Coloring problems on arrangements of pseudolines*

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Abstract

Arrangements of pseudolines are a widely studied generalization of line arrangements. They are defined as a finite family of infinite curves in the Euclidean plane, any two of which intersect at exactly one point. One can state various related coloring problems depending on the number n of pseudolines. In this article, we show that n colors are sufficient for coloring the crossings avoiding twice the same color on the boundary of any cell, or, alternatively, avoiding twice the same color along any pseudoline. We also study the problem of coloring the pseudolines avoiding monochromatic crossings.

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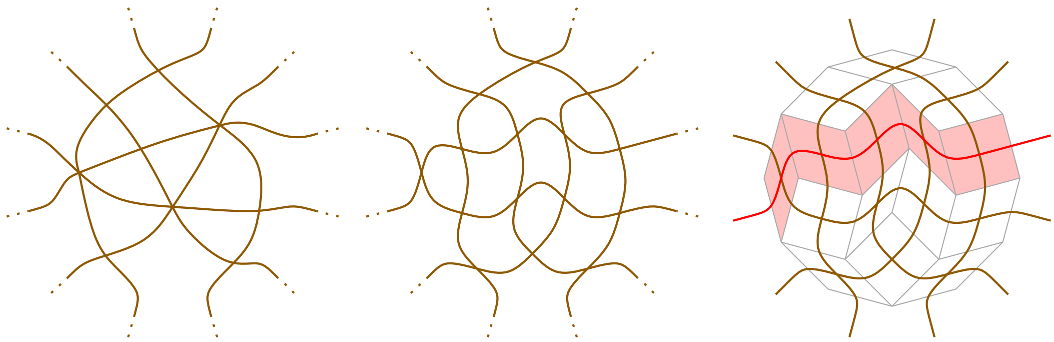
1 Introduction

An *arrangement of pseudolines* or *pseudoline arrangement* is a finite family of simple continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ in the Euclidean plane with

$$\lim_{t\to\infty} \|f_i(t)\| = \lim_{t\to-\infty} \|f_i(t)\| = \infty,$$

and the property that each pair f_i, f_j, $i \neq j$ crosses in exactly one point. A pseudoline arrangement is *simple*, if at most two pseudolines cross in a single point, see Figure 1a and Figure 1b for examples of a non-simple and a simple arrangement of 6 pseudolines.



(a) **(b)** **(c)**

Figure 1 A non-simple (a) and a simple (b) arrangement together with a corresponding tiling (c).

Pseudoline arrangements are widely studied objects. They were first described in 1926 by Levi [18] and were further studied by Ringel [19] and Grünbaum [13]. Every line arrangement is also a pseudoline arrangement. On the other hand, there exist arrangements of at least $n \geq 8$ pseudolines that cannot be „strechted“, i.e. they are not isomorphic to any line arrangement, see [19] and [12]. But pseudoline arrangements are not only a generalization of line arrangements: Isomorphism classes of simple arrangements are in correspondence with a

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rich variety of other objects, such as rhombic tilings of 2-dimensional zonotopes (indicated in Figure 1c), classes of reduced words of permutations and oriented matroids of rank 3. For a general introduction to pseudoline arrangements we refer to [8], [10] and [2, ch. 6].

1.1 Related work

In 2006, Felsner, Hurtado, Noy and Streinu [9] studied the *arrangement graph* $G_{\mathcal{A}}$ of a pseudoline arrangement $\mathcal{A}$, which consists of the crossings in $\mathcal{A}$ as vertices and its edges are formed by the arcs between them. They give a short argument that $G_{\mathcal{A}}$ can be colored using three colors if $\mathcal{A}$ is simple. As $G_{\mathcal{A}}$ is planar, it is clearly 4-colorable, including for non-simple arrangements. In [5] one can find an infinite family of line arrangements that require 4 colors.

In 2013, Bose et al. [3] introduced further coloring problems on line arrangements. An arrangement decomposes the Euclidean plane into *cells*: The example in Figure 1a consists of 7 *bounded cells* and 12 *unbounded cells*. One of the most remarkable results in [3] states that coloring the lines of a simple arrangement of n lines avoiding cells whose bounding lines have all the same color requires at most $\mathcal{O}(\sqrt{n})$ colors. This was improved to $\mathcal{O}(\sqrt{n/\log n})$ by Ackerman, Pach, Pinchasi, Radoičić and Tóth [1], extending it also to non-simple line arrangements. Finding line arrangements that require many colors in such a coloring seems to be a difficult task; in [3] they provide a construction that requires $\Omega\left(\log /n \log\log n\right)$ colors.

1.2 Results

In [3] and [1], the language of hypergraph coloring serves as a common formalization of the different coloring concepts and allows for the use of results from this field. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, a *vertex coloring* of $\mathcal{H}$ is a coloring of the vertices avoiding *monochromatic edges*, i.e. hyperedges whose contained vertices are assigned all the same color, while an *edge coloring* of $\mathcal{H}$ is a coloring of the hyperedges with no vertex being incident to two edges of the same color. The *(vertex) chromatic number* $\chi(\mathcal{H})$ is the minimal number of colors of a vertex coloring, while the *edge chromatic number* $\chi'(\mathcal{H})$ is the minimal number of colors of an edge coloring. Note that vertex coloring is not equivalent to edge coloring of the hypergraph dual.

Our results can all be stated in terms of two hypergraphs: The vertices of $\mathcal{H}_{\text{cell-vertex}}(\mathcal{A})$ are the (bounded and unbounded) cells of $\mathcal{A}$, and each crossing c defines a hyperedge consisting of the cells that contain c on their boundary. At the same time, the hypergraph $\mathcal{H}_{\text{line-vertex}}(\mathcal{A})$ is defined on the set of n pseudolines as vertices and each crossing in $\mathcal{A}$ defines a hyperedge consisting of the pseudolines involved in c. Section 2 is devoted to problems in which the crossings are colored. We show $\chi'(\mathcal{H}_{\text{cell-vertex}}) \leq n$ for every pseudoline arrangement:

▶ **Theorem 1.1.** *Let $\mathcal{A}$ be an arrangement of n pseudolines. The crossings of $\mathcal{A}$ can be colored using n colors so that no color appears twice on the boundary of any cell.*

The abovementioned results in [3] and [1] are bounds on the chromatic number of a hypergraph $\mathcal{H}_{\text{line-cell}}$ restricted to the case of line arrangements. However, none of the coloring problems that are discussed in [3] relates lines with crossings. This is done in the following two theorems, the first one of which shows $\chi'(\mathcal{H}_{\text{line-vertex}}) \leq n$:

▶ **Theorem 1.2.** *Let $\mathcal{A}$ be an arrangement of n pseudolines. The crossings of $\mathcal{A}$ can be colored using n colors so that no color appears twice along any pseudoline.*

Figure 2 shows an example of a coloring as guaranteed in Theorem 1.1 and in Theorem 1.2. In Section 3, we study the number of colors required to color the pseudolines avoiding monochromatic crossings. In addition to several minor results, we prove:

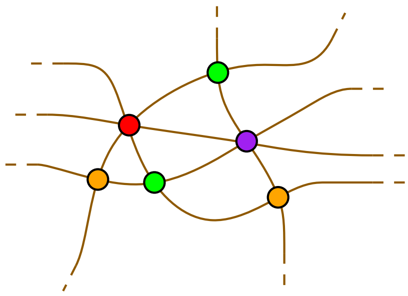


Figure 2 Coloring that fulfills the statements of both Theorem 1.1 and Theorem 1.2.

▶ **Theorem 1.3.** *Let $\mathcal{A}$ be an arrangement of n pseudolines. The pseudolines of $\mathcal{A}$ can be colored using $\mathcal{O}(\sqrt{n})$ colors avoiding monochromatic crossings of degree at least 4.*

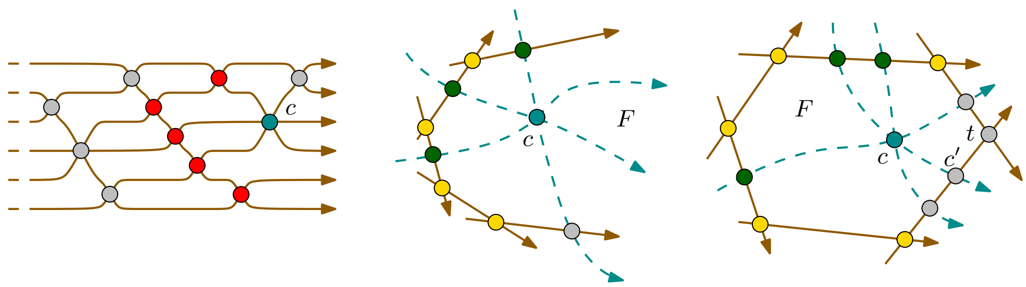
Here, the *degree* of a crossing is the number of pseudolines that intersect in said crossing.

2 Coloring crossings

In this section we sketch the proofs of Theorem 1.1 and Theorem 1.2. For complete proofs we refer to the full version.

2.1 Avoiding twice the same color on the boundary of any cell

A pseudoline arrangement can always be drawn in a way in which the pseudolines are x-monotone curves and no two crossings lie on a vertical line, see Figure 3a. This is also known as a *wiring diagram*, see [10]. We call this a *monotone drawing* and aim for coloring the crossings greedily from left to right. For any crossing c, a *conflict ancestor* is a crossing c' that lies left of c and both c and c' are on the boundary of a common cell. Figure 3a shows an example: The red crossings are conflict ancestors of c.



(a) (b) (c)

Figure 3 (a): Example for conflict ancestors; (b), (c): Case distinction for bounding their number.

Fix some crossing c in $\mathcal{A}$. Consider the arrangement $\mathcal{A}'$ obtained from $\mathcal{A}$ by dropping all pseudolines that contain c. In $\mathcal{A}'$ there is a cell F whose area contains the point c. All conflict ancestors of c lie on the boundary of F. By distinguishing the cases in which F is an unbounded or a bounded cell one can obtain that each crossing has at most $n-1$ conflict ancestors, see Figure 3b and Figure 3c and the proof in the full version. Theorem 1.1 now follows: Color the crossings from left to right. Using n colors, we can always avoid the colors

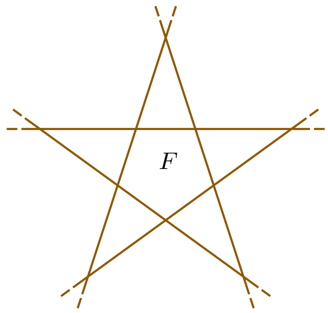


Figure 4 Construction that shows that Theorem 1.1 is tight.

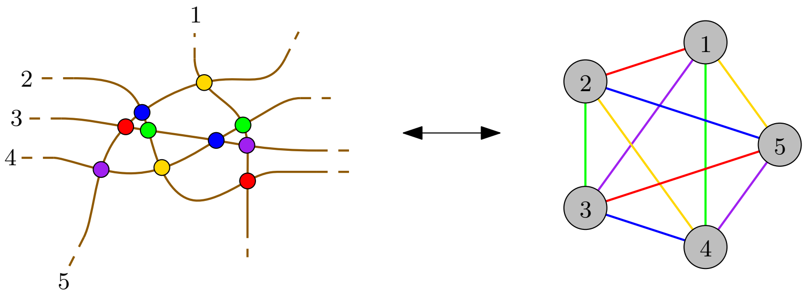


Figure 5 Coloring simple arrangements is equivalent to edge-coloring of K_n.

that were already assigned to conflict ancestors. It is easy to see that Theorem 1.1 is tight: There are arbitrarily large arrangements as in Figure 4 in which a cell F is incident to all pseudolines.

2.2 Avoiding twice the same color along any pseudoline

In view of Theorem 1.2, we now focus on coloring the crossings of a pseudoline arrangement $\mathcal{A}$ avoiding twice the same color along any pseudoline. If $\mathcal{A}$ is simple, this is equivalent to edge-coloring of the complete graph K_n (see Figure 5) and n colors are sufficient. We deduce the general case from the following recent breakthrough result by Kang et al. [15], which solved a longstanding conjecture by Erdős, Faber and Lovász [7].

▶ **Theorem 2.1** (D. Y. Kang, T. Kelly, D. Kühn, A. Methuku & D. Osthus, 2021). *For every simple hypergraph $\mathcal{H}$ with n vertices, $\chi'(\mathcal{H}) \leq n$.*

Here, a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *simple* if all hyperedges have cardinality at least 2 and for all $E_1, E_2 \in \mathcal{E}$, $E_1 \neq E_2$ it holds that $|E_1 \cap E_2| \leq 1$.

Proof of Theorem 1.2. The statement is equivalent to the existence of an edge-coloring of $\mathcal{H}_{\text{line-vertex}}$ using n colors. $\mathcal{H}_{\text{line-vertex}}$ is simple, because there can be at most one pseudoline passing through any pair of crossings, otherwise this would mean a pair of pseudolines crossing twice. Then the statement follows from Theorem 2.1. ◀

It would be nice to have a bound on the required number of colors that also takes into account how far the arrangement is away from being a simple arrangement. For this purpose, we introduce $\text{mx}(\mathcal{A})$, which is defined as the maximal number of crossings along any pseudoline in $\mathcal{A}$. For simple arrangements we have $\text{mx}(\mathcal{A}) = n - 1$. Excluding *trivial arrangements* where all pseudolines cross in a single point ($\text{mx}(\mathcal{A}) = 1$), it was shown

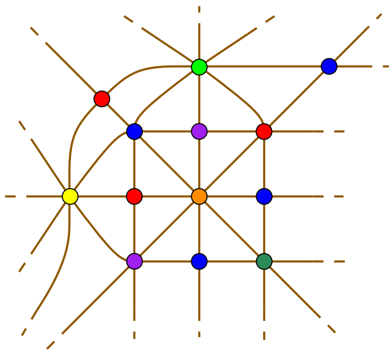


Figure 6 Minimal coloring using 7 colors of an arrangement with $\mathrm{mx} = 4$.

recently in [6] that the number of pseudolines is linearly upper bounded by $\mathrm{mx}(\mathcal{A})$, in particular $n \leq 845 \cdot \mathrm{mx}(\mathcal{A})$ for large values of n. Therefore, $\mathrm{mx}(\mathcal{A})$ can be interpreted as a measure of the size of an arrangement alternatively to the number of pseudolines n.

▶ Conjecture 1. There is a constant c so that the crossings of every pseudoline arrangement $\mathcal{A}$ can be colored using $\mathrm{mx}(\mathcal{A}) + c$ colors so that no color appears twice along any pseudoline.

Figure 6 shows an arrangement $\mathcal{A}$ with $\mathrm{mx}(\mathcal{A}) = 4$ that requires 7 colors. We were unable to find any arrangement where the gap between these numbers is larger than 3. The following proposition is a consequence of a result about hypergraph coloring by Kahn [14, 16]. It shows that Conjecture 1 holds at least asymptotically and under a certain restriction.

▶ Proposition 1. For every $k, \varepsilon > 0$, there exists an $\mathrm{mx}_0 \in \mathbb{N}$ so that the following holds: If a pseudoline arrangement $\mathcal{A}$ only contains crossings of degree at most k and fulfills $\mathrm{mx}(\mathcal{A}) \geq \mathrm{mx}_0$, then its crossings can be colored using $(1+\varepsilon) \cdot \mathrm{mx}(\mathcal{A})$ colors so that no color appears twice along any pseudoline.

3 Coloring pseudolines

Again, complete proofs for this section can be found in the full version of this article. A *pseudoline coloring* of an arrangement $\mathcal{A}$ is defined as a coloring of the pseudolines in $\mathcal{A}$ such that there are no monochromatic crossings, i.e. crossings of pseudolines of a single color class. We let $\chi_{pl}(\mathcal{A})$ denote the minimal number of colors in a pseudoline coloring of $\mathcal{A}$.

3.1 Pseudoline colorings and ordinary points

The study of pseudoline colorings is closely related to the study of *ordinary points*. An ordinary point is defined as a crossing of exactly two pseudolines, also known as *simple crossing*. Every non-trivial pseudoline arrangement contains at least $\lceil 6n/13 \rceil$ ordinary points [17]. Two pseudolines that cross each other in an ordinary point must be assigned different colors. Hence, for simple arrangements $\mathcal{A}$ we have $\chi_{pl}(\mathcal{A}) = n$. In the following we want to take a closer look at the relationship between $\chi_{pl}(\mathcal{A})$ and the structure of the ordinary points in a pseudoline arrangement. For this purpose, we define the *ordinary graph* $G_o(\mathcal{A})$ that has the n pseudolines of $\mathcal{A}$ as its vertices and two of them share an edge if and only if they cross each other in an ordinary point. Clearly, $\chi_{pl}(\mathcal{A}) \geq \chi(G_o(\mathcal{A}))$. Let $\sigma_k(n)$ denote the maximal number of ordinary points that an arrangement of n pseudolines $\mathcal{A}$ with $\chi_{pl}(\mathcal{A}) \leq k$ can have. Turán's theorem, applied on $G_o(\mathcal{A})$, gives us close bounds on $\sigma_k(n)$:

▶ Proposition 2. We have $\sigma_k(n) \in \Theta(n^2)$. More precisely, let $t_k(n)$ denote the Turán number, i.e. the maximum number of edges that a graph on n vertices without containing a $(k+1)$-clique can have. Then we have $t_k(n) - n \leq \sigma_k(n) \leq t_k(n)$.

We would like to know how much $\chi_{pl}(\mathcal{A})$ and $\chi(G_o(\mathcal{A}))$ can differ. We observe:

▶ Proposition 3. There are arbitrarily large arrangements with $\chi_{pl}(\mathcal{A}) = 2 \cdot \chi(G_o(\mathcal{A}))$.

It is unknown to us whether the factor of 2 in Proposition 3 can be further improved.

When it comes to the complexity of computing $\chi_{pl}(\mathcal{A})$ we have the following result:

▶ Proposition 4. Given an arrangement of pseudolines $\mathcal{A}$, it is NP-hard to compute $\chi_{pl}(\mathcal{A})$.

3.2 Avoiding monochromatic crossings of high degrees

Even though $\chi_{pl}(\mathcal{A})$ and $\chi(G_o(\mathcal{A}))$ can differ by a multiplicative factor, as stated in Proposition 3, for most arrangements, $\chi_{pl}(\mathcal{A})$ does not seem to be far away from $\chi(G_o(\mathcal{A}))$. This is why our focus lies now on a variant of pseudoline colorings: Instead of avoiding monochromatic crossings of any degree, including ordinary points, we only forbid crossings of certain degrees to be monochromatic. Lemma 3.1 can be proven using the Lovász Local Lemma.

▶ **Lemma 3.1.** *Let $l, r \in \mathbb{N}$ and let $\mathcal{A}$ be an arrangement of n pseudolines. Then, using*

$$\left(\frac{4(l+r)}{l-1}n\right)^{\frac{1}{l-1}} \in \mathcal{O}(n^{\frac{1}{l-1}})$$

colors, $\mathcal{A}$ can be colored avoiding monochromatic crossings of degree within $\{l, l+1, \cdots, l+r\}$.

Theorem 1.3 follows from Lemma 3.1 by first coloring the crossings of degree at most $\sqrt{n}$. If we only want to avoid monochromatic crossings of a single degree, then we can obtain a stronger result by applying a theorem by Frieze and Mubayi [11].

▶ Proposition 5. Let $\mathcal{A}$ be an arrangement of n pseudolines. Fix some $l \geq 3$. Then, the pseudolines in $\mathcal{A}$ can be colored using

$$c \cdot \left(\frac{\mathrm{mx}(\mathcal{A})}{\log \mathrm{mx}(\mathcal{A})}\right)^{\frac{1}{l-1}} \in \mathcal{O}\left(\left(\frac{n}{\log n}\right)^{\frac{1}{l-1}}\right)$$

colors avoiding monochromatic crossings of degree exactly l, where c only depends on l.

4 Conclusion and Future Work

We consider Theorem 1.1 as our main result. When coloring the crossings avoiding twice the same color along any pseudoline, Theorem 1.2 is a direct application of the recently proven Erdős-Faber-Lovász conjecture. However, for the specific hypergraphs induced by pseudoline arrangements, one could hope for a simple deterministic coloring procedure, like the one proposed in [4] that requires $\lceil (3/2)n - 2 \rceil$ colors.

We mentioned Conjecture 1 as an open problem. One may also ask whether for sufficiently large arrangements there always exists a coloring using n colors that satisfies the conditions of Theorem 1.1 and Theorem 1.2 simultaneously. When it comes to pseudoline colorings, we asked whether $\chi_{pl}(\mathcal{A})$ and $\chi(G_o(\mathcal{A}))$ can differ by a factor larger than 2. Finally, in view of Lemma 3.1, Theorem 1.3 and Proposition 5, we expect it to be possible to color the pseudolines of every arrangement using $\mathcal{O}(n^{\frac{1}{l-1}})$ colors avoiding monochromatic crossings of degree at least l.

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