Bipartite Dichotomous Ordinal Graphs[*] [†]

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Abstract

A *dichotomous ordinal graph* consists of an undirected graph $G = (V, E_s \cup E_\ell)$ with an ordered partition of the set of edges into a set E_s of *short* edges and a set E_ℓ of *long* edges. A *geometric representation* of a dichotomous ordinal graph is a straight-line drawing Γ of G such that the short edges of G are exactly those edges that have length at most one in Γ.

We characterize for which bipartite graphs all ordered partitions of the edge set admit a geometric representation as a dichotomous ordinal graph. On the one hand, such a representation always exists if the graph is a subgraph of $K_{3,m}$, for an arbitrary m, or a subgraph of $K_{4,6}$. On the other hand, there exist dichotomous ordinal $K_{4,7}$ and $K_{5,5}$ that do not admit a geometric representation. Moreover, any bipartite dichotomous ordinal graph admits a geometric representation if the short edges induce an outerplanar graph and any dichotomous ordinal graph admits a geometric representation if the short edges induce a subgraph of the rectangular grid.

1 Introduction

A *dichotomous ordinal graph* consists of an undirected graph $G = (V, E_s \cup E_\ell)$ with a partition of the edges into a set E_s of *short* edges and a set E_ℓ of *long* edges. A *geometric representation* of a dichotomous ordinal graph is a straight-line drawing Γ of G such that the short edges of G are exactly those edges that have length at most one in Γ. Fig. 1 shows two straight-line drawings of the same dichotomous ordinal graph. The drawing in (a) is a geometric representation of it, whereas the drawing in (b) is not.

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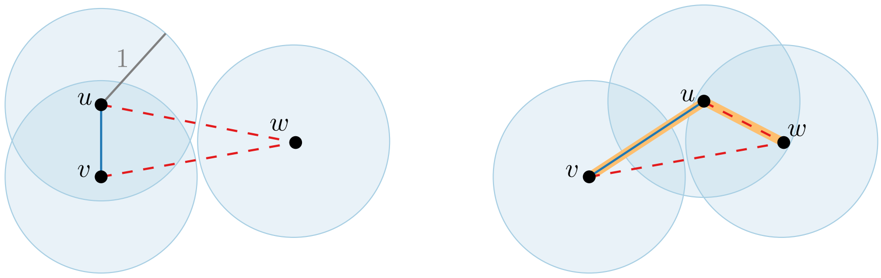


(a) valid drawing of $\triangle uvw$ **(b)** invalid drawing of $\triangle uvw$

Figure 1 valid (a) and invalid (b) drawing of a dichotomous ordinal triangle $\triangle uvw$; short edge uv (blue) and long edges uw and vw (red/dashed)

Related Results. It is NP-hard to decide whether a dichotomous ordinal graph admits a geometric representation, even if the underlying graph is a complete graph and the short edges induce a planar graph [1, Lemma 1] or the underlying graph is a complete bipartite graph [10, Theorem 4]. In the latter case, the problem is even known to be $\exists\mathbb{R}$-complete [10].

Angelini et al. [2] investigated for which graphs G any ordered partition of the edges admits a geometric representation as a dichotomous ordinal graph. This is the case if G is a *double-wheel* (a simple cycle and two additional vertices connected to all vertices of the cycle), *2-degenerate* (can be reduced to the empty graph by repeatedly removing vertices of degree at most two), *subcubic* (each vertex has degree at most three), or 4-colorable and the short edges induce a *caterpillar* (tree such that the removal of degree one vertices yields a path). On the negative side, they [2] proved that if G is the double-wheel plus one edge, then there exists a partition of the edge set of G into short and long edges that doesn't admit a geometric representation as a dichotomous ordinal graph.

Closely related is the notion of *ordinal embeddings*. Given a set of objects $x_1, \dots, x_n$ in an abstract space together with a set of *ordinal constraints* of the form $dist(x_i, x_j) < dist(x_k, x_l)$, we are asked to compute a set of points $p_1, \dots, p_n$ in the d-dimensional Euclidean space $\mathbb{R}^d$ such that, by preserving as many ordinal constraints as possible, it returns a good approximation of the displacement of $x_1, \dots, x_n$. Ordinal embeddings were first studied in the 60's by Shepard [11, 12] and Kruskal [8, 9] in the context of psychometric data analysis. Recently, there have been applications in the field of Machine Learning [14]. The computation of ordinal embeddings is also known in the literature as *non-metric multi-dimensional scaling*. For an extensive literature review on ordinal embeddings refer to [15].

Of particular interest in relation to our work is the application of ordinal embeddings in the problem of recognizing Euclidean Multidimensional preferences [3, 5, 10] in the field of Computational Social Science. The objects are either *voters* or *alternatives*, which, together with the ordinal constraints (i.e., the voters' preferences), naturally define a bipartite graph. However, the goal is to find an embedding in $\mathbb{R}^d$ where all constraints are satisfied rather than to seek for an approximation. Efficient algorithms exist when $d = 1$ [4, 5], while for any $d \geq 2$ the problem is as hard as the existential theory of the reals [10]. The case where a voter either *likes* or *dislikes* a preference has also been studied [6, 10]. Note that, in this setting, an embedding that employs short and long edges can fully represent the likeness/dislikeness of voters to alternatives. This is precisely the problem this paper is devoted to.

Our Results. A dichotomous ordinal graph $G = (U \cup W, E_s \cup E_\ell)$ is *bipartite* if $E_s \cup E_\ell \subseteq U \times W$. We study in particular complete bipartite dichotomous ordinal $K_{n,m}$, i.e., bipartite graphs $G = (U \cup W, E_s \cup E_\ell)$ with $|U| = n$, $|W| = m$, and $E_s \cup E_\ell = U \times W$. We show that subgraphs of dichotomous ordinal $K_{3,m}$, $m \in \mathbb{N}$ or $K_{4,6}$ always admit a geometric representation (Theorem 2.1) while there are dichotomous ordinal $K_{4,7}$ (Theorem 2.2) and $K_{5,5}$ (Theorem 2.3) that do not admit a geometric representation. Further, a bipartite dichotomous ordinal graph always admits a geometric representation if the short edges induce an outerplanar graph (Theorem 3.1) or a subgraph of the grid (Theorem 3.2). In both cases, the subgraph of short edges can even be drawn without edge crossings. However, there are bipartite dichotomous ordinal graphs that do not admit a geometric representation even though the subgraph of short edges is planar (Theorems 2.2 and 2.3).

Preliminaries. Let $G = (V, E_S \cup E_\ell)$ be a dichotomous ordinal graph and assume that there exists a set of long edges whose removal creates different connected components. We can draw these connected components far apart. Now the long edges of G between different connected components will be drawn with a length greater than one. This yields the following.

▶ Observation 1. A dichotomous ordinal graph admits a geometric representation if and only if each subgraph induced by a connected component of the short edges does.

2 Complete Bipartite Graphs — A Characterization

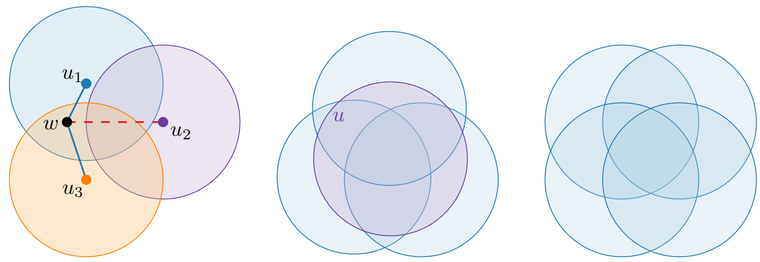
A convenient way to reason about geometric representations for bipartite graphs is in terms of arrangements of unit circles. Consider a bipartite dichotomous ordinal graph $G = (U \cup W, E)$ and suppose that the vertices of U are already drawn as points in the plane. Then, to obtain a geometric representation for G the task is to place each $w \in W$ such that for each $u \in U$ the point w lies in the unit disk centered at u if and only if the edge uw is short; see Fig. 2a.

A related question is the existence of a representation of a graph as a *unit disk graph*, where vertices are represented by unit disks, and they are connected by an edge if and only if the corresponding disks intersect. The main difference compared to dichotomous ordinal graphs lies in the different types of edges. In a unit disk representation, there are only two types: edge and non-edge, and all of them have to be faithfully represented. In a geometric realization of dichotomous ordinal graphs, there are three types of edges: long, short, and non-edges, and we have no constraints concerning the last type.

Let $U = \{u_1, \ldots, u_n\}$, let C_i denote the unit circle centered at u_i, and let D_i denote the corresponding unit disk. Let $\mathcal{C}$ denote the arrangement of $C_1, \ldots, C_n$. With every vertex $w \in W$ we associate a subset $V(w) \subseteq U$ such that $u \in V(w)$ if and only if the edge uw is short. We refer to $V(w)$ as a *singleton*, a *pair*, or a *triple* if $V(w)$ contains one, two, or three vertices, respectively. A subset $X \subseteq U$ is *realized* by a drawing of U if there is a cell r in $\mathcal{C}$ such that $r \subseteq D_i$ if and only if $u_i \in X$. Then there exists a geometric realization for G if and only if there exists a drawing/placement of U such that $V(w)$ is realized for all $w \in W$.

▶ **Theorem 2.1.** *Every dichotomous ordinal $K_{3,m}$, for $m \in \mathbb{N}$, and every dichotomous ordinal $K_{4,m}$, for $m \leq 6$, admits a geometric representation.*

Proof. For $K_{3,m}$ we can draw $U = \{u_1, u_2, u_3\}$ so that all eight subsets of U are realized; see Fig. 2a. For $|U| \geq 4$ such a universal placement is not possible because an arrangement of n circles has at most $n(n-1) + 2$ cells [13]. So an arrangement of four circles has at most 14 cells, whereas a four-element set has 16 subsets. However, for $|U| = 4$ and $|W| \leq 6$ we can always obtain a geometric representation as follows. Let $V(W) \subset 2^U$ denote the set of subsets of U that are associated to some vertex of W.



(a) $K_{3,m}$ (b) $K_{4,m}$ all pairs (c) $K_{4,m}$ all triples

Figure 2 Regions for the other side.

If there are at least three pairs in $V(W)$, then, given that $|V(W)| \leq |W| \leq 6$, the number of triples plus the number of singletons in $V(W)$ together is at most three. Thus, as $|U| = 4$, there exists a vertex $u \in U$ such that $\{u\} \notin V(W)$ and $U \setminus \{u\} \notin V(W)$. So we can use the drawing depicted in Fig. 2b, where we assign u to the central circle. As all subsets of U other than $\{u\}$ and $U \setminus \{u\}$ are realized, this is a valid geometric representation of G.

Otherwise, there are at most two pairs in $V(W)$. We use the drawing depicted in Fig. 2c, where we assign the vertices of U to the circles so that both pairs in $V(W)$ appear consecutively in the circular order of circles. (This works regardless of whether or not these pairs share a vertex.) As all subsets of U other than the two pairs that correspond to opposite circles in the drawing are realized, this is a valid geometric representation of G. ◀

▶ **Theorem 2.2.** *There is a dichotomous ordinal $K_{4,7}$ that does not admit a geometric representation.*

Proof Sketch. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, \ldots, w_7\}$ denote the vertex partition. For each w_i, we can specify an associated set $U_i \subseteq U$ (such that exactly the edges between w_i and U_i are short; see Fig. 3a). We choose all four subsets of size three and the three subsets of size two that contain u_4, and distribute them among the vertices of W arbitrarily. In any geometric representation, each set U_i corresponds to a cell in the induced arrangement $\mathcal{C}$ of unit circles. Two more cells are required implicitly: The outer cell, which corresponds to $\emptyset \subset U$, and a cell that corresponds to the whole set U and is required by Helly's Theorem [7] because disks are convex and we specified all triples to be among the sets U_i. Using these properties of $\mathcal{C}$ we can show that it cannot be realized using unit circles. ◀

▶ **Theorem 2.3.** *There is a dichotomous ordinal $K_{5,5}$ that does not admit a geometric representation.*

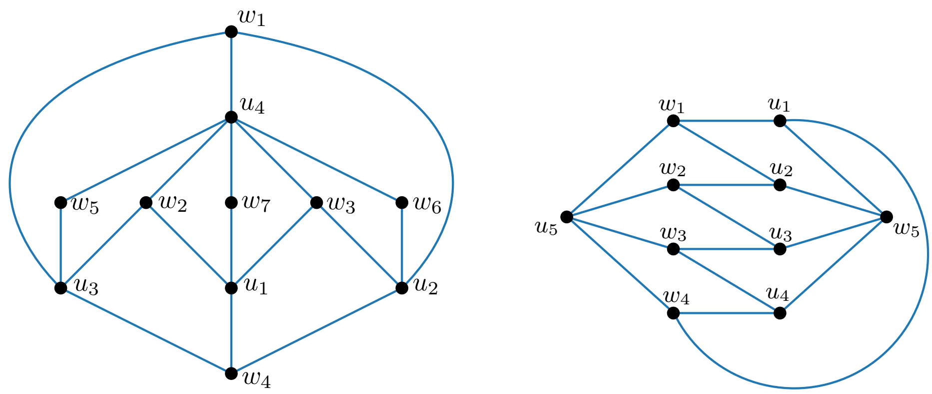
Proof Sketch. Let $U = \{u_1, \ldots, u_5\}$ and $W = \{w_1, \ldots, w_5\}$ denote the vertex partition. To each $w_i \in W$, we associate a set $U_i \subseteq U$ of "short neighbors" (see Fig. 3b):

$$U_i = \{u_i, u_{i \oplus 1}, u_5\}, \text{for } 1 \leq i \leq 4, \text{ and } U_5 = U \setminus \{u_5\},$$

where $i \oplus 1 = (i \bmod 4) + 1$. In any geometric representation, each set U_i corresponds to a cell in the induced arrangement $\mathcal{C}$ of unit circles. Using the existence of these cells we can analyze $\mathcal{C}$ geometrically and show that it cannot be realized using unit circles. ◀

3 Short Outerplanar Graphs and Short Subgraphs of the Grid

We show that every bipartite dichotomous ordinal graph admits a geometric representation if the subgraph G_s induced by the short edges is outerplanar or a subgraph of the grid. In



(a) short edges of $K_{4,7}$ **(b)** short edges of $K_{5,5}$

Figure 3 A dichotomous ordinal $K_{4,7}$ and $K_{5,5}$, respectively, that does not admit a geometric representation. The drawn edges are the short edges. Edges between vertices labeled u on one hand and w on the other hand, that are not drawn, are long.

the first case, we construct a planar drawing of G_s in which the BFS-layers are drawn on horizontal lines. See Fig. 4b. In the second case, we suitably perturb the grid. See Fig. 5.

▶ **Theorem 3.1.** *A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.*

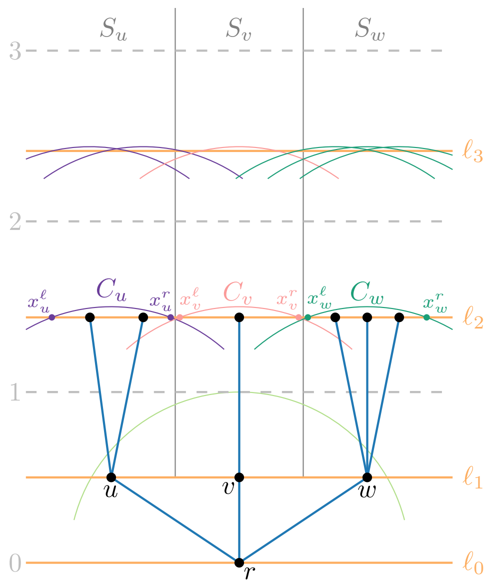
Proof Sketch. Let $G = (V, E_s \cup E_\ell)$ be a bipartite dichotomous ordinal graph such that $G_s = (V, E_s)$ is outerplanar. By Observation 1, we may assume that G_s is connected. We root G_s at an arbitrary vertex r. Let V_k, $k = 0, \ldots$ be the BFS layers of G_s rooted at r, i.e., $V_0 = \{r\}$, V_1 is the set of neighbors of r, and V_{k+1}, $k \geq 1$ is the set of neighbors of the vertices in V_k that are not already in V_{k-1}. We say that w is a *child* of v and v is a *parent* of w if vw is an edge of G_s, $v \in V_k$ and $w \in V_{k+1}$ for some k. By outerplanarity, each vertex has at most two parents. We construct a planar drawing of G_s with the following properties.

- The root r is drawn with y-coordinate $y_0 = 0$. All vertices in layer V_k, $k > 0$ are on a horizontal line ℓ_k with y-coordinate y_k strictly between $k - 1$ and k.
- The distance between a vertex and its children is at most 1 while the distance between two vertices on consecutive layers is greater than 1 if they are not adjacent in G_s.
- For each vertex v there is a vertical strip S_v such that (a) v is in S_v, (b) S_w is contained in the union of the strips of w's parents, (c) S_u and S_v are internally disjoint if u and v are on the same layer.

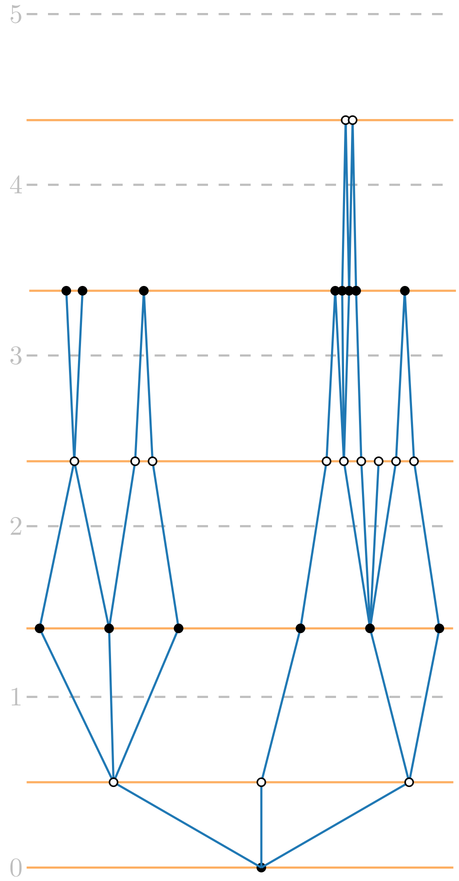
Special care has to be taken if a vertex $w \in V_{k+1}$ has two parents u and v, i.e., if w closes an internal face. In that case, we want to draw w on line ℓ_{k+1}, on the common boundary of S_u and S_v, and with distance exactly one to both u and v. ◀

▶ **Theorem 3.2.** *A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.*

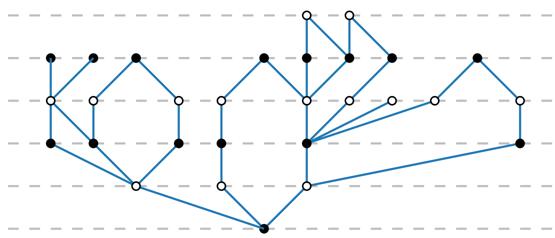
Proof Sketch. Extend $G_s = (V, E_s)$ by the remaining grid edges and require the new edges to be long. Use the construction in Fig. 5 to place the vertices. Then the short edges are shorter than $n^2 + 1/2$, while the long edges have length at least $n^2 + 1$. Scale the drawing. ◀



(a) construction for a tree



(b) geometric realization when G_s is the graph in (c)



(c) bipartite outerplanar graph

Figure 4 How to construct a geometric realization of a bipartite dichotomous ordinal graph if the short edges induce an outerplanar graph.

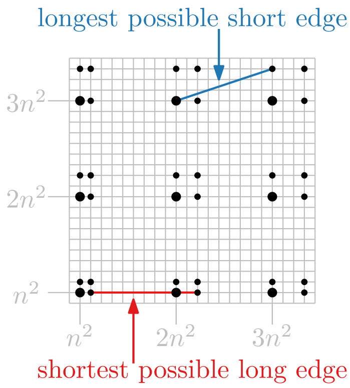


Figure 5 For each grid point (i, j), $1 \le i \le n$, $1 \le j \le n$ there are four possible points. If $i > 1$, the x-coordinate is in^2 if the edge between $(i-1, j)$ and (i, j) is short and $in^2 + i$ otherwise. If $j > 1$, the y-coordinate is jn^2 if the edge between $(i, j-1)$ and (i, j) is short and $jn^2 + j$ otherwise.

4 Conclusion

We leave open the questions whether bipartite dichotomous ordinal graphs always admit a geometric realization in any of the following cases: (i) the underlying graph is planar; (ii) the underlying graph is 3-degenerate; or (iii) the graph induced by the short edges is a 2-tree. Questions (i) and (ii) are open even for non-bipartite dichotomous ordinal graphs.

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