonally) having the same color. In this paper, we prove that deciding solvability of a given Nondango puzzle is NP-complete.

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1 Introduction

Nondango is a pencil puzzle published by Nikoli. The puzzle consists of a rectangular grid partitioned into polyominoes called *regions*, with some cells containing a white circle. The player has to color some circles black to satisfy the following constraints [20].

1. Every region contains exactly one black circle.
2. There are no three consecutive circles (horizontally, vertically, or diagonally) having the same color (see Figure 1).

In this paper, we show that it is NP-complete to decide whether a given Nondango puzzle has a solution.

▶ **Theorem 1.1.** *Deciding solvability of a given Nondango instance is NP-complete.*

As the problem clearly belongs to NP, the nontrivial part is to prove the NP-hardness. We do so by constructing a reduction from the 1-in-3-SAT+ problem (deciding whether there is a Boolean assignment such that every clause has exactly one literal that evaluates to true, in a setting where each clause contains exactly three positive literals), which is known to be NP-complete [27].

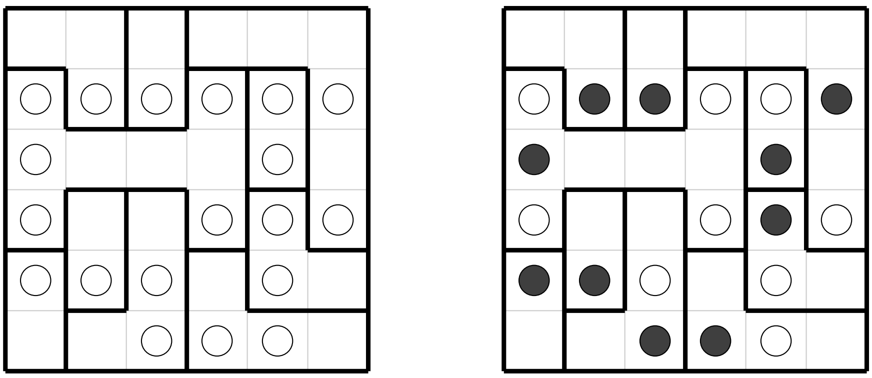

Figure 1 An example of a 6 × 6 Nondango puzzle (left) and its solution (right)

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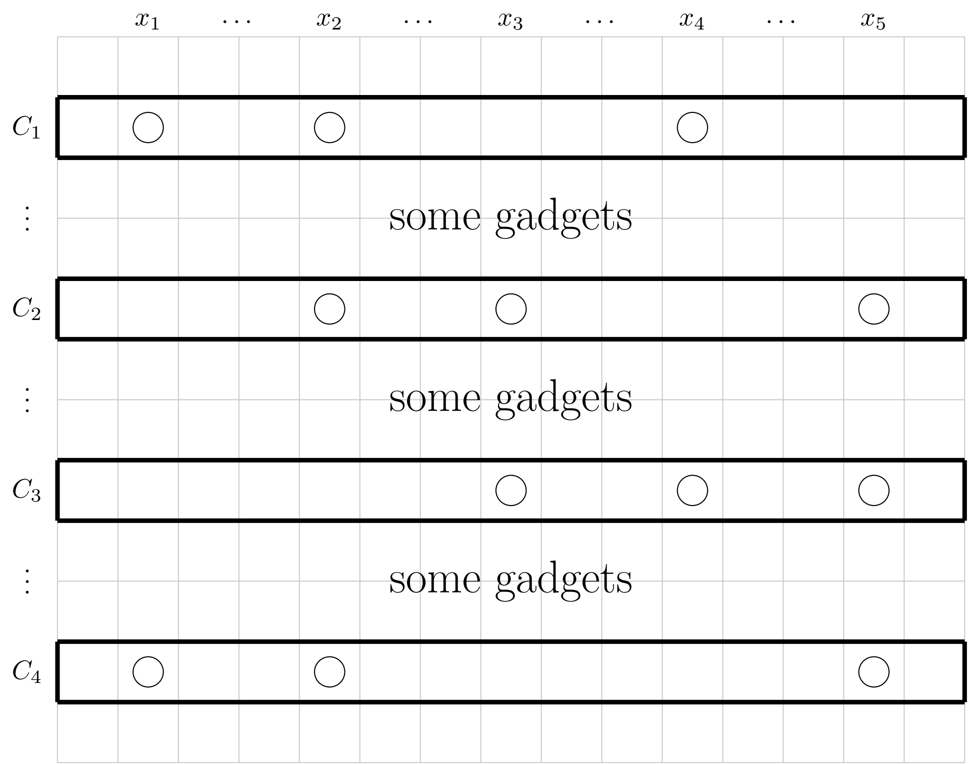

Figure 2 Basic structure of a Nondango instance transformed from a formula consisting of clauses $C_1 = x_1 \vee x_2 \vee x_4$, $C_2 = x_2 \vee x_3 \vee x_5$, $C_3 = x_3 \vee x_4 \vee x_5$, and $C_4 = x_1 \vee x_2 \vee x_5$

1.1 Related Work

Many pencil puzzles have been proved to be NP-complete, including Dosun-Fuwari [15], Fillmat [30], Five Cells [18], Goishi Hiroi [5], Hashiwokakero [4], Herugolf [14], Heyawake [11], Juosan [16], Kakuro [31], Kurodoko [21], Kurotto [16], LITS [6], Makaro [14], Moon-on-Sun [19], Nagareru [19], Nonogram [29], Norinori [6], Numberlink [1], Nurikabe [10], Nurimeizu [19], Nurimisaki [17], Pencils [23], Ripple Effect [28], Roma [9], Sashigane [17], Shakashaka [8], Shikaku [28], Slitherlink [31], Sto-Stone [3], Sudoku [31], Suguru [24], Sumplete [25], Tatamibari [2], Tilepaint [31], Toichika [26], Usowan [13], Yin-Yang [7], and Yosenabe [12].

2 Idea of the Proof

Given a 1-in-3-SAT+ formula, we will transform it into a Nondango puzzle. In the puzzle grid, each variable and each clause is represented by a column (called a *variable column*) and a row (called a *clause row*), respectively. In each clause row, a rectangular region of height 1 consists of the whole row. Inside the region, we place three circles (called *variable circles*) at columns corresponding to the three variables appearing in that clause (see Figure 2).

We interpret a black (resp. white) circle in a Nondango solution as a true (resp. false) literal. The constraint that exactly one literal in each clause is true is equivalent to that exactly one circle in that region is black. However, a more challenging task is to force every circle in each variable column to have the same color (which is equivalent to that each variable must have the same truth value in every clause it appears). We will show how to construct gadgets to enforce this constraint in the next section.

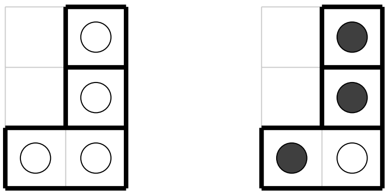

Figure 3 A gadget for creating a forced white circle (left) and its only solution (right)

3 Reduction

We use the following gadgets to construct components of the Nondango puzzle with certain properties.

3.1 Enforcing a Black Circle

Creating a circle that must be black in the solution is trivial; in a region with exactly one circle, that circle must be black in the solution. We call such circle a *forced black circle*.

3.2 Enforcing a White Circle

We can create a circle that must be white in the solution by using a gadget represented in Figure 3. As the bottom-right circle is placed next to two vertically consecutive forced black circles, it must be white in the solution (otherwise there will be three consecutive black circles in the solution). Analogously, we call such circle a *forced white circle*.

3.3 Enforcing Two Circles with Different Colors

For any region with exactly two circles, the colors of these two circles in the solution are always different. However, if we want to force two circles *in different regions* to have different colors in the solution, we can do so by using a gadget represented in Figure 4. The idea is that there are four consecutive circles arranged diagonally, with a circle at one end being forced white and at the other end being forced black. As a result, the two middle circles must have different colors in the solution (otherwise there will be three consecutive circles with the same color in the solution).

3.4 Connecting Two Consecutive Clause Rows with Common Variables

For two consecutive clause rows (e.g. clause rows for C_2 and C_3) which the corresponding clauses share a variable, we use a gadget represented in Figure 5 (see also Figure 6 for its solutions) to connect the two variable circles corresponding to that common variable. This gadget forces these two variable circles to have the same color in the solution, thus ensuring that the variable has the same truth value in both clauses. The idea behind this gadget is to use multiple copies of the gadget in Section 3.3.

3.5 Skipping a Clause Row

As the gadget in the Section 3.4 can only connect consecutive clause rows, we also provide a method to skip a clause row that does not contain the given variable. For example, if x_1

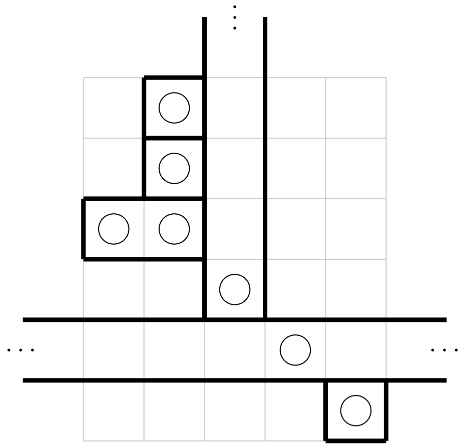

Figure 4 A gadget for enforcing two circles with different colors (left) and its only two solutions (right), where $\{A, B\} = \{\text{black}, \text{white}\}$

appears in C_2 and C_4, but not in C_3, we need to connect the clause rows of C_2 and C_4 while skipping the one of C_3[1].

We can do so by using a gadget represented in Figure 7 (see also Figure 8 for its solutions). The idea behind this gadget is that we put a forced white circle as a variable circle in the clause row we want to skip, so that the color constraint for other circles in that row will not be affected.

3.6 Filling Empty Area

We can simply make each connected empty area into one region, with one circle placed inside it, not touching any boundary. That circle is forced to be black without affecting other regions.

Recall that we interpret a black (resp. white) circle in a Nondango solution as a true (resp. false) literal. We can see that the Nondango puzzle we construct has a solution if and only if the original 1-in-3-SAT+ problem is satisfiable. As the reduction is clearly parsimonious, we can conclude that deciding solvability of a given Nondango puzzle is NP-complete.

[1] In fact, this gadget is unnecessary if we instead construct a reduction from the planar positive rectilinear 1-in-3-SAT problem, which is also NP-complete [22], where the clause rows can be arranged such that we do not need to skip a clause row. However, we include this gadget for the sake of completeness.

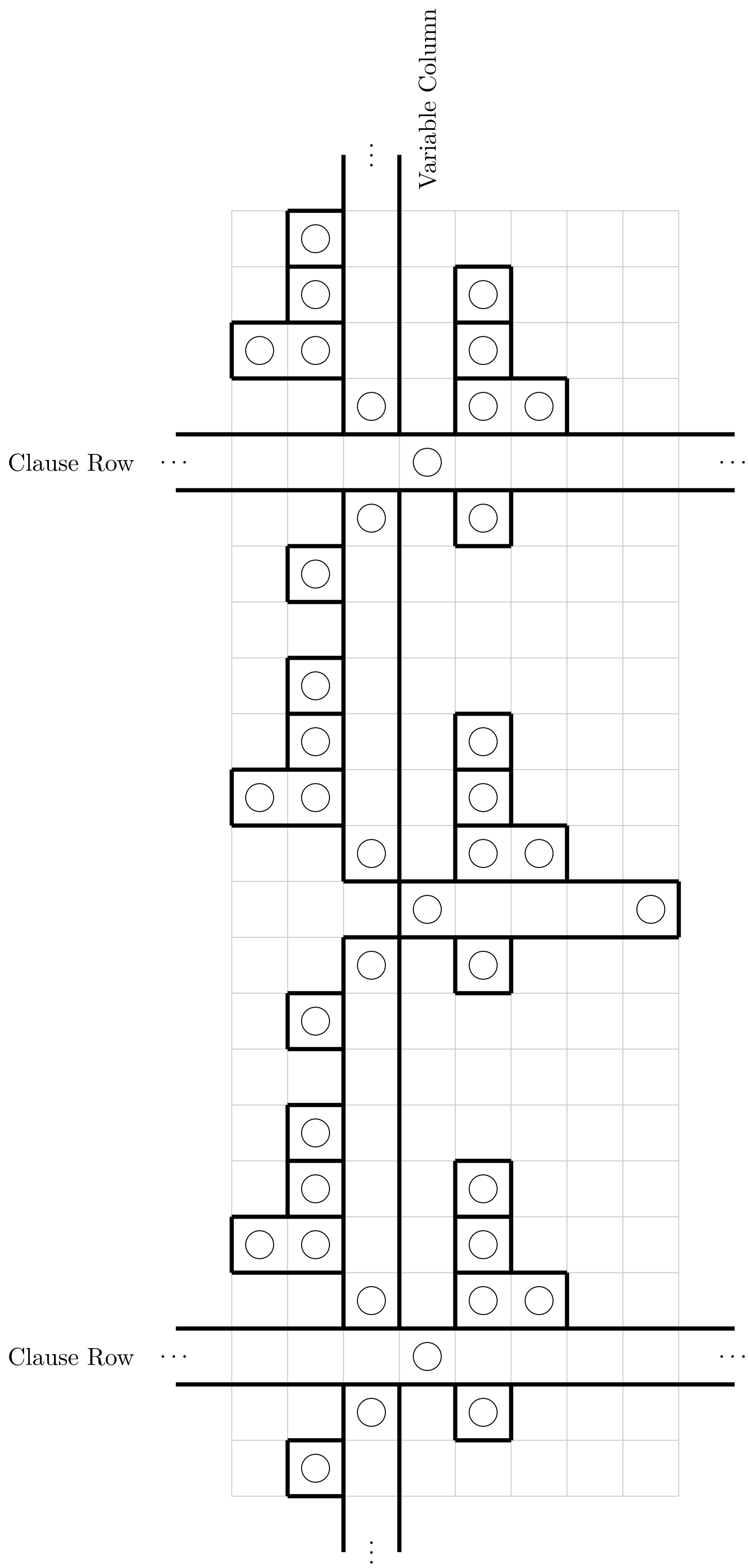

Figure 5 A gadget connecting two consecutive clause rows, forcing the two variable circles to have the same color

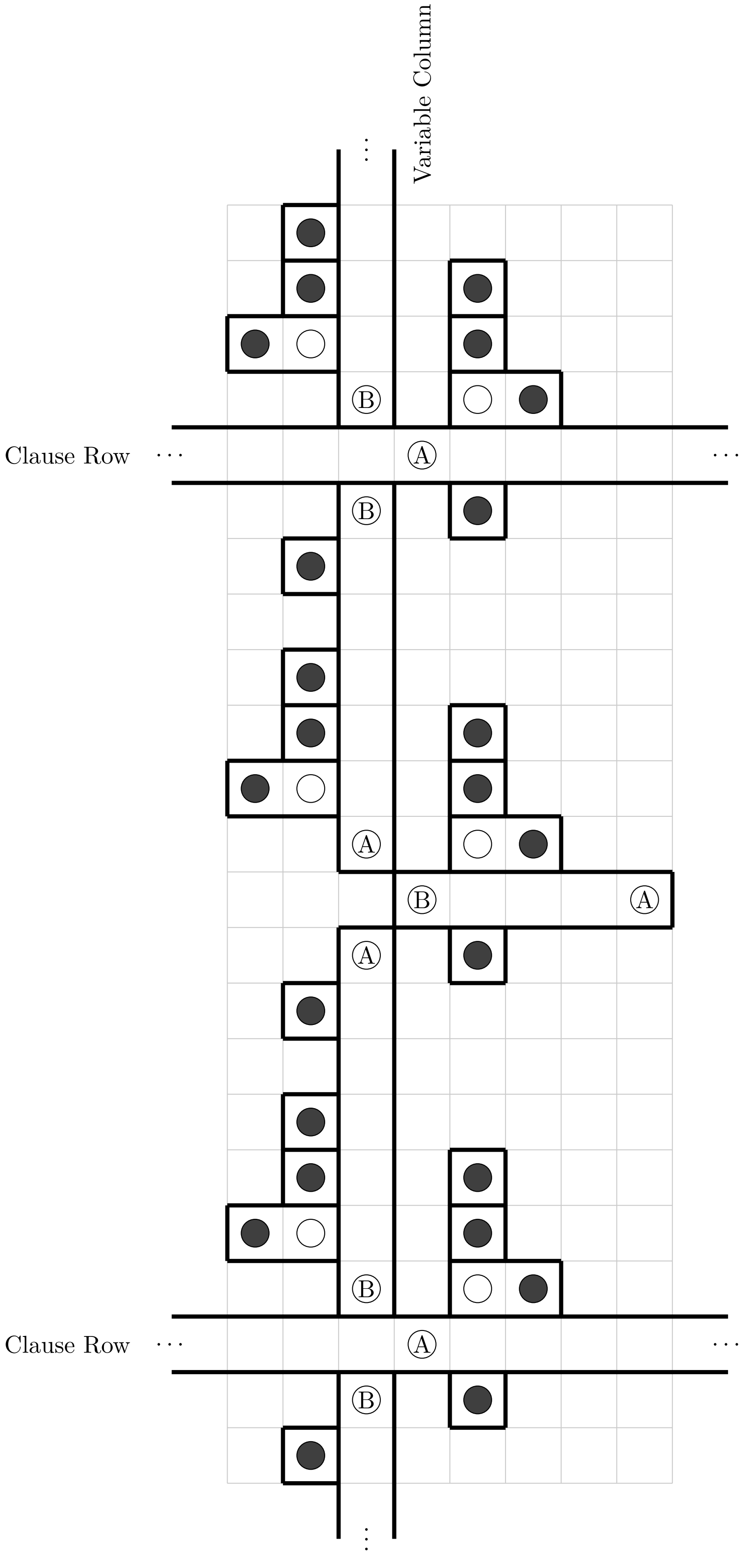

Figure 6 The only two solutions of the puzzle in Figure 5, where $\{A, B\} = \{\text{black}, \text{white}\}$

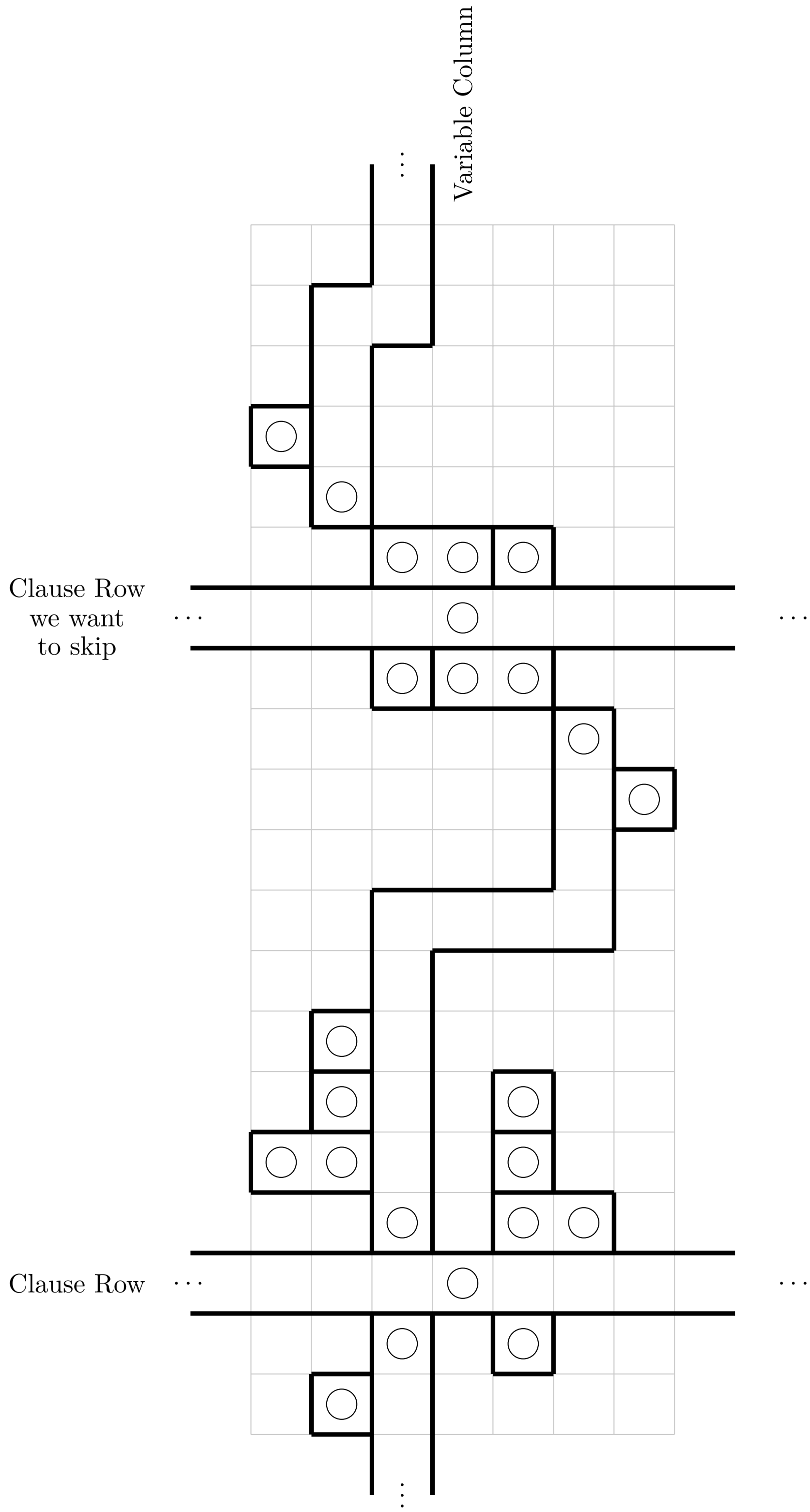

Figure 7 A gadget to skip a clause row, starting from the lower clause row and skipping the upper clause row to connect to some clause row beyond it

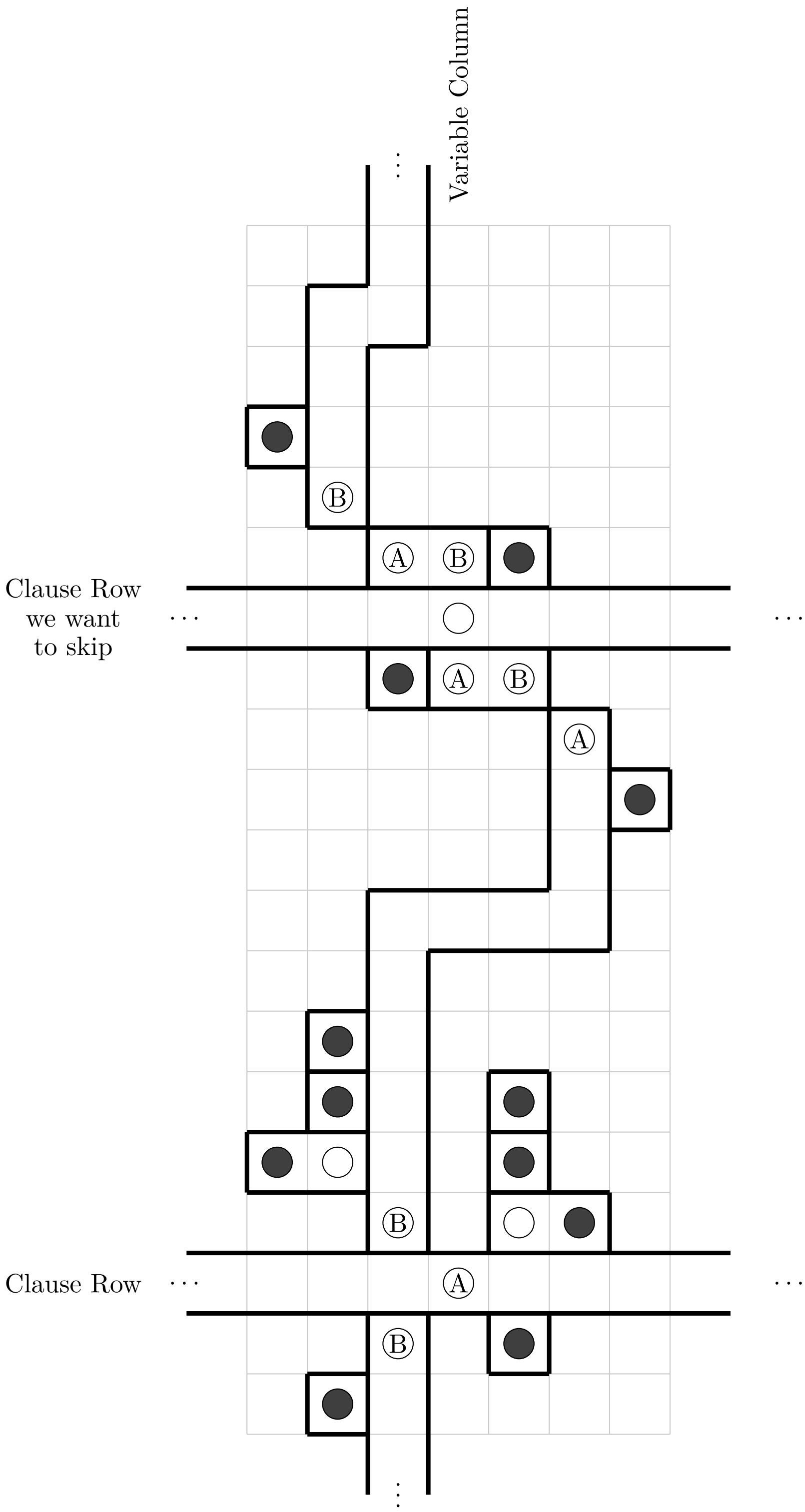

Figure 8 The only two solutions of the puzzle in Figure 7, where $\{A, B\} = \{\text{black}, \text{white}\}$

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