# Bounds on the Edge-length Ratio of 2-outerplanar Graphs

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## Abstract

The edge-length ratio of a planar straight-line drawing $\Gamma$ of a graph $G$ is the largest ratio between the lengths of every pair of edges of $\Gamma$. If the ratio is measured by considering only pairs of edges that are incident to a common vertex, we talk about local edge-length ratio. The (local) edge-length ratio of a planar graph is the infimum over all (local) edge-length ratios of its planar straight-line drawings. It is known that the edge length ratio of outerplanar graphs is upper bounded by a constant, while there exist graph families with non-constant outerplanarity that have non-constant lower bounds to their edge-length ratios. In this paper we prove an $\Omega(\sqrt{n})$ lower bound on the local edge-length ratio (and hence on the edge-length ratio) of the $n$-vertex 2-outerplanar graphs. We also prove a constant upper bound to the edge length ratio of Halin graphs.

## 1 Introduction

Let $\Gamma$ be a planar straight-line drawing of a planar graph $G = (V, E)$. For any edge $e \in E$, let $|e|_\Gamma$ be the length of the segment representing $e$ in $\Gamma$. The *edge-length ratio of* $\Gamma$, denoted as $\rho(\Gamma)$, is the maximum ratio between the lengths of every two edges in $\Gamma$; the *local edge-length ratio* $\rho_\ell(\Gamma)$ *of* $\Gamma$ is the maximum ratio between the lengths of two adjacent edges. Formally,

$$\rho(\Gamma) = \max_{(u,v),(z,w)\in E} \frac{|(u,v)|_\Gamma}{|(z,w)|_\Gamma}, \qquad \rho_\ell(\Gamma) = \max_{(u,v),(v,w)\in E} \frac{|(u,v)|_\Gamma}{|(v,w)|_\Gamma}.$$

The *edge-length ratio* $\rho(G)$ *of* $G$ is the infimum of $\rho(\Gamma)$ over the set $\mathcal{D}(G)$ of all planar straight-line drawings $\Gamma$ of $G$, i.e., $\rho(G) = \inf_{\Gamma\in\mathcal{D}(G)} \rho(\Gamma)$. Analogously, the *local edge-length ratio* $\rho_\ell(G)$ *of* $G$ is defined as $\rho_\ell(G) = \inf_{\Gamma\in\mathcal{D}(G)} \rho_\ell(\Gamma)$.

We remark that since the publication of the first book on graph drawing [7], minimizing the maximum edge length provided that the shortest edge has length one (i.e. minimizing the edge-length ratio) is among the most relevant optimization goals, because of its impact on the readability of the computed visualization. Eades and Wormald [9] prove that deciding whether a biconnected planar graph has edge-length ratio one is NP-hard, and Cabello et al. [5] extend this result to triconnected instances. Borrazzo and Frati [4] prove that the edge-length ratio of $n$-vertex planar 3-trees is $\Omega(n)$. As for $n$-vertex planar 2-trees, Blažej et al. [3] prove an $\Omega(\log n)$ lower bound. Notably, both the lower bound by Borrazzo and Frati and the lower bound by Blažej et al. use graph families whose outerplanarity grows as a function of $n$. In contrast, Lazard et al. [10] show that graphs with outerplanarity one (i.e. outerplanar graphs) have a constant upper bound to their edge-length ratio.

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

A natural question that stems from the previous literature is whether outerplanarity one is a hard cutoff for achieving constant edge-length ratio. We answer this question in the affirmative, proving that graphs with outerplanarity two, i.e. the *2-outerplanar graphs*, have unbounded edge-length ratio. Nonetheless we prove a constant upper bound on the edge-length ratio of a well-studied family of 2-outerplanar graphs. Our results are as follows.

- We describe a family of $n$-vertex 2-outerplanar graphs whose *local* edge-length ratio is in $\Omega(\sqrt{n})$ which implies a lower bound also for the edge-length ratio of these graphs. It is worth noticing that while graph families with $O(1)$ local edge-length ratios are known [3], no family with $\omega(1)$ local edge-length ratio was previously known.
- We show that Halin graphs have edge-length ratio at most 3. We remark that Halin graphs are well-known subjects of study in the graph drawing literature; see e.g. [2, 6, 8].

Our approach for the lower bound builds upon ideas of Borrazzo and Frati [4]. Our upper bound is proved by translating the problem of computing drawings with bounded edge-length ratio to a topological question about (a variant of) level planarity with limited edge span. As a byproduct, the proof regarding the edge-length ratio upper bound of Halin graphs fixes an imprecision in the literature about the span of their weakly leveled planar drawings. For reasons of space some proofs are omitted or sketched.

## 2 Lower Bound

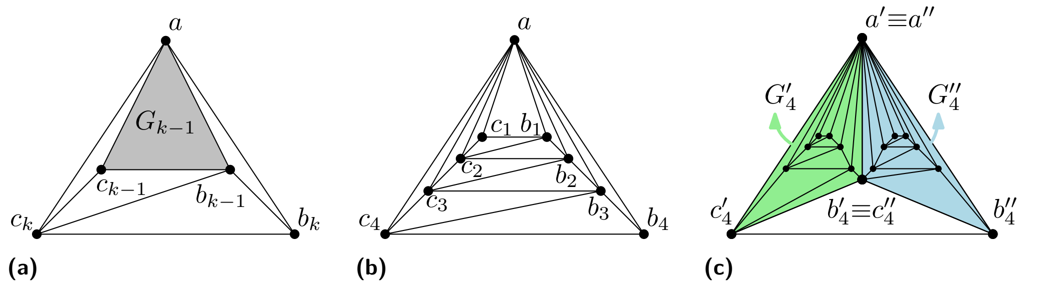



**Figure 1** (a) Definition of the graph $G_k$. (b) Graph $G_4$ (c) Example of a graph $G$ of Theorem 2.3.

We define a family of 2-outerplanar graphs $G_k$, for every $k \geq 1$, such that $G_k$ has $n = 2k + 1$ vertices. The graph $G_1$ is a 3-cycle $C_1$. Assume that $G_{k-1}$ has been defined and that its outer face is a 3-cycle $C_{k-1}$ whose vertices are denoted as $a$, $b_{k-1}$, and $c_{k-1}$; then $G_k$ is obtained by adding two vertices $b_k$ and $c_k$, and the edges $(a, b_k)$, $(a, c_k)$, $(b_k, c_k)$, $(b_k, b_{k-1})$, $(c_k, c_{k-1})$, and $(c_k, b_{k-1})$, embedded as shown in Fig. 1a. Note that $G_k$ is 2-outerplanar and has $2k + 1$ vertices (see Fig. 1b for an example with $k = 4$). Let $\Gamma$ be an embedding-preserving planar straight-line drawing of $G_k$. For $i = 1, 2, \ldots, k$, we denote by $\Delta_i$ the triangle that represents $C_i$ in $\Gamma$ and by $p(\Delta_i)$ its perimeter. We assume that the shortest edge over all triangles $\Delta_i$ has length 1; if not, we scale the drawing so to achieve this condition. The next lemma is a consequence of results by Borrazzo and Frati [4, pp.140-142].

▶ **Lemma 2.1.** *Let $\Gamma$ be an embedding-preserving planar straight-line drawing of $G_k$, for $k \geq 2$; then $p(\Delta_i) > p(\Delta_{i-1}) + \gamma$, with $\gamma = 0.3$.*

We first prove a lower bound on $\rho_\ell(G_k)$ that holds if we consider only drawings that preserve the planar embedding of $G_k$. We then remove this restriction.

▶ **Lemma 2.2** ($\star$). *Let $\Gamma$ be an embedding-preserving planar straight-line drawing of $G_k$, for $k \geq 2$; then $\rho_\ell(\Gamma) \geq \sqrt{\frac{3k}{40}}$.*

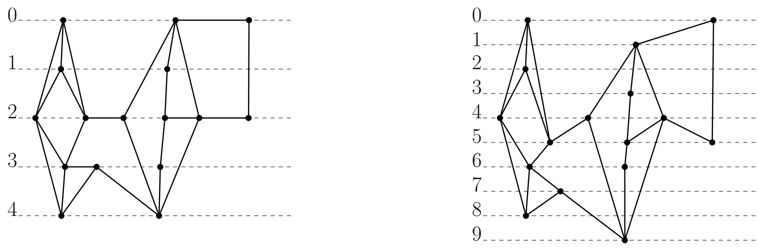
**Sketch.** If $k \leq \frac{40}{3}$ the the statement is trivially true, since $\rho_\ell(\Gamma) \geq 1$. Thus, we can assume that $k > \frac{40}{3}$. By using induction, Lemma 2.1, and the fact that $p(\Delta_1) > \gamma$, we can prove that $p(\Delta_i) > \gamma \cdot i$, for every $i = 1, 2, \ldots k$. Let $L$ denote the length of the longest edge $e_1$ of the triangle $\Delta_k$ incident to $a$. We have that $L \geq \frac{p(\Delta_k)}{4} > \frac{\gamma \cdot k}{4} = \frac{3k}{40}$. Let $e_2$ be the shortest edge of $\Gamma$. Edge $e_2$ is incident to vertex $a$ or to a neighbor $v$ of $a$. If $e_2$ is incident to $a$, then $\rho_\ell(\Gamma) \geq \frac{|e_1|_\Gamma}{|e_2|_\Gamma} \geq \frac{3k}{40}$. Otherwise, edge $e_3 = (v, a)$ has vertex $a$ in common with $e_1$ and vertex $v$ in common with $e_2$. By definition we have $|e_1|_\Gamma \leq \rho_\ell(\Gamma)|e_3|_\Gamma \leq \rho_\ell(\Gamma)^2|e_2|_\Gamma = \rho_\ell(\Gamma)^2$ and $\rho_\ell(\Gamma) \geq \sqrt{|e_1|_\Gamma} \geq \sqrt{\frac{3k}{40}}$. ◀

To prove the next theorem we construct a 2-outerplanar graph with $n = 4k$ vertices such that, in every embedding, it contains a copy of $G_k$ embedded as in Lemma 2.2 (see Fig. 1c).

▶ **Theorem 2.3** ($\star$). *For every integer $k \geq 2$, there exists a 2-outerplanar graph $G$ with $n = 4k$ vertices such that $\rho_\ell(G) \geq \sqrt{\frac{3n}{160}}$.*

## 3 Edge-length Ratio of Halin Graphs

A *$k$-span weakly level planar drawing* (*$k$-SWLP drawing*) $\Gamma$ is a straight-line planar drawing whose vertices lie on a set of horizontal equispaced lines, called *levels*, and whose edges intersect at most $k+1$ levels. Notice that, in a $k$-SWLP drawing edges between vertices that are consecutive in the same level are allowed. We assume that the levels are numbered from top to bottom and that the distance between consecutive levels is 1. An edge that intersects $k+1$ levels has *span* $k$. A graph is *$k$-SWPL* if it has a $k$-SWLP drawing.




**(a)** **(b)**

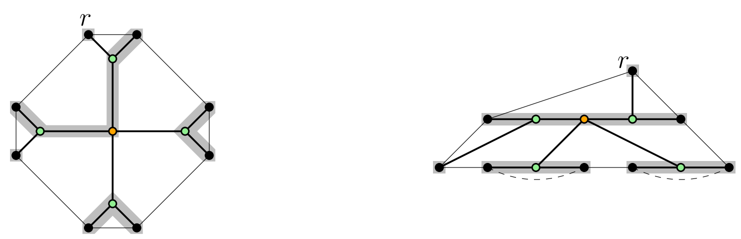
**Figure 2** (a) A 2-SWLP drawing $\Gamma$; (b) The 5-SWLP drawing $\Gamma'$ obtained from $\Gamma$.

▶ **Lemma 3.1** ($\star$). *If $G$ is a $k$-SWLP graph for some $k \geq 1$, then $\rho(G) \leq 2k+1$.*

**Proof.** Let $\Gamma$ be a $k$-SWLP drawing of $G$. We first transform $\Gamma$ into a $(2k+1)$-SWLP drawing such that every edge has span at least one, i.e., no edge has both end-vertices on the same level. To this aim we split each level $i$ into two levels, numbered $2i$ and $2i+1$, and assign the vertices of level $i$ alternating between $2i$ and $2i+1$. Let $\Gamma'$ be the resulting drawing (see Fig. 2 for an example). For an arbitrarily chosen value $\varepsilon > 0$, we squeeze horizontally the drawing $\Gamma'$ so that its width is $\varepsilon$. After this squeezing, for every edge $e$ we have $1 \leq |e|_{\Gamma'} \leq 2k+1+\varepsilon$. It follows that $\rho(\Gamma) \leq 2k+1+\varepsilon$ and $\rho(G) \leq 2k+1$. ◀

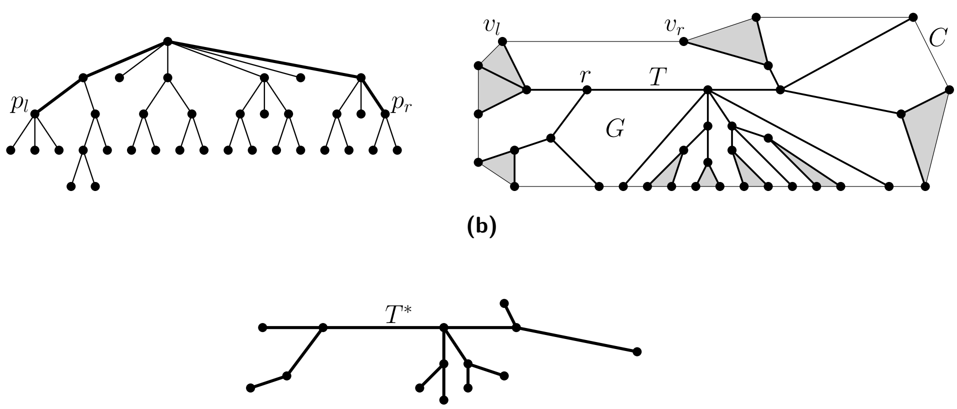
In the remainder we exploit Lemma 3.1 to prove a constant upper bound on the edge-length ratio of *Halin graphs*. A *Halin graph* (see Figs. 3a and 4b) is a 3-connected embedded planar graph $G$ such that, by removing the edges along the boundary $C$ of its outerface, one gets a tree $T$ whose internal vertices have degree at least 3 and whose leaves are incident to the outerface of $G$. We call $T$ the *characteristic tree* of $G$ and we call $C$ the *adjoint cycle* of $G$. A Halin graph $G$ is *trivial* if it is a wheel graph, i.e., if $T$ has only one non-leaf vertex.

Bannister et al. [1, Thm. 17] state that all Halin graphs are 1-SWLP which, together with Lemma 3.1, would imply an upper bound of 3 to their edge-length ratio. Unfortunately, $K_4$ is a Halin graph that is not 1-SWLP and the proof technique of [1] fails even for instances of Halin graphs different from $K_4$. Namely, let $T$ be the characteristic tree of the Halin graph of Fig. 3a. According to the proof of Theorem 17 of [1] a leveling of $T$ is computed as follows: Choose a leaf of $T$ as the root and assign it to level 0; at Step $i$, assign to level $i+1$ the previously-unassigned nodes that are either children of nodes at level $i$ or that belong to a path from one such children to its leftmost or rightmost leaf descendant in $T$. However, for any possible choice of the root of $T$, one obtains the leveling of Fig. 3b that has an edge between two non-consecutive vertices on a same level.




**(a)** **(b)**

**Figure 3** (a) A Halin graph; the vertices are grouped according to a leveling obtained with the technique in [1] choosing $r$ as the root; (b) the corresponding level drawing.




**(a)** **(b)** **(c)**

**Figure 4** (a) The external path of a tree $T$; (b) A Halin graph $G$; the thick edges form the characteristic tree $T$, while the thin edges from the adjoint cycle $C$; the tufts of $T$ are highlighted with gray areas. (c) The pruned tree $T^*$ of $T$.

We now prove that all Halin graphs except $K_4$ are 1-SWLP (Lemmas 3.3 and 3.4), which by Lemma 3.1 implies an upper bound of 3 to the edge-length ratio of these graphs. Let $T$ be an ordered rooted tree. The *external path* of $T$ is defined as follows. If $T$ is a single vertex $r$, then the external path of $T$ coincides with $r$; otherwise it is the path connecting the parent $p_l$ of the leftmost leaf of $T$ and the parent $p_r$ of the rightmost leaf of $T$ (see Fig. 4a).

Let $G$ be a non-trivial Halin graph. A *tuft* of the characteristic tree $T$ of $G$ is a maximal set of at least two leaves having the same parent, and such that this parent is adjacent to exactly one other internal vertex of $T$ (in Fig. 4b the tufts are highlighted with gray areas). The *pruned tree* $T^*$ of $T$ is obtained by removing all leaves from $T$ (see Fig. 4c).

▶ **Lemma 3.2** ($\star$). *Let $G$ be a Halin graph distinct from $K_4$ and let $T$ be the characteristic tree of $G$. The number of tufts of $T$ is equal to the number of leaves of the pruned tree of $T$.*

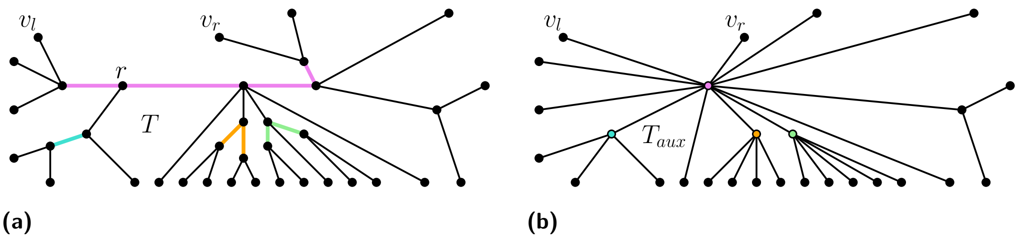



**Figure 5** (a) A decomposition in characteristic paths of the characteristic tree of the Halin graph of Fig. 4b; (b) The corresponding auxiliary tree $T_{aux}$.

▶ **Lemma 3.3** ($\star$). *Let $G$ be a non-trivial Halin graph and let $T$ be the characteristic tree of $G$ rooted at any non-leaf vertex. Let $v_l$ be the leftmost leaf and $v_r$ be the rightmost leaf of $T$. If both $v_l$ and $v_r$ belong to a tuft, then $G \setminus (v_l, v_r)$ has a 1-SWLP drawing $\Gamma$ such that $v_l$ is the first vertex of the topmost level and $v_r$ is the last vertex of the same level.*

**Sketch.** We simplify the characteristic tree $T$ of $G$ by collapsing into single vertices a set of suitably defined paths called *characteristic paths* and illustrated in Fig. 5. The external path of $T$ is a characteristic path. For each vertex $v$ of a characteristic path $\pi$ and for each child $w$ of $v$ that is not in $\pi$, let $T'$ be the tree rooted at $w$. The external path of $T'$ is a characteristic path of $T$. Denote by $T_{aux}$ the tree obtained by collapsing the characteristic paths into vertices; for a vertex $v$ of $T_{aux}$ that corresponds to a path $\pi$ of $T$, we say that $\pi$ is the *pertinent path* of $v$. We compute first a 1-SWLP drawing $\Gamma_{aux}$ of $T_{aux}$ (see Fig. 6a). The level of each vertex is equal to its depth in $T_{aux}$ and the order of the vertices in each level is given by the left-to-right order of $T_{aux}$. We now replace each vertex of $T_{aux}$ by its pertinent path, thus obtaining a 1-SWLP drawing $\Gamma_T$ of $T$ (see Figs. 6b and 6c). It is easy to see that all the edges of the adjoint cycle that are distinct from $(v_l, v_r)$ can be added to the drawing without crossings and with no span increase. We finally move $v_l$ and $v_r$ to the topmost level. Let $v_l = v_1, v_2 \ldots, v_k = v_r$ be the leaves of $T$ in the order they appear counterclockwise along the adjoint cycle $C$ of $G$. Since $v_l$ belongs to a tuft, $v_2$ is a sibling of $v_l$ and they are both drawn on level 1. Also, their parent is the first vertex on level 0. Thus, $v_l$ can be moved to the left of the leftmost vertex of level 0 without crossings and with no span increase. By a symmetric argument, $v_r$ can be moved to the right of the righmost vertex of level 0 (see Fig. 6c). ◀

Lemma 3.3 allows us to compute a drawing of a Halin graph except for one edge. In the next lemma we explain how to cope with this issue.

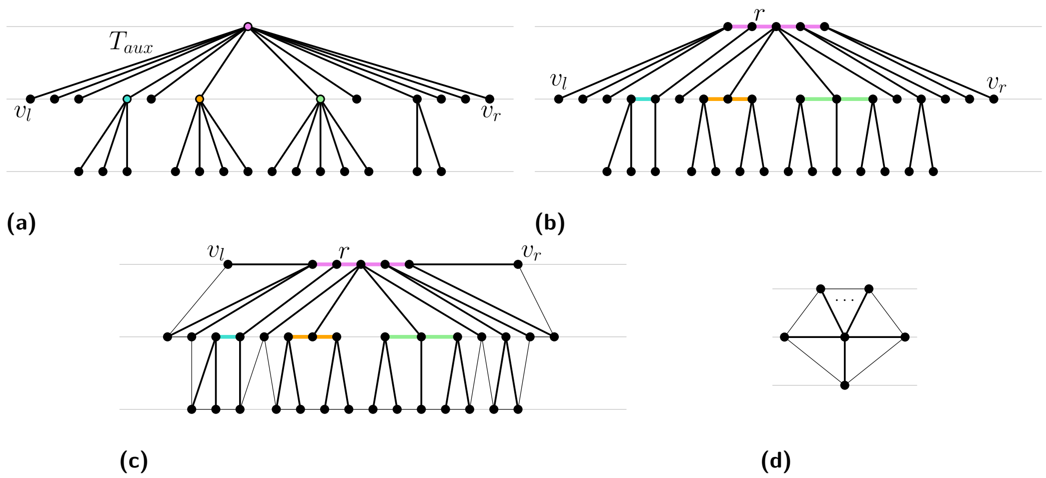



**Figure 6** (a) A 1-SWLP drawing $\Gamma_{aux}$ of $T_{aux}$; (b) A 1-SWLP drawing of $T$ obtained from $\Gamma_{aux}$ by replacing each vertex with its pertinent path; (c) A 1-SWLP drawing of $G \setminus (v_l, v_r)$ with the properties of Lemma 3.3; (d) A 1-SWLP drawing of a trivial Halin graph.

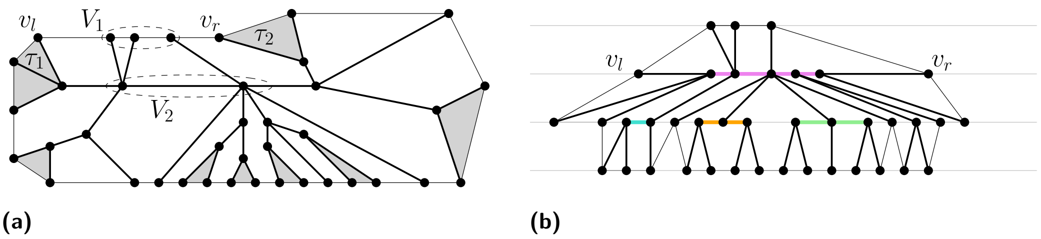



**Figure 7** Illustration of Lemma 3.4, Case 1: (a) A Halin graph $G$; (b) A 1-SWLP drawing of $G$.

▶ **Lemma 3.4** ($\star$). *Every Halin graph $G$ distinct from $K_4$ has a 1-SWLP drawing.*

**Proof.** Let $T$ be the characteristic tree of $G$ and $C$ be its adjoint cycle. If $G$ is trivial, a 1-SWLP drawing of $G$ is computed as in Fig. 6d. Otherwise, $T$ has at least one edge and, by Lemma 3.2, at least two tufts. A leaf not belonging to any tuft is a *single leaf.*

**Case 1: $T$ has at least one single leaf.** We remove a maximal set $V_1$ of consecutive single leaves along $C$ (see Fig. 7a). By Lemma 3.3 we compute a drawing of the resulting graph such that the leaf $v_l$ preceding $V_1$ walking clockwise along $C$ and the leaf $v_r$ following $V_1$ walking clockwise along $C$, are the first and the last vertex, respectively, on the topmost level. To construct a 1-SWLP drawing of $G$, we put the single leaves of $V_1$ on a new level above the topmost in the order they appear along $C$. See Fig. 7b.

**Case 2: $T$ has no single leaves.** If $T^*$ has at most 1 internal vertex, then $T^*$ is a single edge or it is a star with at least three edges; a 1-SWLP drawing of $G$ can be constructed as in Figs. 8a and 8b. Otherwise, $T^*$ has one edge $e^*$ whose end-vertices are both non-leaves. Further, $e^*$ is shared by two faces each having an edge belonging to $C$. Removing these two edges and $e^*$ (possibly smoothing the end-vertices of $e^*$ if they have degree two after the removal) we get two subgraphs $G_a$ and $G_b$ of $G$ (Fig. 8c) for which we compute two 1-SWLP drawings according to Lemma 3.3 (Figs. 8d and 8e). We then combine the two drawings by mirroring vertically and horizontally one of them. This allows us to add the three removed edges without crossings and without span increase (Fig. 8f). ◀

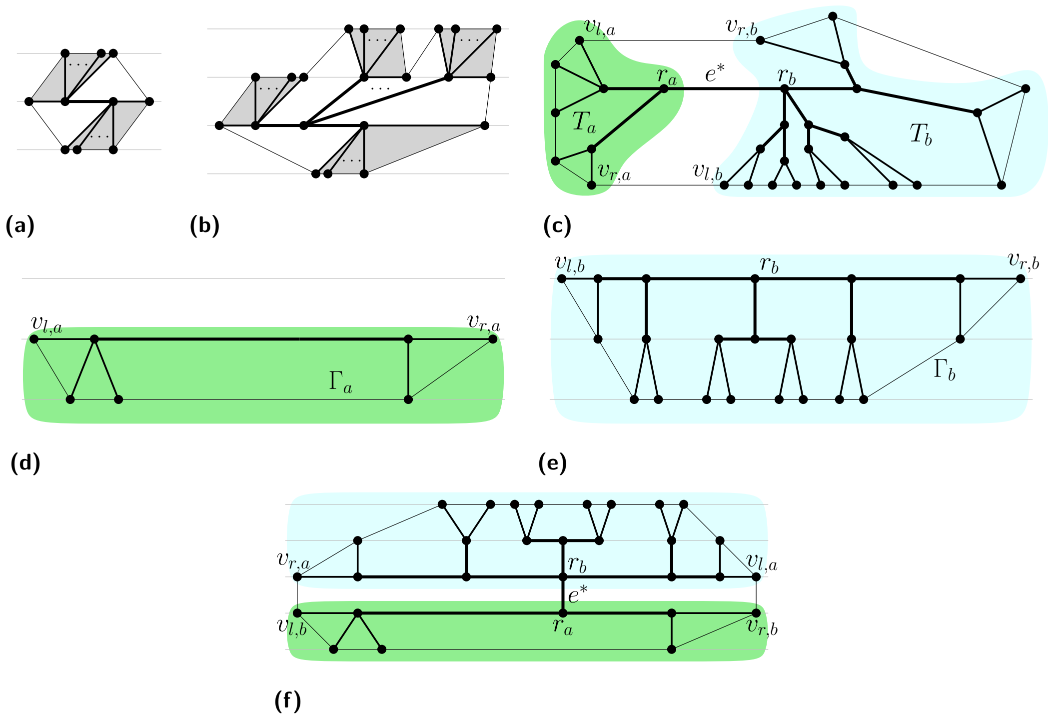



**Figure 8** Illustration of Lemma 3.4, (a)-(b) Case 2a; (c)-(f) Case 2b.

▶ **Theorem 3.5.** *If $G$ is a Halin graph, then $\rho(G) \leq 3$.*

## 4 Open Problems

(i) Is the bound of Theorem 2.3 asymptotically tight? (ii) Study other sub-families of 2-outerplanar graphs that have constant (local) edge-length ratio.

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