Simplified and Improved Bounds on the VC-Dimension for Elastic Distance Measures*

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Abstract

We study the Vapnik–Chervonenkis dimension (VC-dimension) of range spaces, where the ground set consists of either polygonal curves in $\mathbb{R}^d$ or polygonal regions in the plane that may contain holes and the ranges are balls defined by an elastic distance measure, such as the Hausdorff distance, the Fréchet distance and the dynamic time warping distance (DTW). We show for the Fréchet distance of polygonal curves and the Hausdorff distance of polygonal curves and planar polygonal regions that the VC-dimension is upper-bounded by $O(dk\log(km))$, where k is the complexity of the center of a ball, m is the complexity of the polygonal curve or region in the ground set, and d is the ambient dimension. For $d \geq 4$ this bound is tight in each of the parameters d, k and m separately. For DTW of polygonal curves, our analysis directly yields an upper-bound of $O(\min(dk^2\log(m), dkm\log(k)))$.

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1 Introduction

The Vapnik–Chervonenkis dimension (VC-dimension) [15] is a measure of complexity for range spaces. Knowing the VC-dimension of a range space can be used to determine sample bounds for various computational tasks. These include sample bounds on the test error of a classification model in statistical learning theory [14] or sample bounds for an ε-net [11] or an (η, ε)-approximation [10] in computational geometry. Sample bounds based on the VC-dimension have been successfully applied in the context of kernel density estimation [12], neural networks [2, 13], coresets [5, 8, 9], clustering [1, 3] and other data analysis tasks.

We study range spaces, where the ground set consists of polygonal curves or polygonal regions and the ranges consist of balls defined by the Hausdorff distance. Previous to our work, Driemel, Nusser, Phillips and Psarros [7] derived almost tight bounds on the VC-dimension in the setting of polygonal curves. At the heart of their approach lies the definition of a set of boolean functions (predicates) which can be used to determine if a query curve is contained in a ball of given radius around a center curve. Their proof of the VC-dimension bound uses the worst-case number of operations needed to determine the truth values of each predicate.

In this paper, we extend the known set of predicates to be able to decide the Hausdorff distance between polygonal regions with holes in the plane. We give an improved analysis for the VC-dimension that considers each predicate as a combination of sign values of polynomials. This approach does not use the computational complexity of the distance evaluation, but instead uses the underlying structure of the range space defined by a system of polynomials directly. Our techniques extend to other elastic distance measures, such as the Fréchet distance and the dynamic time warping distance (DTW).

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1.1 Preliminaries

Let X be a set. We call a set $\mathcal{R}$ where any $r \in \mathcal{R}$ is of the form $r \subseteq X$ a ***range space*** with ***ground set*** X. We say a subset $A \subseteq X$ is ***shattered*** by $\mathcal{R}$ if for any $A' \subseteq A$ there exists an $r \in \mathcal{R}$ such that $A' = r \cap A$. The ***VC-dimension*** of $\mathcal{R}$ (denoted by $VCdim(\mathcal{R})$) is the maximal size of a set $A \subseteq X$ that is shattered by $\mathcal{R}$. We define the ball with radius Δ and center c under the distance measure d_ρ on a set X as $b_\rho(c, \Delta) = \{x \in X \mid d_\rho(x, c) \leq \Delta\}$. We study range spaces with ground set $(\mathbb{R}^d)^m$ of the form

$$\mathcal{R}_{\rho,k} = \{b_\rho(c, \Delta) \mid \Delta \in \mathbb{R}_+, c \in (\mathbb{R}^d)^k\}.$$

Let $\mathcal{R}$ be a range space with ground set X, and F be a class of real-valued functions defined on $\mathbb{R}^d \times X$. For $a \in \mathbb{R}$ let ***sgn(a)*** $= 1$ if $a \geq 0$ and ***sgn(a)*** $= 0$ if $a < 0$. We say that $\mathcal{R}$ is a ***t-combination*** of $sgn(F)$ if there is a boolean function $g : \{0,1\}^t \to \{0,1\}$ and functions $f_1, \ldots, f_t \in F$ such that for all $r \in \mathcal{R}$ there is a parameter vector $y \in \mathbb{R}^d$ such that $r = \{x \in X \mid g(sgn(f_1(y, x)), \ldots, sgn(f_t(y, x))) = 1\}$.

Central to our approach is the following well-known theorem for bounding the VC-dimension of such range spaces. The theorem can be proven by investigating the complexity of arrangements of zero sets of polynomials (see full version [4]).

▶ **Theorem 1.1** ([2], Theorem 8.3). *Let F be a class of functions mapping from $\mathbb{R}^d \times X$ to $\mathbb{R}$ so that, for all $x \in X$ and $f \in F$ the function $y \to f(y, x)$ is a polynomial on $\mathbb{R}^d$ of degree no more than l. Suppose that $\mathcal{R}$ is a t-combination of $sgn(F)$. Then we have*

$$VCdim(\mathcal{R}) \leq 2d \log_2(12tl).$$

Let $\|\cdot\|$ denote the standard Euclidean norm. Let $X, Y \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$. The ***directed Hausdorff distance*** from X to Y is defined as $d_{\overrightarrow{H}}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|$ and the ***Hausdorff distance*** between X and Y is defined as $d_H(X, Y) = \max\{d_{\overrightarrow{H}}(X, Y), d_{\overrightarrow{H}}(Y, X)\}$. If a set X consists of a single point $p \in \mathbb{R}^d$, we may write p instead of $\{p\}$ to simplify the notation, e.g. $d_H(p, Y)$ instead of $d_H(\{p\}, Y)$. Let $d, m \in \mathbb{N}$. A sequence of vertices $p_1, \ldots, p_m \in \mathbb{R}^d$ defines a ***polygonal curve*** P by connecting consecutive vertices to create the edges $\overline{p_1, p_2}, \ldots, \overline{p_{m-1}, p_m}$. We may think of P as an element of $\mathbb{X}^d_m := (\mathbb{R}^d)^m$ and write $P \in \mathbb{X}^d_m$. We may also think of P as a continuous function $P : [0,1] \to \mathbb{R}^d$ by fixing m values $0 = t_1 < \ldots < t_m = 1$, and defining $P(t) = \lambda p_{i+1} + (1 - \lambda)p_i$ where $\lambda = \frac{t - t_i}{t_{i+1} - t_i}$ for $t_i \leq t \leq t_{i+1}$. We call P a ***closed curve*** if $p_1 = p_m$ and we call P ***self-intersecting*** if there exist $s \in [0,1]$, $t \in (0,1)$ with $s \neq t$ such that $P(s) = P(t)$. In the case that P is a closed curve in $\mathbb{R}^2$ which is not self-intersecting, we call the union of P with its interior a ***simple polygonal region*** S (without holes). We denote with ∂S the boundary of S, which is P. Given a simple polygonal region S_0 and a set of pairwise disjoint simple polygonal regions $S_1, \ldots, S_h$ in the interior of S_0, we also consider the set $S = S_0 - (S_1 \cup \cdots \cup S_h)$ a polygonal region and we call $S_1, \ldots, S_h$ the ***holes*** of S.

Let $s, t \in \mathbb{R}^d$. We denote with $\ell(\overline{st})$ the line supporting $\overline{st}$. We define the stadium centered at $\overline{st}$ with radius $\Delta \in \mathbb{R}_+$ as $D_\Delta(\overline{st}) = \{x \in \mathbb{R}^d \mid \exists p \in \overline{st}, \|p - x\| \leq \Delta\}$. Let $e_1, e_2 \in \mathbb{X}^d_2$ be two edges. We define the double stadium of the edges e_1 and e_2 with radius Δ as $D_{\Delta,2}(e_1, e_2) = D_\Delta(e_1) \cap D_\Delta(e_2)$.

Let X be a set of subsets (called sites) of $\mathbb{R}^2$. The ***Voronoi region*** $reg(A)$ consists of all points $p \in \mathbb{R}^2$ for which A is the closest among all sites in X, i.e. $reg(A) = \{p \in \mathbb{R}^2 \mid d_{\overrightarrow{H}}(p, A) < d_{\overrightarrow{H}}(p, U)$ for all $U \in X \setminus \{A\}\}$. The ***Voronoi diagram*** is $vd(X) = \mathbb{R}^2 \setminus \cup_{A \in X} reg(A)$. We call the set $bisec(A, B) = \{p \in \mathbb{R}^2 \mid d_{\overrightarrow{H}}(p, A) = d_{\overrightarrow{H}}(p, B)\}$ the ***bisector*** of A and B. The ***Voronoi edge*** of A, B is defined as $ve(A, B) = vd(X) \cap bisec(A, B)$ and the ***Voronoi vertices*** of A, B, C are defined as $vv(A, B, C) = vd(X) \cap bisec(A, B) \cap bisec(B, C)$.

		new	ref.	old
discrete polygonal curves	DTW	$O(dk^2 \log(m))$	Thm. 4, [4]	-
		$O(dkm \log(k))$	Thm. 4, [4]	
	Hausdorff	$O(dk \log(km))$	Thm. 2, [4]	$O(dk \log(dkm))$ [7]
	Fréchet	$O(dk \log(km))^{(*)}$	Thm. 3, [4]	
continuous polygonal curves	**Hausdorff**	$O(dk \log(km))$	**Thm. 3.5**	$O(d^2k^2 \log(dkm))$ [7]
	Fréchet	$O(dk \log(km))^{(*)}$	Thm. 27, [4]	
	weak Fréchet	$O(dk \log(km))^{(*)}$	Thm. 27, [4]	$O(d^2k \log(dkm))$ [7]
polygons $\mathbb{R}^2$	**Hausdorff**	$O(k \log(km))$	**Thm. 3.6**	-

Table 1 Overview of VC-dimension bounds. Results marked with $^{(*)}$ were independently obtained by Cheng and Huang [6].

2 Results

For the ***Hausdorff distance*** of ***polygonal regions*** (with holes) in the plane, we show that the VC-dimension of $\mathcal{R}_{d_H,k}$ is bounded by $O(k \log(km))$. For the ***Fréchet distance*** and the ***Hausdorff distance*** of ***polygonal curves***, in the discrete and the continuous case, we show that for the VC-dimension of $\mathcal{R}_{\rho,k}$ our techniques imply the same bound of $O(dk \log(km))$. Parallel and independent to our work, Cheng and Huang [6] obtained the same result for the Fréchet distance using very similar techniques. The bounds improve upon the upper bounds of [7] in all of the considered cases. An overview of our results with references to theorems and comparison to [7] and the independent results from [6] is given in Table 1. By the lower bound $\Omega(\max(dk \log(k), \log(dm)))$ for $d \geq 4$ in [7], the new bounds for polygonal curves are tight in each of the parameters k, m and d separately. For the ***Dynamic time warping distance***, we show a new bound of $O(\min(dk^2 \log(m), dkm \log(k)))$. The proofs for Fréchet and DTW are very similar to the ones used for the Hausdorff distance and we discuss them in the full version [4].

3 Analysis for the Hausdorff distance

To bound the VC-dimension of range spaces of the form $\mathcal{R}_{d_H,k}$, we define geometric predicates. The truth values of these predicates have to uniquely determine distance queries with d_H. We give predicates such that the directed Hausdorff distance query $d_{\overrightarrow{H}}(P, Q) \leq \Delta$ is determined by them. The other direction $d_{\overrightarrow{H}}(Q, P) \leq \Delta$ is analogous. We will show that our predicates can be viewed as combinations of simple predicates.

▶ **Definition 3.1.** Let F be a class of functions mapping from $\mathbb{R}^{dm} \times \mathbb{R}^{dk+1}$ to $\mathbb{R}$ so that, for all $f \in F$ the function $(x, y) \to f(x, y)$ is a polynomial of constant degree. Let $\mathcal{P}$ be a function from $\mathbb{R}^{dm} \times \mathbb{R}^{dk+1}$ to $\{0, 1\}$. We say that the predicate $\mathcal{P}$ is ***simple*** if $\mathcal{P}$ is a t-combination of $sgn(F)$ with $t \in O(1)$.

In our proof of the VC-dimension bounds we will use the following corollary to Theorem 1.1.

▶ **Corollary 3.2.** *Suppose that for a given d_ρ there exists a polynomial $p(k, m)$ such that for any $k, m \in \mathbb{N}$ the space $\mathcal{R}_{\rho,k}$ with ground set $\mathbb{R}^{dm}$ is a t-combination of simple predicates with $t = p(k, m)$. Then $VCdim(\mathcal{R}_{\rho,k})$ is in $O(dk \log(km))$.*

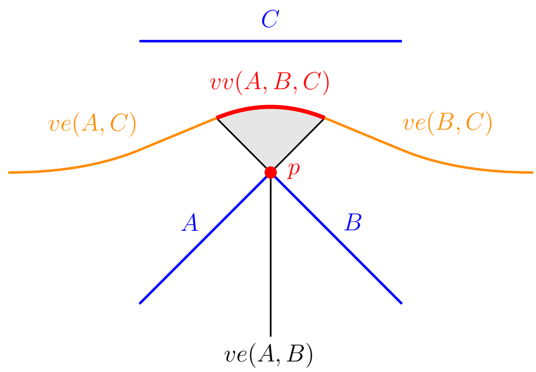

Figure 1 Degenerate case: $vv(A,B,C)$ consist of a whole arc and $ve(A,B)$ contains a region.

Let $P \in \mathbb{X}_m^d$ with vertices $p_1,\dots,p_m$ and $Q \in \mathbb{X}_k^d$ with vertices $q_1,\dots,q_k$ be two polygonal curves. Let further $\Delta \in \mathbb{R}_+$. By [7] the directed Hausdorff distance query $d_{\overrightarrow{H}}(P,Q) \leq \Delta$ is uniquely determined by the following predicates.

- ($\mathcal{P}_1$): Given an edge e_1 of Q and a vertex p of P, this predicate returns true iff there exists a point q on e_1, such that $\|q-p\| \leq \Delta$.
- ($\mathcal{P}_2$): Given an edge of P, $\overline{p_j p_{j+1}}$, and two edges e_1, e_2 of Q, this predicate is equal to $\ell(\overline{p_j p_{j+1}}) \cap D_{\Delta,2}(e_1,e_2) \neq \emptyset$.

Examples for the predicates $\mathcal{P}_1$ and $\mathcal{P}_2$ are depicted in Figure 3 (they are also used for polygonal regions).

▶ **Lemma 3.3** (Lemma 7.1, [7]). *For any two polygonal curves P,Q, given the truth values of all predicates of the type $\mathcal{P}_1,\mathcal{P}_2$ one can determine whether $d_{\overrightarrow{H}}(P,Q) \leq \Delta$.*

In the case of polygonal regions that may contain holes, we define some of the predicates based on the Voronoi vertices of the edges of the boundary of the polygonal region. Since degenerate situations can occur if Voronoi sites intersect in a point p (see Figure 1), we restrict the predicates to the subset of the Voronoi vertices that are relevant to our analysis.

▶ **Definition 3.4.** Let $a = \overline{a_1 a_2}, b = \overline{b_1 b_2}$ and $c = \overline{c_1 c_2}$ be edges of a polygonal region that may contain holes. Consider their vertices and supporting lines $A = \{\{a_1\},\{a_2\},\ell(a)\}$, $B = \{\{b_1\},\{b_2\},\ell(b)\}$ and $C = \{\{c_1\},\{c_2\},\ell(c)\}$. Let $X \in A$, $Y \in B$ and $Z \in C$. If either X, Y or Z is a subset of one of the others, we set $V_0(X,Y,Z) = \emptyset$ otherwise let

$$V_0(X,Y,Z) = \{v \in \mathbb{R}^2 \mid d_{\overrightarrow{H}}(v,X) = d_{\overrightarrow{H}}(v,Y) = d_{\overrightarrow{H}}(v,Z)\}$$

be the set of points with the same distance to all sets X, Y and Z. The ***set of Voronoi-vertex-candidates*** $V(a,b,c)$ of the line segments a, b and c is defined as

$$V(a,b,c) = \bigcup_{X\in A, Y\in B, Z\in C} V_0(X,Y,Z).$$

By only considering Voronoi-vertex-candidates, we restrict ourselves to a finite set of vertices that includes all relevant Voronoi vertices and does not include the degenerate cases. Let P and Q be two polygonal regions that may contain holes. Let further $\Delta \in \mathbb{R}_+$. The distance $d_{\overrightarrow{H}}(p,Q)$ for points $p \in P$ can be maximized at points in the interior of P or at points at the boundary of P (see Figure 2 for the two cases). Since these cases are different to analyze, we split the query into two general predicates.

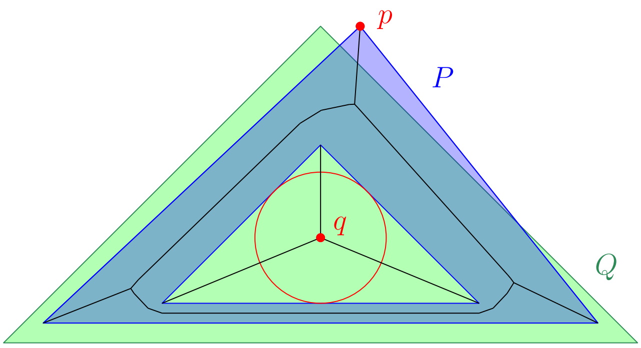

Figure 2 Illustration of the two cases: The point p on the boundary of P maximizes $d_{\overrightarrow{H}}(p,Q)$. The point q in the interior of Q that is a Voronoi vertex of the edges of P maximizes $d_{\overrightarrow{H}}(q,P)$.

- $(\mathcal{B})$ (Boundary): This predicate returns true if and only if $d_{\overrightarrow{H}}(\partial P, Q) \leq \Delta$.
- $(\mathcal{I})$ (Interior): This predicate returns true if $d_{\overrightarrow{H}}(P,Q) \leq \Delta$. This predicate returns false if $d_{\overrightarrow{H}}(P,Q) > d_{\overrightarrow{H}}(\partial P, Q)$ and $d_{\overrightarrow{H}}(P,Q) > \Delta$.

Note that it is not defined what $(\mathcal{I})$ returns if $d_{\overrightarrow{H}}(P,Q) = d_{\overrightarrow{H}}(\partial P, Q)$ and $d_{\overrightarrow{H}}(P,Q) > \Delta$. This does not matter, since the correctness of $d_{\overrightarrow{H}}(P,Q) \leq \Delta$ is still equivalent to both $(\mathcal{B})$ and $(\mathcal{I})$ being true.

Since $(\mathcal{B})$ and $(\mathcal{I})$ are very general, we define more detailed predicates that can be used to determine feasible truth values of $(\mathcal{B})$ and $(\mathcal{I})$. To determine $(\mathcal{B})$, we need the following predicates in combination with $\mathcal{P}_1$ and $\mathcal{P}_2$ (defined earlier for curves):

- $(\mathcal{P}_3)$: Given a vertex p of P, this predicate returns true if and only if $p \in Q$.
- $(\mathcal{P}_4)$: Given an edge e_1 of P and an edge e_2 of Q, this predicate is equal to $e_1 \cap e_2 \neq \emptyset$.
- $(\mathcal{P}_5)$: Given a directed edge e_1 of P and two edges e_2 and e_3 of Q, this predicate is true if and only if $e_1 \cap e_2 \neq \emptyset$, $e_1 \cap e_3 \neq \emptyset$ and e_1 intersects e_2 before or at the same point that it intersects e_3.
- $(\mathcal{P}_6)$: Given a directed edge e_1 of P and two edges e_2 and e_3 of Q, this predicate is true if and only if $e_1 \cap e_2 \neq \emptyset$ and if there exists a point b on e_3 such that $\|a-b\| \leq \Delta$ where a is the first intersection point of $e_1 \cap e_2$.
- $(\mathcal{P}_7)$: Given a directed edge e_1 of P and two edges e_2 and e_3 of Q, this predicate is true if and only if $e_1 \cap e_2 \neq \emptyset$ and if there exists a point b on e_3 such that $\|a-b\| \leq \Delta$ where a is the last intersection point of $e_1 \cap e_2$.

Using Voronoi-vertex-candidates, we define the detailed predicates for determining $(\mathcal{I})$:

- $(\mathcal{P}_8)$: Given 4 edges e_1, e_2, e_3, e_4 of Q and a point v from the set of Voronoi-vertex-candidates $V(e_1, e_2, e_3)$, this predicate returns true if and only if there exists a point $p \in e_4$, such that $\|v-p\| \leq \Delta$.
- $(\mathcal{P}_9)$: Given 3 edges e_1, e_2, e_3 of Q and a point v from the set of Voronoi-vertex-candidates $V(e_1, e_2, e_3)$, this predicate returns true if and only if $v \in Q$.
- $(\mathcal{P}_{10})$: Given 3 edges e_1, e_2, e_3 of Q and a point v from the set of Voronoi-vertex-candidates $V(e_1, e_2, e_3)$, this predicate returns true if and only if $v \in P$.

Examples for the predicates $\mathcal{P}_3, \ldots \mathcal{P}_{10}$ are depicted in Figure 3.

In the full version [4], we show that given the truth values of all these predicates one can determine a feasible truth value for predicates of the type $(\mathcal{B})$ and $(\mathcal{I})$. The proof for $(\mathcal{B})$ is very similiar to the proof of Lemma 7.1 in [7] for polygonal curves. In the proof for $(\mathcal{I})$, we show by contradiction that if the distance is realized only in the interior and at no Voronoi

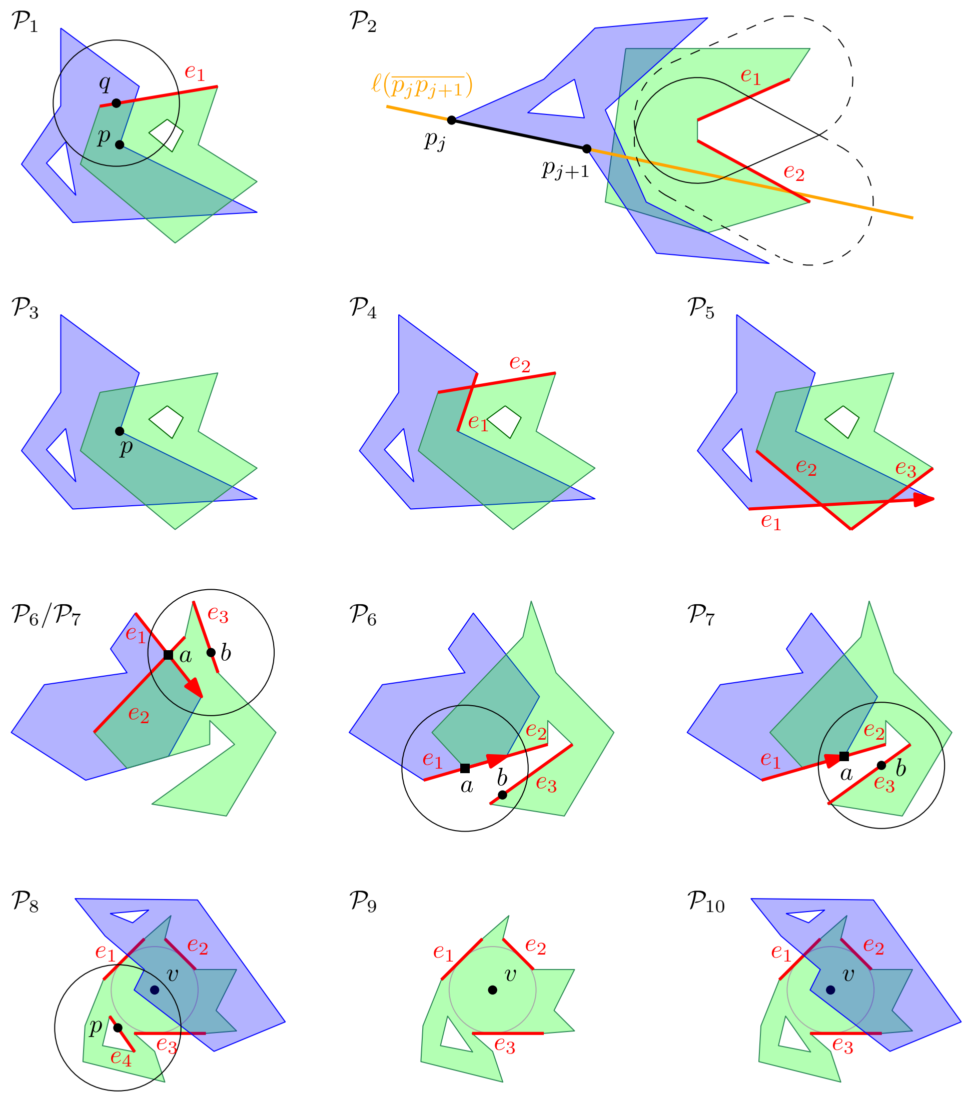

Figure 3 Illustration of the predicates $\mathcal{P}_1, \ldots, \mathcal{P}_{10}$: In all depicted cases the predicates are true.

vertex, then you can always increase the distance of Q to a point p in the interior of P by moving p to a Voronoi vertex or to the boundary.

Furthermore, we give a detailed proof in the full version [4], that all predicates $\mathcal{P}_1, \ldots, \mathcal{P}_{10}$ can be determined by a polynomial number of simple predicates. In that technical proof, we explicitly determine for each predicate, how it can be divided into sign values of polynomials. Corollary 3.2 then implies the following bounds on the VC-dimension.

▶ **Theorem 3.5.** *Let $\mathcal{R}_{d_H,k}$ be the range space of balls centered at polygonal curves in $\mathbb{X}_k^d$ with ground set $\mathbb{X}_m^d$. $VCdim(\mathcal{R}_{d_H,k})$ is in $O(dk\log(km))$.*

▶ **Theorem 3.6.** *Let $\mathcal{R}_{d_H,k}$ be the range space of balls centered at polygonal regions that may contain holes in $(\mathbb{R}^{2+1})^k$ with ground set $(\mathbb{R}^{2+1})^m$. The third dimension encodes a label that associates each vertex with its boundary component. $VCdim(\mathcal{R}_{d_H,k})$ is in $O(k\log(km))$.*

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