On Totally-Concave Polyominoes

Gill Barequet[1], Noga Keren[1], Johann Peters[2], and Adi Rivkin[1]

1 **Dept. of Computer Science, Techion—Israel Inst. of Technology, Haifa, Israel**
barequet@cs.technion.ac.il, [noga.keren,adi.rivkin]@campus.technion.ac.il

2 **Dept. of Mathematics, Univ. of Waterloo, ON, Canada**
j8peters@uwaterloo.ca

Abstract

A polyomino is an edge-connected set of cells on the square lattice. Every row or column of a *totally-concave* (TC) polyomino consists of more than one sequence of consecutive cells of the polyomino. We show that the minimum area (number of cells) of a TC polyomino is 21 cells. We also suggest, implement, and run an efficient algorithm for counting TC polyominoes. Finally, we prove that the associated sequence $(\kappa(n))$ has a finite growth constant λ_κ, and prove the lower bound $\lambda_\kappa > 2.4474$.

1 Introduction

A *polyomino* of area n is a connected set of n cells on the square lattice $\mathbb{Z}^2$, where connectivity is through edges. Two (fixed) polyominoes are considered equivalent if one can be transformed into the other by a translation.

Counting polyominoes is a long-standing problem in discrete geometry, originating in statistical physics in the context of percolation processes [8] and popularized in Golomb's pioneering book [9] and by M. Gardner's columns in *Scientific American*; The sequence $A(n)$, which lists the number of fixed polyominoes, is currently known up to $n = 70$ [1].

The growth constant of polyominoes has also attracted much attention in the literature. Klarner [13] showed that the limit (a.k.a. *Klarner's constant*) $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists. The convergence of $A(n+1)/A(n)$ to λ, as $n \to \infty$, was proved only three decades later by Madras [14]. The best-known lower [4] and upper [5] bounds on λ are 4.0025 and 4.5252, respectively. By applying numerical methods to the known values of $A(n)$, it is widely believed that $\lambda \approx 4.06$, and the currently best estimate of λ is 4.0625696 ± 0.0000005 [11]. (Based on the new counts of $A(n)$ till $n = 70$, a better estimate is 4.06256912(2) [12].)

In a *totally-concave* (TC) polyomino, each row and column consists of at least *two* maximal continuous sequences of cells, as is shown in Figure 1. It is hinted in Ref. [7, §14, p. 369, problem 14.5.4] that the minimum possible area of a TC polyomino is 21. Let $\kappa(n)$ be the number of TC polyominoes of size (area) n. An algorithm for computing $\kappa(n)$, for a given n, is also sought as an open problem [Ibid., problem 14.5.5]. Among other results, we settle the minimality conjecture and suggest an efficient algorithm.

In this paper, we investigate a few problems related to TC polyominoes. We prove that the minimum possible area of such a polyomino is indeed 21; suggest an efficient algorithm for counting TC polyominoes, and report the values of $\kappa(n)$ till $n = 35$; show that the seqeunce $(\kappa(n))$ has a growth constant λ_κ; and finally, prove that $\lambda_\kappa > 2.4474$.

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Symbolic Eatable With marked edges
(a) Area 21 (b) Area 24

Figure 1 TC polyominoes of various areas and flavors. The symbolic representation in (b) distinguishes between *hidden* edges (green), *inside* edges (blue), and *outside* edges (red).

2 Minimum Area

▶ **Theorem 2.1.** *The minimum area of a TC polyomino is 21.*

The proof of this theorem follows a necessity-sufficiency format. Necessity is shown by deducing upper and lower bounds on the area of TC polyominoes in $m \times \ell$ bounding boxes; These bounds contradict each other for areas less than 21. Sufficiency is evident by example.

Proof. A lower bound on the area of a TC polyomino within an $m \times \ell$ bounding box is achieved by partitioning the edges of such a polyomino into *hidden*, *outside*, and *inside* edges, as shown in Figure 1(b). The top (resp., right/bottom/left) edge of a cell c is *hidden* if there is a cell of the polyomino immediately above (resp., to the right of/below/to the left of) c. An edge is *outside* if it is not facing any other edge. An *inside* edge is an edge facing another edge, but not immediately, that is, with a gap of at least one cell. Consider a TC polyomino. Denote by n its area, and by h, o, and i the number of hidden, outside, and inside edges, respectively, of the polyomino. For example, by these definitions, the "U-pentomino" () has $i = 2$, $o = 10$, and $h = 8$. For the area-24 TC-polyomino depicted in Figure 1(b), we have $i = 24$, $o = 24$, and $h = 46$. By duplicity of inside and outside edges in rows and columns, we have that $o = 2m + 2\ell$ and $i \geq 2m + 2\ell$. We also have that $h \geq 2n - 2$ since the polyomino is connected and, hence, it must include at least $n-1$ cell adjacencies. Since $h + o + i = 4n$, we have that $n \geq 2m + 2\ell - 1$.

For an upper bound on n, we may assume without loss of generality that $m \leq \ell$. Then, a TC polyomino within an $m \times \ell$ bounding box must be missing at least one cell from each of the ℓ columns, none of which is in the top or bottom row (for guaranteeing concavity of the columns), as well as at least two further cells, one in the top and one in the bottom row (for guaranteeing concavity of these rows). Therefore, $n \leq m\ell - \ell - 2$.

Altogether, we have that $2m + 2\ell - 1 \leq n \leq m\ell - \ell - 2$, with $m \leq \ell$. A simple case analysis shows that the smallest n satisfying these constraints is 21, with $m = 5$ and $\ell = 6$.

Hence, $n \geq 21$ is a necessary condition for a TC polyomino. On the other hand, the existence of a TC polyomino of area 21 is evident by Fig. 1(a). This completes the proof. ◀

This result was verified by our TC-polyomino counting programs (see Section 3). Figure 2 shows representatives of the 152 TC polyominoes of area 21. (None of these polyominoes have any symmetries, hence, each of the 19 drawn polyominoes has eight distinct orientations.)

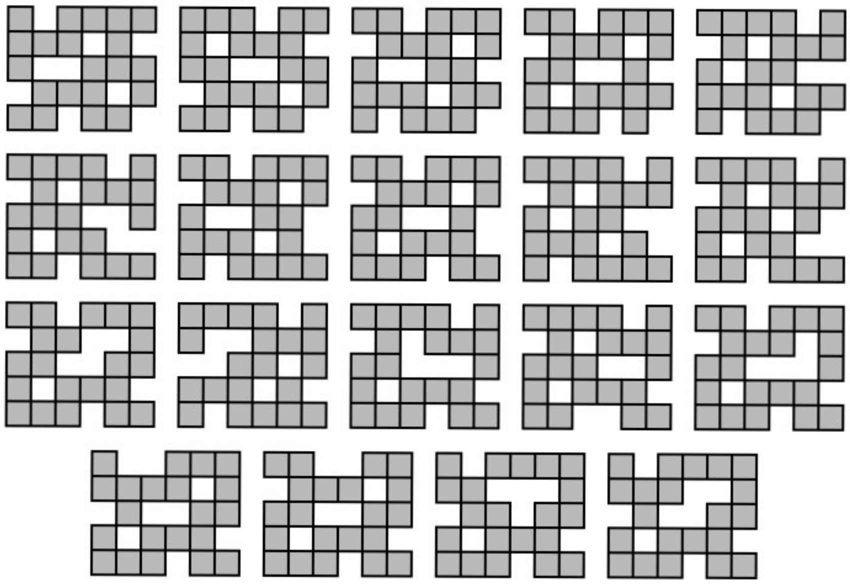

Figure 2 The 19 TC polyominoes of area 21, up to rotation and mirroring.

3 An Efficient Counting Algorithm

3.1 Algorithm

We first implemented a prototype backtracking algorithm for counting TC polyominoes. The program recursively concatenated concave columns to a growing polyomino. A branch of this procedure was abandoned when the area of the polyomino grew too large or if it was no longer possible for it to become connected with the addition of further columns. (This happened when a component of the polyomino became permanently detached.)

We then designed a much more efficient algorithm, based on Jensen's algorithm for counting all polyominoes [10, 11]. In a nutshell, Jensen's algorithm counts polyominoes within horizontal bounding strips of height h, where $1 \leq h \leq \lceil n/2 \rceil$. The algorithm considers column by column from left to right, and cell by cell from top to bottom within each column. At each cell, the algorithm considers either to have it occupied (belonging to the polyomino) or empty (not belonging). At all stages, the algorithm does not keep in memory all polyominoes but all possible *right boundaries* of polyominoes, that is, all combinations of the last h cells considered. The algorithm maintains a database whose entries have keys that are the different *signatures*, where a signature consists of a boundary plus all possible connections between cells on the boundary by cells found to the left of it. In other words, the keys reflect all possible splits of boundary cells into connected components, where the connections are to the left of the boundary. In addition, a signature also includes two bits that indicate whether or not the polyominoes associated with that entry touch the top and/or bottom of the strip. The contents of each entry in the database is statistics of all partially-built polyominoes ("partially" means that polyominoes may still consist of more than one connected component), that is, the *counts* of all polyominoes parameterized by area, having that specific signature. When the currently considered cell is chosen to be occupied, the counts of polyominoes are updated by adding the numbers of fully-built polyominoes, that is, polyominoes that consist of exactly one connected component and touch the top and bottom of the strip.

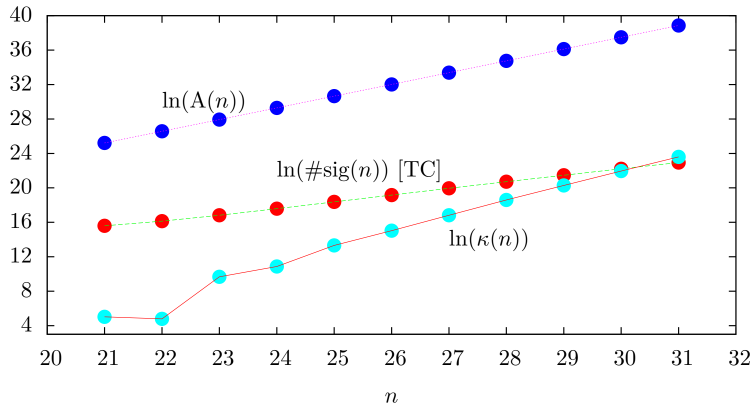

Figure 3 Plots of the number of signatures (while counting TC polyominoes), all poyominoes, and TC polyominoes.

For counting TC polyominoes, we also need to ensure that each column and each row consists of more than one consecutive sequence of cells. This is simple to achieve for columns: At the end of processing a column, we discard from the database all entries that correspond to columns that contain less than two sequences of occupied cells. For rows, we enhance the signatures by splitting each one into at most 4^h subsignatures: For each row, we keep a number as follows: '0' indicates that the first sequence of occupied cells has not been met yet; '1' means that we are in the middle of the first sequence; '2' states that we are between the first and second sequences; and '3' signifies that we have already entered the second sequence. (Once we reach '3,' we do not need to update this indicator any more.) Then, we count only polyominoes with signatures whose line indicators are all '3.' Note that the indicators of the top and bottom rows make the two bits described above redundant.

Jensen's algorithm is efficient in the sense that it's the only known algorithm whose running time, $\tilde{O}(1.732^n)$ [3], is smaller than the total number of polyominoes, $\tilde{\Theta}(\lambda^n)$. (Recall that $\lambda \approx 4.063$.) Our modification splits every signatures into at most $4^{n/2} = 2^n$ subsignatures (in practice, into much less than that), thus, the running time of the modified algorithm is $\tilde{O}(3.464^n)$, which is still much smaller than the total number of polyominoees. Figure 3 plots in a semi-logarithmic scale the number of distinct signatures encountered by the algorithm while computing $\kappa(n)$) (in red circles), together with the number of TC polyominoes (cyan) and the total number of polyominoes (blue), all as functions of n, for $21 \le n \le 31$.

3.2 Results

Our prototype program, implemented in Python, computed in 90 hours (elapsed time) $\kappa(n)$ up to $n = 26$ on a PC with a 64-bit system operating an i5-9400F Intel Core CPU at 2.90GHz with 12GB of RAM.

The modified version of Jensen's algorithm was implemented in `C++` and run on a 12th generation Intel(R) i9-12900KF with 128GiB of RAM. Using about 41 hours of CPU, the program computed $\kappa(n)$ up to $n = 35$, obtaining the values reported in Table 1 and agreeing with all values computed by the prototype program.

Table 1 Counts of TC polyominoes.

n	$\kappa(n)$	n	$\kappa(n)$	n	$\kappa(n)$	n	$\kappa(n)$
1–20	0	24	52,306	28	119,309,768	32	88,476,873,440
21	152	25	606,636	29	641,447,812	33	435,921,253,072
22	120	26	3,376,528	30	3,403,173,276	34	2,113,011,155,472
23	15,820	27	20,204,672	31	17,634,751,456	35	10,065,872,407,536

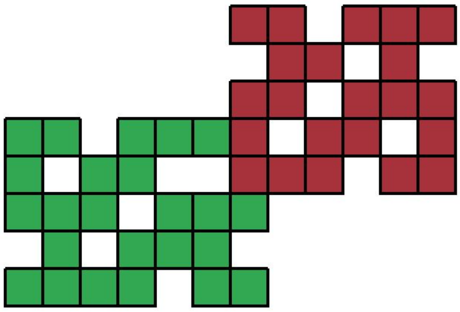

Figure 4 The concatenation of two TC polyominoes is always a TC polyomino.

4 Growth Constant

4.1 Existence

▶ **Definition 4.1.** (lexicographic order) For cells c_1, c_2, we say that $c_1 \prec c_2$ if c_1 lies in a column which is to the left of the column of c_2, *or* if c_1 lies below c_2 in the same column.

▶ **Definition 4.2.** (concatenation) Let P_1, P_2 be two polyominoes, and let c_1 (resp., c_2) be the biggest (resp., smallest) cell of P_1 (resp., P_2). The *concatenation* of P_1 and P_2 is the placement of P_2 relative to P_1, such that c_2 is found immediately on top of c_1.

▶ **Theorem 4.3.** *The limit* $\lambda_\kappa := \lim_{n\to\infty} \sqrt[n]{\kappa(n)}$ *exists and is finite.*

Proof. We follow closely the proof of existence and finiteness of Klarner's constant λ [13]. First, the sequence $\kappa(n)$ is supermultiplicative, that is, $\kappa(n)\kappa(m) \le \kappa(n+m)$ for all $m, n \in \mathbb{N}$. This is justified by a simple concatenation argument. Indeed, all TC polyominoes of area n can be concatenated with all TC polyominoes of area m (see, *e.g.*, Figure 4), yielding distinct TC polyominoes of area $n+m$. Second, there exists a constant $\mu > 0$ for which $\kappa(n) \le \mu^n$ for all $n \in \mathbb{N}$. For example, $\mu = \lambda$, the growth constant of all polyominoes. (This follows immediately from the fact that $\kappa(n) \le A(n) \le \lambda^n$.) By a lemma of Fekete (Klarner cites Ref. [15, p. 852] for similar results), the claim follows. ◀

Remark In fact, it makes more sense (see Section 4.2) to explore $((4\kappa(n))^{1/n})$ instead of $((\kappa(n))^{1/n})$. Figure 5 shows plots of the known values of $(4\kappa(n))^{1/n}$ and $\kappa(n)/\kappa(n-1)$. Surprisingly, the ratio sequence seems empirically to be monotone *decreasing* (except some low-order fluctuations), a property rarely found in other families of polyominoes.

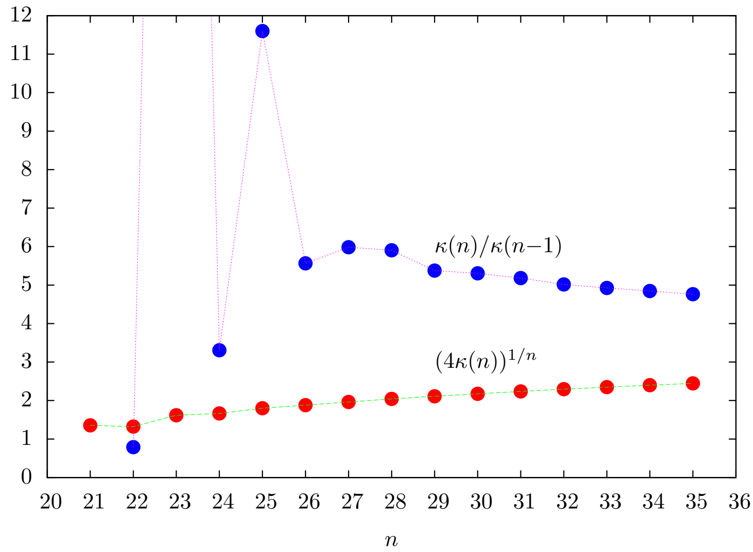

Figure 5 Plots of known values of $(4\kappa(n))^{1/n}$ and $\kappa(n)/\kappa(n-1)$.

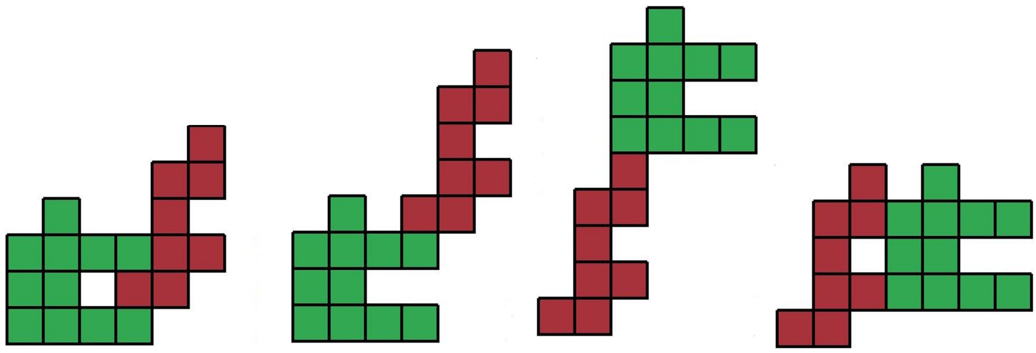

Figure 6 A few compositions of a sample pair of polyominoes.

4.2 A Lower Bound on λ_κ

We now present a computer-assisted proof of a lower bound on λ_κ.

▶ **Definition 4.4.** (composition) A *composition* of two polyominoes is a relative placement of the two polyominoes, such that they touch (edge to edge), possibly in multiple places, but do not overlap.

Figure 6 shows a few compositions of a pair of polyominoes P, Q. Note that some compositions have the property that all cells of P are smaller than all cells of Q (or vice versa), and some compositions do not. It is easy to observe that a composition of two TC polyominoes is not always a TC polyomino.

▶ **Lemma 4.5.** *(A simplified version of Theorem 1(a) in Ref. [2, p. 3]) Assume that the limit* $\mu := \lim_{n\to\infty} \sqrt[n]{Z(n)}$ *exists for a sequence* $(Z(n))$*. Let* $c_1 \neq 0, c_2$ *be some constants. Then, if* $c_1 n^{c_2} Z^2(n) \leq Z(2n)$ $\forall n \in \mathbb{N}$*, then* $\sqrt[n]{c_1 (2n)^{c_2} Z(n)} \leq \mu$ $\forall n \in \mathbb{N}$.

▶ **Theorem 4.6.** $\lambda_\kappa > 2.4474$.

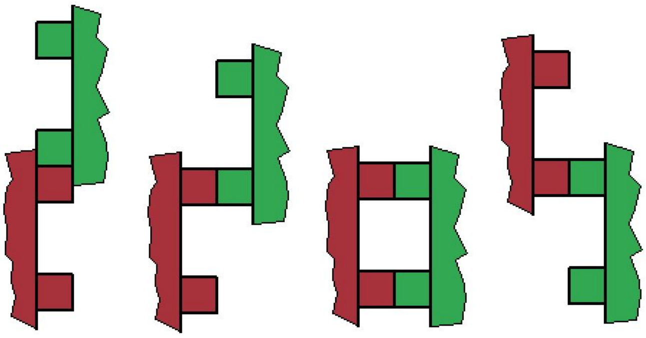

Figure 7 The at least four order-preserving compositions of a pair of TC polyominoes.

Proof. We use a composition argument, using the property that the extreme (rightmost and leftmost) columns of any TC polyomino have at least two cells. This property allows at least four lexicographic compositions of any pair of TC polyominoes P, Q that yield TC polyominoes, that is, compositions in which all cells of P are lexicographically smaller than all cells of Q. It can easily be verified that the minimum number of such compositions is obtained when both the rightmost column of P and the leftmost column of Q contain exactly two cells, with the same vertical gap between them. For such pairs of TC polyominoes, we have the four lexicographic compositions shown in Figure 7.

Consequently, we have that $4(\kappa(n))^2 \leq \kappa(2n)$. Then, Lemma 4.5 implies that any term of the form $(4\kappa(n))^{1/n}$ is a lower bound on λ_κ. Checking the known values of $\kappa(n)$, we see that $n = 35$ provides the best lower bound $\lambda_\kappa \geq (4\kappa(35))^{1/35} > 2.4474$. ◀

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