Faces in Rectilinear Drawings of Complete Graphs*

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Abstract

We study extremal problems about faces in *convex rectilinear drawings* of K_n, that is, drawings where vertices are represented by points in the plane in convex position and edges by line segments between the points representing the end-vertices. We show that if a convex rectilinear drawing of K_n does not contain a common interior point of at least three edges, then there is always a face forming a convex 5-gon while there are such drawings without any face forming a convex k-gon with $k \geq 6$.

A convex rectilinear drawing of K_n is *regular* if its vertices correspond to vertices of a regular convex n-gon. We characterize positive integers n for which regular drawings of K_n contain a face forming a convex 5-gon.

To our knowledge, this type of problems has not been considered in the literature before and so we also pose several new natural open problems.

1 Introduction

Let G be a graph with no loops nor multiple edges. In a *rectilinear drawing of* G the vertices are represented by distinct points in the plane and each edge corresponds to a line segment connecting the images of its end-vertices. We consider only drawings where no three points representing vertices lie on a common line. As usual, we identify the vertices and their images, as well as the edges and the line segments representing them.

A *crossing* in a rectilinear drawing D of G is a common interior point of at least two edges of D where they properly cross. A *heavy crossing* in D is a common interior point of at least three edges of D where they properly cross. We say that D is *generic* if there are no heavy crossings in D. That is, crossings in a generic drawing D are the points where exactly two edges of D cross.

We focus on rectilinear drawings of complete graphs K_n on n vertices. We say that a rectilinear drawing D of a graph K_n is *convex* if the points representing the vertices of

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K_n are in convex position. We say that a convex drawing D of K_n is *regular* if the points representing the vertices of K_n form a regular n-gon; see Figure 1 for regular drawings of K_8 and K_{12}.

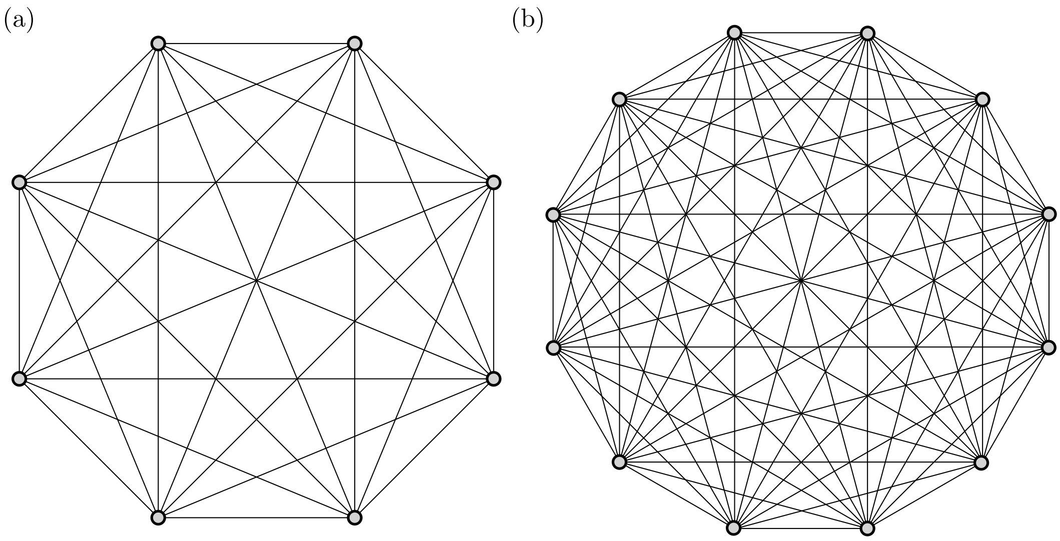

Figure 1 Regular drawings of K_8 (part (a)) and K_{12} (part (b)). Observe that none of these drawings contains a 5-face.

A *face* in a rectilinear drawing D of K_n is a non-empty connected component of $\mathbb{R}^2 \setminus D$. Note that exactly one face of D is unbounded and that every bounded face of D is a convex polygon. Thus, we can define the *size* of a bounded face F of D to be the number of vertices of the polygon that forms F. If the size of F equals k, then we call F a k*-face* of D.

In this paper, we study extremal problems about the bounded faces of a given size in convex drawings of K_n. To our knowledge, there has been no systematic study of this topic despite the fact that it offers an abundance of natural and interesting problems. For example, what is the largest face we can always find in a convex drawing of K_n for large n? What if we restrict ourselves to generic convex drawings of K_n? Or to regular drawings of K_n? In this paper, we address these questions and we pose several natural open problems.

2 Previous Work

Despite the fact that these problems are very natural and that rectilinear drawings of K_n have been studied extensively, we did not find any relevant reference in the literature. The existence of faces of a given size in regular drawings of K_n was recently considered by Shannon and Sloane [14], who computed the values from Table 1, but we are not aware of any publication. The total number of faces in a regular drawing of K_n was considered by Harborth [8] and Poonen and Rubinstein [12], but these results do not distinguish faces of different sizes and do not apply to all convex drawings of K_n. Finally, Hall [7] studied large faces in convex drawings of K_n where the vertices are points from the integer lattice.

Concerning other graph classes, Griffiths [6] calculated the number of regions enclosed by the edges of so-called regular drawings of the complete bipartite graphs $K_{n,n}$. There are also various results about the complexity of faces in the more general setting of line arrangements; for example [1, 2, 4, 5, 11]. However, we do not know any result that would imply the existence of large bounded faces in all convex drawings of sufficiently large K_n.

Closely related to our paper is the work of Poonen and Rubinstein [12] who gave a formula for the number of crossings in regular drawings of K_n and used it to count the number of faces in regular drawings of K_n. In particular, it follows from their formula that all regular drawings of K_n with odd n have $\binom{n}{4}$ crossings and thus are generic. They also showed that, apart from the center, no point is the intersection of more than 7 edges of a regular drawing of K_n for any positive integer n. We also note that these results are connected to the well-known *Blocking conjecture*; see [10, 13].

3 Our Results

First, we address the question about the maximum size of a face that we can always find in convex or regular drawings of K_n for large n. We observe that finding faces of size 3 or 4 in convex drawings of K_n is not difficult.

▶ **Proposition 3.1.** *Let n be a positive integer and D a convex drawing of K_n. Then, D contains a 3-face if and only if $n \geq 3$. Moreover, D contains a 4-face if and only if $n \geq 6$.*

To find larger faces, we restrict ourselves to generic convex drawings of K_n. In this case, we can show that a 5-face always exists if we have at least five vertices.

▶ **Theorem 3.2.** *For every positive integer n and every generic convex drawing D of K_n, the drawing D contains a 5-face if and only if $n \geq 5$.*

On the other hand, we can provide examples of generic convex drawings of K_n with arbitrarily large n that do not contain any k-face with $k \geq 6$.

▶ **Theorem 3.3.** *For every positive integer n, there is a generic convex drawing of K_n that does not contain any k-face with $k \geq 6$.*

Thus, in the case of generic convex drawings of K_n, we can settle the question about the largest face we can always find completely. A k-face with $k \in \{3, 4, 5\}$ is guaranteed in all sufficiently large drawings, while faces of sizes larger than 5 can be avoided (even simultaneously). The problem, however, becomes significantly more difficult if we allow heavy crossings.

We were not able to find a k-face with $k \geq 5$ in every sufficiently large convex drawing of K_n. In fact, finding larger faces becomes surprisingly difficult already for regular drawings of K_n. Here, however, we can at least show that a 5-face always exists in all sufficiently large regular drawings of K_n. In fact, we can even precisely characterize the values of n for which a regular drawing of K_n contains a 5-face.

▶ **Theorem 3.4.** *For a positive integer n, a regular drawing of K_n contains a 5-face if and only if $n \notin \{1, 2, 3, 4, 6, 8, 12\}$.*

The proof of Theorem 3.4 is quite involved and is based on the results obtained by Poonen and Rubinstein [12].

Finally, although we were not able to find a 5-face in all sufficiently large convex drawings of K_n, we can at least show that every convex drawing of K_7 contains at least one.

▶ **Proposition 3.5.** *Every convex drawing of K_7 contains a 5-face.*

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$a(k)$	3	6	5	9	7	13	9	29	11	40	13	43	15	212	17	231	19

Table 1 The values of $a(k)$, the smallest n such that the regular drawing of K_n contains a k-face, computed by Shannon and Sloane [14].

4 Open Problems and Discussion

The study of extremal questions about faces of a given size in convex drawings of K_n offers plenty of interesting and natural problems. Here, we draw attention to some of them.

Although we were able to determine the largest size of a bounded face that appears in every sufficiently large generic convex drawing of K_n, the same question remains unsolved for general convex drawings of K_n. In particular, the following problem is open.

▶ Problem 4.1. Is there a positive integer n_0 such that for every $n \geq n_0$ every convex drawing of K_n contains a 5-face?

Since the regular drawing of K_{12} does not contain a 5-face, we have $n_0 \geq 13$, if it exists. An affirmative answer to Problem 4.1 would imply that every sufficiently large *regular* drawing of K_n contains a 5-face, a fact that was quite difficult to prove.

Considering the regular drawings of K_n, although we proved that all sufficiently large regular drawings of K_n contain a 5-face, we do not know much about larger faces. It seems plausible that we can find arbitrarily large faces in regular drawings of K_n as n grows.

▶ Problem 4.2. Is it true that for every integer $k \geq 3$ there is an integer $n(k)$ such that every regular drawing of K_n with $n \geq n(k)$ contains a k-face?

For every integer k with $3 \leq k \leq 19$, Shannon and Sloane [14] computed the value $a(k)$, which is the smallest n such that the regular drawing of K_n contains a k-face; see Table 1. Note that even if $a(k)$ exists, $n(k)$ might not. Those computations suggest that the answer to Problem 4.2 might be positive. In such a case, it would be interesting to determine the growth rate of $n(k)$ with respect to k. It follows from Proposition 3.1 and Theorem 3.4 that $n(3) = 3$, $n(4) = 6$, and $n(5) = 13$. We encourage the reader to visit website[1] to see the regular drawings for themselves.

For k odd, we trivially have $a(k) = k$ as the regular drawing of K_n with n odd contains an n-face in the center. It might be interesting to explore the size of the largest faces in such drawings if we exclude this n-face.

A more difficult version of Problem 4.2 would be to determine, for a given $k \geq 3$, all values of n such that every regular drawing of K_n contains a k-face.

Another possible direction is to count the minimum number of k-faces in a convex drawing of K_n. For example, regarding 3-faces, it is simple to show that there are always at least $n(n-3)$ by considering the area of a convex drawing around its 3-face as long as $n \geq 3$, but what is the growth rate of the minimum number of 3-faces with respect to n?

▶ Problem 4.3. What is the minimum number of 3-faces in a convex drawing of K_n? What if the drawing is generic or regular?

In the whole paper, we focused on convex drawings. The problems we considered can also be stated for all rectilinear drawings of K_n. Here, we can show that every generic rectilinear drawing of K_n with $n \geq 10$ contains a k-face with $k \geq 5$. This follows easily since, by a result

[1] fklute.com/regularkn.html

of Harborth [9], every set P of at least 10 points in the plane without three collinear contains a *5-hole*, that is, a set H of 5 points in convex position with no point of P in the interior of the convex hull of H. If we then apply this result on the vertex set of a generic rectilinear drawing of K_n and use a similar reasoning as in the proof of Theorem 3.2 on the drawing induced by the resulting 5-hole in D, then we find a bounded face of size at least 5 in D.

Finally, we considered the problem of finding a bounded face of size exactly k for a given integer k, but it also makes sense to consider more relaxed variants of the above problems where we want to find a bounded face of size at least k for a given integer k. In particular, this leads to the following potentially simpler variant of Problem 4.1.

▶ Problem 4.4. Is there a positive integer n_1 such that for every $n \geq n_1$ every convex drawing of K_n contains a bounded face of size at least 5?

We note that a simple double-counting argument based on Euler's formula yields the existence of k-faces in generic convex drawings of K_n with $k \geq 4$. If we knew that there are many 3-faces in such drawings, then the argument gives the existence of k-faces with $k \geq 5$. This also illustrates that some insight for Problem 4.3 might have consequences for our original questions.

5 Proof of Theorem 3.3

We prove that, for every positive integer n, there is a generic convex drawing of K_n that does not contain a k-face with $k \geq 6$. We apply a similar construction to the one used by Balko et al. [3].

First, we state some auxiliary definitions. For an integer $k \geq 3$, a set of k points in the plane is a *k-cup* if all its points lie on the graph of a convex function. Similarly, a set of k points is a *k-cap* if all its points lie on the graph of a concave function. Clearly, k-cups and k-caps are sets of points in convex position. A convex polygon P is *k-cap free* if no k vertices of P form a k–cap. Note that P is k-cap free if and only if it is bounded from above by at most $k-2$ segments (edges of P). Analogously, P is *k-cup free* if no k vertices of P form a k–cup. Observe that vertices of a k-face determine an a-cap and a u-cup that share the leftmost and the rightmost vertex and satisfy $a+u=k+2$. We use $e(P)$ to denote the leftmost edge bounding P from above; see part (a) of Figure 2.

We inductively construct a certain generic convex drawing D_n of K_n with vertices represented by points $p_1,\dots,p_n$ that form an n-cup in the plane and their x-coordinates satisfy $x(p_i)=i$; see part (b) of Figure 2. Let $V(D_n)$ denote the vertex set of D_n. We recall that we identify the vertices of K_n and the points from D_n representing them. We let $V(D_1)=\{(1,0)\}$ and $V(D_2)=\{(1,0),(2,0)\}$. Now, assume that we have already constructed the drawing D_{n-1} with $V(D_{n-1})=\{p_1,\dots,p_{n-1}\}$ for some integer $n \geq 3$. We choose a sufficiently large number y_n, and we let p_n be the point (n,y_n). We then set $V(D_n)=V(D_{n-1})\cup\{p_n\}$ and we let D_n be the drawing of K_n on this vertex set. The number y_n is chosen large enough so that the following three conditions are satisfied:

1. for every $i=1,\dots,n-1$, every intersection point of two line segments spanned by points from $V(D_{n-1})$ lies on the left side of the line $\overline{p_ip_n}$ if and only if it lies to the left of the vertical line $x=i$ containing the point p_i,
2. if F is a 4-cap free face of D_n that is not 3-cap free, then there is no point p_i below the (relative) interior of $e(F)$,
3. no crossing of two edges of D_n lies on the vertical line containing some point p_i.

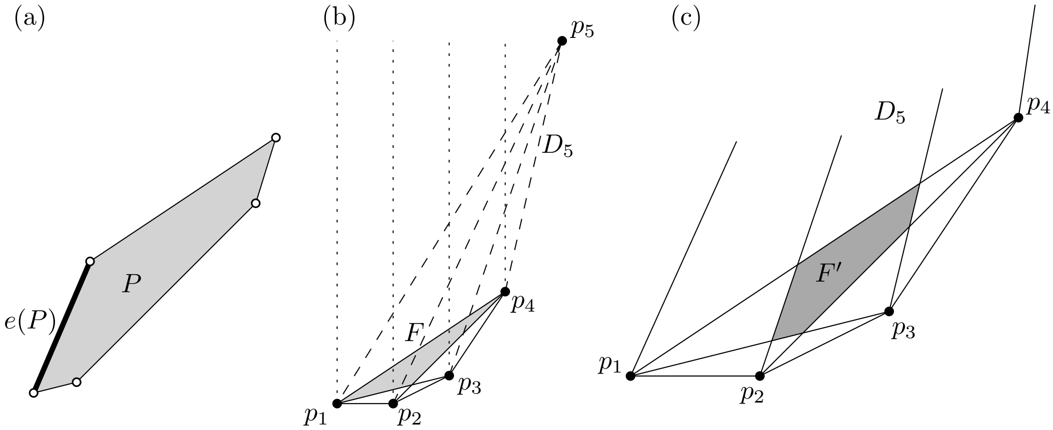

Figure 2 (a) A 4-cap free and 5-cup free polygon P that is not 3-cap free nor 4-cup free. (b) A construction of the drawing D_n for $n = 5$. If the point p_n is chosen sufficiently high above $V(D_{n-1})$, then each line segment $\overline{p_i p_n}$ with $i < n$ is very close to the vertical line containing p_i and thus all faces of D_n will be 4-cap free and 5-cup free. (c) The face F of D_{n-1} is split into new faces of D_n and contains the face F' that is 4-cap free and 5-cup free but not 3-cap free nor 4-cup free.

Choosing the point p_n is indeed possible as for a sufficiently large y-coordinate y_n of p_n we get that for each i, all the intersections of the line segments $p_i p_n$ with line segments of D_{n-1} lie very close to the vertical line $x = i$ containing the point p_i. Note that no line segment of D_n is vertical and that there are no heavy crossings in D_n. Since $p_1, \ldots, p_n$ form an n-cup, they are in convex position and D_n is a generic convex drawing of K_n.

It remains to prove that there are no k-faces with $k \geq 6$ in D. To show that, we use the following lemma.

▶ **Lemma 5.1.** *Each bounded face of D_n is a 4-cap free and 5-cup free convex polygon.*

Now, suppose for contradiction that there is a k-face F in D_n for some integer $k \geq 6$. By Lemma 5.1, the face F is a 4-cap free and 5-cup free convex polygon. On the other hand, the vertex set of F is in convex position and thus determines an a-cap and a u-cup that share the leftmost and the rightmost vertex and satisfy $a + u \geq 8$. Therefore, we either have $a \geq 4$ or $u \geq 5$, However, this contradicts the fact that F is 4-cap free and 5-cup free.

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