Counting Pseudoline Arrangements*

Fernando Cortés Kühnast, Stefan Felsner, and Manfred Scheucher

Institut für Mathematik, Technische Universität Berlin, Germany
lastname@math.tu-berlin.de

Abstract

Arrangements of pseudolines are classic objects in discrete and computational geometry. They have been studied with increasing intensity since their introduction almost 100 years ago. The study of the number B_n of non-isomorphic simple arrangements of n pseudolines goes back to Goodman and Pollack, Knuth, and others. It is known that B_n is in the order of $2^{\Theta(n^2)}$ and finding asymptotic bounds on $b_n = \frac{\log_2(B_n)}{n^2}$ remains a challenging task. In 2011, Felsner and Valtr showed that $0.1887 \leq b_n \leq 0.6571$ for sufficiently large n. The upper bound remains untouched but in 2020 Dumitrescu and Mandal improved the lower bound constant to 0.2083. Their approach utilizes the known values of B_n for up to $n = 12$.

We tackle the lower bound with a dynamic programming scheme. Our new bound is $b_n \geq 0.2526$ for sufficiently large n. The result is based on a delicate interplay of theoretical ideas and computer assistance.

1 Introduction

Levi [12] introduced arrangements of pseudolines as a natural generalization of line arrangements in 1926. An *arrangement of pseudolines* in the Euclidean plane $\mathbb{R}^2$ is a finite family of simple curves, called pseudolines, such that each curve approaches infinity in both directions and every pair intersects in exactly one point where the two curves cross. More generally, we call a collection of pseudolines *partial arrangement* if every pair intersects in at most one crossing-point. Pseudolines which do not intersect are said to be *parallel*. Note that, while for partial arrangements of *proper* lines the relation 'parallel' is transitive, this is no longer true in partial pseudoline arrangements.

In this article, the focus will be on *simple* arrangements, that is, no three or more pseudolines intersect in a common point (called *multicrossing*). Moreover, we consider all arrangements to be *marked*, that is, they have a unique marked unbounded cell, which is called *north-cell*. Two arrangements are *isomorphic* if one can be mapped to the other by an orientation preserving homeomorphism of the plane that also preserves the north-cell.

While it is known that the number B_n of non-isomorphic arrangements of n pseudolines grows as $2^{\Theta(n^2)}$, it remains a challenging problem to bound the multiplicative factor of the leading term of $\log_2 B_n = \Theta(n^2)$. Our focus will be on finding better estimates on the lower bound constant $c^- := \liminf_{n\to\infty} \frac{\log_2 B_n}{n^2}$. One can analogously define the upper bound constant $c^+ := \limsup_{n\to\infty} \frac{\log_2 B_n}{n^2}$ but it seems to be open whether c^+ and c^- coincide.

In the 1980's Goodman and Pollak [9] investigated pseudopoint configurations, which are dual to pseudoline arrangements, and established the lower bound $c^- \geq \frac{1}{8}$. An alternative and simpler construction for $c^- \geq \frac{1}{12}$ can be found in Matoušek's textbook [13, Chapter 6].

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

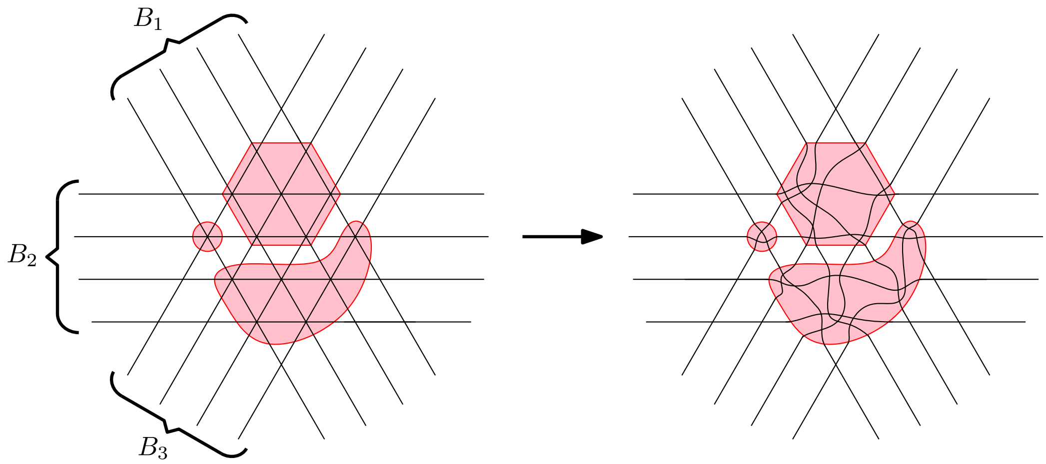


Figure 1 Left: An arrangement of 3 bundles of parallel lines and a collection of interior-disjoint patches (highlighted red) such that each multicrossing point is covered by a patch. Right: A partial pseudoline arrangement with the same parallel bundles obtained by rerouting within the patches.

Concerning the upper bound, Edelsbrunner, O'Rourke and Seidel [6] showed $c^+ < \infty$. In the 1990's Knuth [10, Section 9] improved the bounds to $c^- \geq \frac{1}{6}$ and $c^+ < 0.7925$, and he conjectured that $c^+ \leq 0.5$. The upper bound was lowered to $c^+ < 0.6974$ by Felsner [7], and in 2011, Felsner and Valtr [8] further narrowed the gap by showing $c^- > 0.1887$ and $c^+ < 0.6571$. In 2020 Dumitrescu and Mandal [5] proved the currently best lower bound $c^- > 0.2083$.

In this article, we make a substantial step on the lower bound by proving $c^- > 0.2526$.

▶ **Theorem 1.1.** *The number B_n of non-isomorphic simple arrangements of n pseudolines satisfies the inequality $B_n \geq 2^{cn^2 - O(n \log n)}$ with $c > 0.2526$.*

2 Outline

Our approach is in the spirit of several previous bounds. We consider a specific partial arrangement $\mathcal{L}$ of n lines consisting of k *bundles* $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of parallel lines. We then define a class of local perturbations to $\mathcal{L}$ and consider the number of arrangements that can be obtained by these perturbations. This number is a lower bound on B_n, and it can be improved by recursively applying the same construction to each of the parallel classes $\mathcal{L}_i$.

The main difference between the approaches lies in the number of bundles k and the notion of locality. Matoušek and also Felsner and Valtr used three bundles but the locality was increased from considering just a triple intersection with its two simple resolutions to the full intersection pattern of three bundles. Dumitrescu and Mandal [5] increased the number k of bundles to up to 12 but still restricted to local resolutions of multicrossings.

Our approach combines higher values of k with an increased locality for the perturbations. As illustrated in Figure 1, we allow reroutings of the arrangement within designated regions, which we call *patches*. When rerouting the arrangement within a patch P, the order of the crossings along the pseudolines may change. The boundary information of P fully determines which pairs of pseudolines cross within P, but the order of crossings along the pseudolines is not determined in general. Outside of P, the arrangement remains unaffected, which allows us to count the number of reroutings for each patch independently. The total number of perturbations is obtained as the product of the numbers computed for the individual

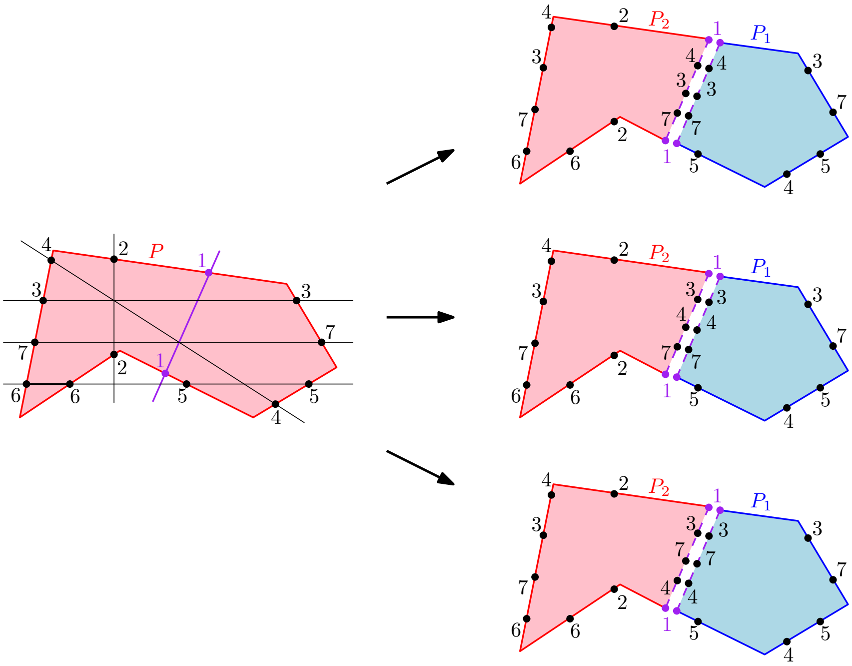


Figure 2 An illustration of how to recursively compute the number of reroutings for a patch P. When cutting along segment 1, highlighted purple, there are intersections with the segments 3, 4, and 7. As the segments 3 and 7 do no cross within P, there are only three possibilities for placing the three crossings along the segment 1, namely 4–3–7 (right top), 3–4–7 (right center) and 3–7–4 (right bottom).

patches. The number of possibilities within a patch are computed recursively via dynamic programming; Figure 2 gives an illustration. Details are given in [11].

To eventually use computer assistance, we choose patches of high regularity and reasonably small complexity. In fact, since our construction is highly regular, it is sufficient to determine the rerouting possibilities only for a small number of patch-types. Only a negligible fraction of patches along the boundaries are different. As we only want to find an asymptotic lower bound on B_n, the small number of irregular patches along the boundary of the regions will not be used in the counting.

To eventually prove Theorem 1.1, we perform the following two steps:

- In the first step (Section 3) we specify the parameters of the construction: We construct $k = 6$ bundles of $\lfloor \frac{n}{k} \rfloor$ parallel lines (see [11] for a description of the approach with $k = 4$ bundles) and cover the multicrossing points by patches. By resolving the multicrossing points within the patches, and taking the product over all patches we obtain an improved lower bound on the number $F_k(n)$ of partial arrangements with k bundles of $\lfloor \frac{n}{k} \rfloor$ parallel pseudolines.
- In the second step (Section 4), we account for crossings in bundles of pseudolines which had been parallel before. The product of the so-computed possibilities yields the improved lower bound on the number B_n of simple arrangements on n pseudolines.

3 Step 1: bundles of parallel lines, patches, and perturbations

For the start we fix an integer k and construct an arrangement $\mathcal{L}$ of k bundles of $\lfloor \frac{n}{k} \rfloor$ parallel lines as in [5]. If n is not a multiple of k, the remaining lines are discarded, or not used in the counting. We then cover all multicrossing points by a family of disjoint regions, called *patches*, and reroute the line segments within the patches so that all multicrossing points will eventually be resolved and the arrangement becomes simple.

3.1 Construction with 6 bundles

In this section we consider a partial arrangement $\mathcal{L}$ of n lines consisting of 6 bundles of $\lfloor \frac{n}{6} \rfloor$ parallel lines $\mathcal{L}_1, \ldots, \mathcal{L}_6$ following [5]. See Figure 3 for an illustration. The construction comes with four types of regions with multicrossings:

- R_i for $i \in \{3, 4, 5\}$ only contains multicrossings of order i and
- R_6 contains multicrossings of order 3 and 6.

Note that multicrossings of order 3 occur in R_3 and R_6.

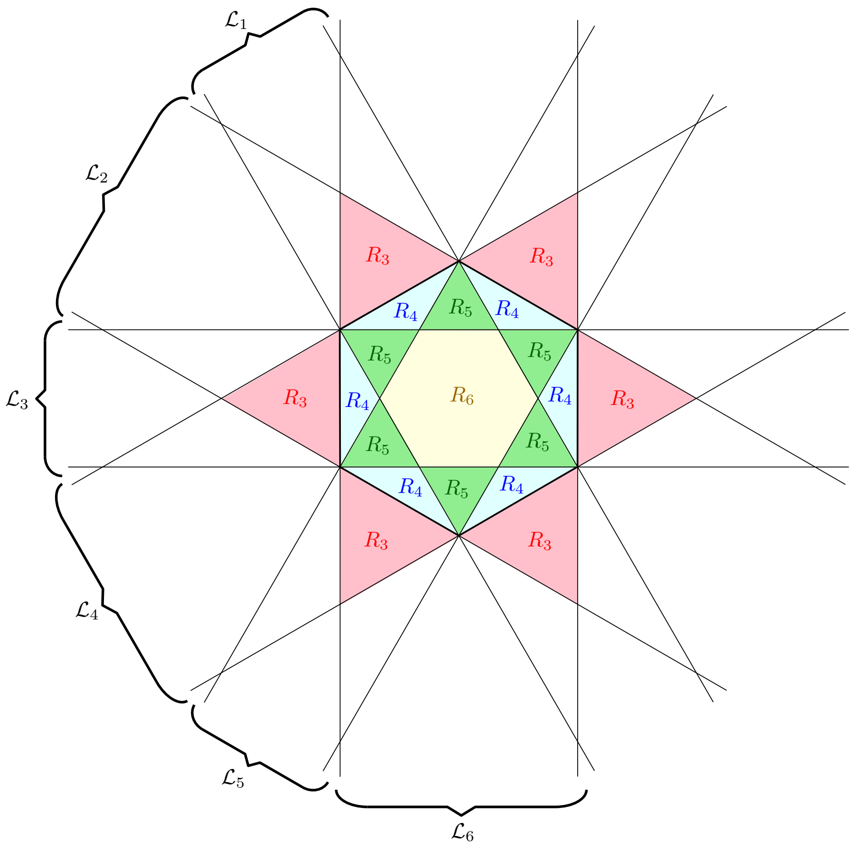


Figure 3 Construction with 6 bundles as in [5].

For each of the four regions R_i we will use a different type of patch P_i that is based on a regular tiling of the plane to ensure regularity; see Figure 4.

We have to determine the number μ_i of patches of type i. Since the number of crossings of each order is asymptotically quadratic in n and each patch contains only a constant number of crossings, the number μ_i of patches of type i is also quadratic. Again, it is important to note that the patches along the boundary of R_i behave differently. However, since there are only linearly many of these deformed patches, they only affect lower order error terms. Hence we can omit them in the calculations.

To obtain asymptotically tight estimates on the μ_i's, we make use of the numbers $\lambda_i(n)$ of i-crossing points, which were determined by Dumitrescu and Mandal [5, Table 2]:

$$\lambda_3(n) = \tfrac{5n^2-O(n)}{144}, \qquad \lambda_4(n) = \tfrac{n^2-O(n)}{144}, \qquad \lambda_5(n) = \tfrac{n^2-O(n)}{144}, \qquad \lambda_6(n) = \tfrac{n^2-O(n)}{144}.$$

For $i = 4, 5, 6$, the number λ_i coincides with $\mu_i \cdot \#\{i\text{-fold crossings in } P_i\} + O(n)$ because only the region P_i contains i-crossings for $i = 4, 5, 6$. For $i = 3$, however, the situation is a bit more complicated because P_3 and P_6 both contains 3-crossings. More specifically, P_6 contains twice as many 3-crossings as 6-crossings. With the multiplicities given in the caption of Figure 4 we obtain:

- $\mu_3(P_3, n) = \frac{\lambda_3(n)-2\cdot\lambda_6(n)}{\#\{3\text{-crossings in } P_3\}} - O(n) = \frac{3n^2}{144\cdot 100} - O(n)$
- $\mu_4(P_4, n) = \frac{\lambda_4(n)}{\#\{4\text{-crossings in } P_4\}} - O(n) = \frac{n^2}{144\cdot 32} - O(n)$
- $\mu_5(P_5, n) = \frac{\lambda_5(n)}{\#\{5\text{-crossings in } P_5\}} - O(n) = \frac{n^2}{144\cdot 12} - O(n)$
- $\mu_6(P_6, n) = \frac{\lambda_6(n)}{\#\{6\text{-crossings in } P_6\}} - O(n) = \frac{n^2}{144\cdot 7} - O(n)$

To compute the numbers $F(P_i)$ of all possible perturbations within the patch type P_i for $i = 3, 4, 5, 6$, we ran our program and obtained:

- $F(P_3) = 1956055471674766249002559523437101670400$
- $F(P_4) = 10233480626615962155895931163981261674$
- $F(P_5) = 3220707785549754650813274O267$
- $F(P_6) = 8129606100972933137253330355173$

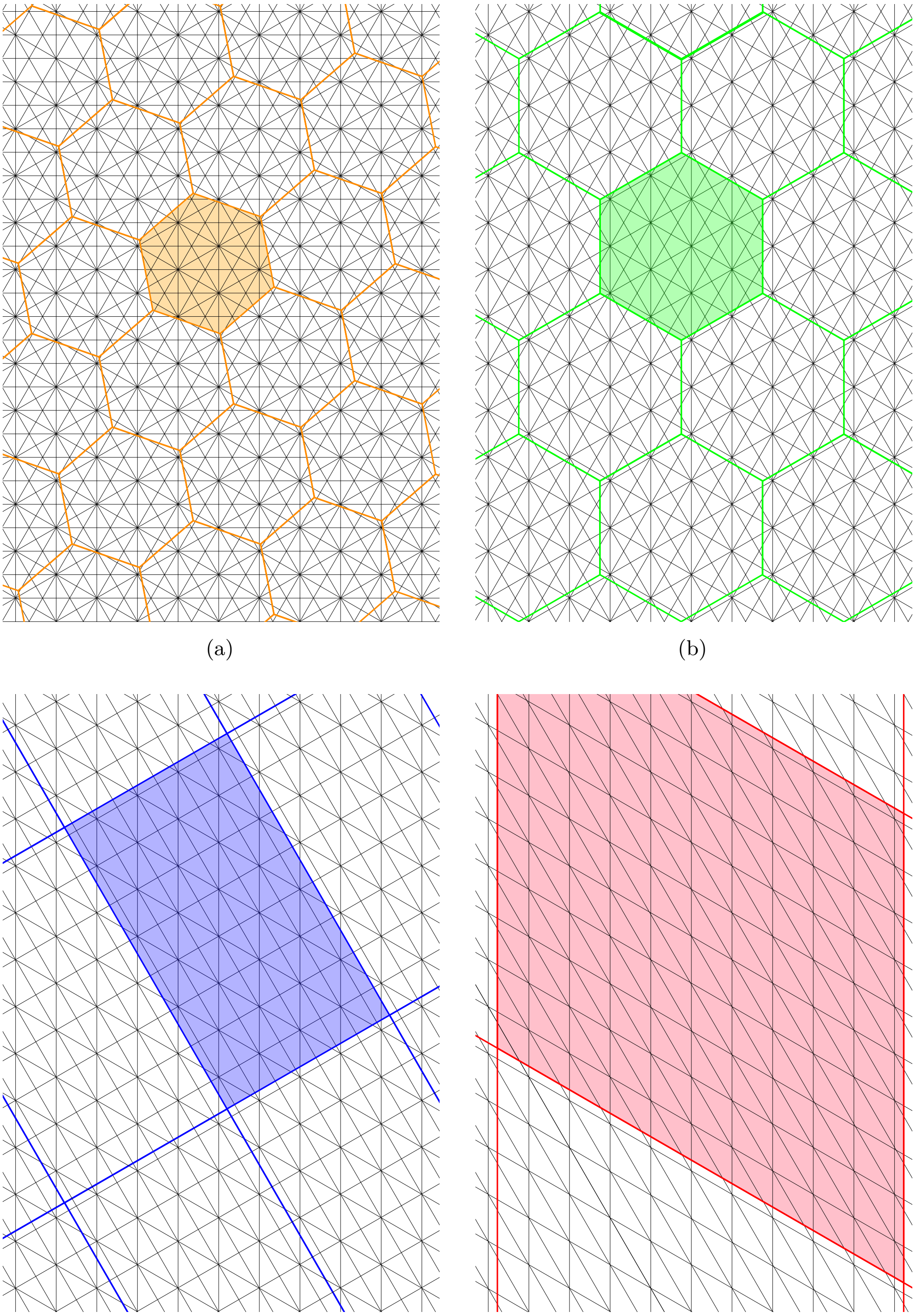
We provide a computer-assisted framework [1] that can fully automatically compute $F(P)$ for a given patch P, which is given as an IPE input file [2]. See [11] for more details. The presented terms were computed within a few CPU hours on cluster nodes of TU Berlin with up to 1TB of RAM. We also provide simpler patches for which the program only needs few CPU seconds and low RAM. Those, however, give slightly worse bounds.

From $F_k(n) \geq \prod_{i=3}^{k} F(P_i)^{\mu_i(n)}$, we can now derive:

▶ **Proposition 3.1.** *$F_6(n) \geq 2^{cn^2-O(n)}$ with $c > 0.2105$.*

More specifically, by writing $c_i := \lim_{n\to\infty} \frac{\mu_i(n)}{n^2} \cdot \log_2(F(P_i))$, we can see the contributions of the patches P_3, P_4, P_5 and P_6 to the leading constant $c = c_3+c_4+c_5+c_6$ from Proposition 3.1:

$$c_3 \approx 0.0272, \qquad c_4 \approx 0.0267, \qquad c_5 \approx 0.0548, \qquad c_6 \approx 0.1019.$$



(a) (b) (c) (d)

Figure 4 The four types of patches for our construction on $k = 6$ bundles:
(a) For R_6 we use a hexagonal tiling where each patch P_6 contains exactly 7 crossings of order 6 and 14 crossings of order 3.
(b) For R_5 we use a hexagonal tiling where each patch P_5 contains exactly 12 crossings of order 5.
(c) For R_4 we use a rectangular tiling where each patch P_4 contains exactly 32 crossings of order 4.
(d) For R_3 we use a rhombic tiling where each patch P_3 contains exactly 100 crossings of order 3.

4 Step 2: resolving parallel bundles

With the second and final step, we want to obtain a simple arrangement of pairwise intersecting pseudolines from a partial arrangement of k bundles of $m = \lfloor \frac{n}{k} \rfloor$ parallel pseudolines. To do so, we use a recursive scheme as in [8, 5] to make each pair of parallel pseudolines cross: For each $i = 1, \ldots, k$, we consider a disk D_i such that

(1) D_i intersects all parallel pseudolines of the bundle $\mathcal{L}_i$ and no other pseudolines, and
(2) no two disks overlap.

Within each disk D_i we can place any of the B_m arrangements of m pseudolines. This makes all the pseudolines of a bundle cross. Figure 5 gives an illustration for the case $k = 3$.

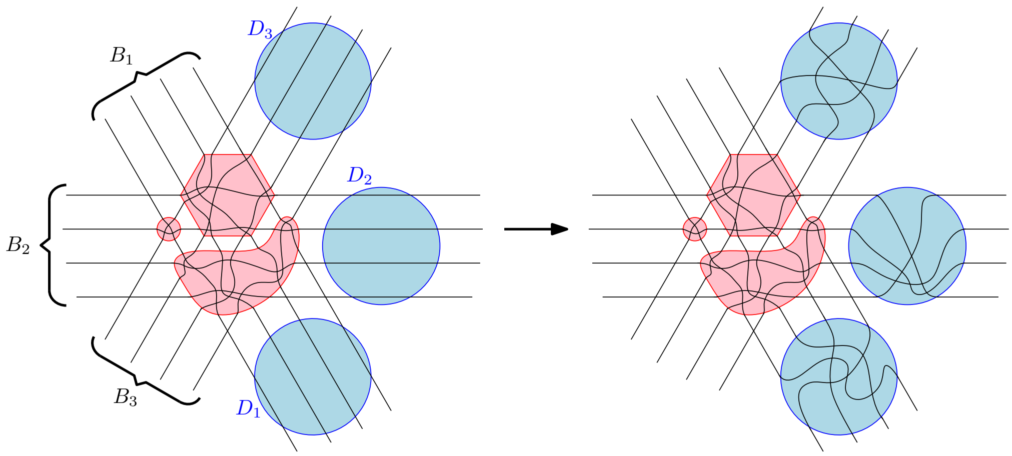


Figure 5 Left: A partial arrangement of 3 bundles of parallel pseudolines and a collection of interior-disjoint disks (highlighted blue) such that each bundle is covered by one disk.
Right: A proper pseudoline arrangement obtained by rerouting within the disks.

Since all $D_i's$ are independent and there are B_m possibilities to reroute within each D_i,

$$B_n \geq \underbrace{F_k(n)}_{\text{Step 1}} \cdot \underbrace{(B_m)^k}_{\text{Step 2}}$$

holds, where $m = \lfloor \frac{n}{k} \rfloor$. With the following lemma we can derive $c^- \geq \frac{k}{k-1}c$ where c is the constant obtained in Section 3. The construction with $k = 6$ bundles gives the lower bound $c^- > 0.2526$, and therefore completes the proof of Theorem 1.1.

▶ **Lemma 4.1.** *If $F_k(n) \geq 2^{cn^2 - O(n)}$ for some $c > 0$ then $B_n \geq 2^{\frac{k}{k-1}cn^2 - O(n \log n)}$.*

5 Discussion

We performed quite some experiments to optimize the set of parameters. To obtain the new lower bound constant $c^- > 0.2526$ presented in Theorem 1.1, we started with the $k = 6$ parallel bundles construction from [5] and covered the multicrossings with a specific selection of patches, which were inspired by regular tilings. Already the construction with $k = 4$ bundles gives $F_4(n) \geq 2^{cn^2 - O(n)}$ with $c > 0.1608$ and $c^- > 0.2144$ (see [11]), which is already an improvement to the previous best bound by Dumitrescu and Mandal [5]. While the results from [5] suggest that larger values of k give better bounds, the computations get more and more complex. In fact, as the number k increases, the complexity of the patches increases. Since our program can only deal with patches containing about 30 to 40 segments in reasonable time, depending on the structure of crossings within it, there is a trade-off between the number of crossings within a patch and the number of bundles k in practice. This was also the reason why we use different types of patch for the four regions.

In the future we plan to investigate constructions with $k = 8$ and $k = 12$ bundles which as depicted in [5, Figures 9 and 13] come with more types of regions. It remains a challenging part to find a good tiling/patches for each of them.

Also note that as long as one fixes k, the counting approach is implicitly limited by $F_k(n)$, which is much smaller than B_n. Since $F_3(n) = 2^{cn^2+o(n^2)}$ with $c = \frac{\log_2(3)}{2} - \frac{2}{3} \approx 0.1258$ is known [8], it would be interesting to determine $\lim_{n\to\infty} \frac{\log_2 F_i(n)}{n^2}$ for $i = 4, \ldots, 12$. In particular, we wonder how far from the truth the constant in Proposition 3.1 is.

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