Sibson's formula for higher order Voronoi diagrams

Mercè Claverol[1], Andrea de las Heras-Parrilla[1], Clemens Huemer[1], and Dolores Lara[2]

1 **Universitat Politècnica de Catalunya**
merce.claverol@upc.edu, andrea.de.las.heras@upc.edu, clemens.huemer@upc.edu

2 **Centro de Investigación y de Estudios Avanzados**
dlara@cs.cinvestav.mx

Abstract

Let S be a set of n points in general position in $\mathbb{R}^d$. The order-k Voronoi diagram of S, $V_k(S)$, is a subdivision of $\mathbb{R}^d$ into cells whose points have the same k nearest points of S. Sibson, in his seminal paper from 1980 (A vector identity for the Dirichlet tessellation), gives a formula to express a point Q of S as a convex combination of other points of S by using ratios of volumes of the intersection of cells of $V_2(S)$ and the cell of Q in $V_1(S)$. The natural neighbour interpolation method is based on Sibson's formula. We generalize his result to express Q as a convex combination of other points of S by using ratios of volumes from Voronoi diagrams of any given order.

1 Introduction

Let S be a set of n points in general position in $\mathbb{R}^d$, meaning no m of them lie in a $(m-2)$-dimensional flat for $m = 2, 3, ..., d+1$ and no $d+2$ of them lie in the same d-sphere, and let k be a natural number with $1 \leq k \leq n-1$. Let σ_d denote the Lebesgue measure on $\mathbb{R}^d$, to simplify we just write σ.

The order-k Voronoi diagram of S, $V_k(S)$, is a subdivision of $\mathbb{R}^d$ into cells such that points in the same cell have the same k nearest points of S. Thus, each cell $f(P_k)$ of $V_k(S)$ is defined by a subset P_k of S of k elements, where each point of $f(P_k)$ has P_k as its k closest points from S. Similarly, the ordered Voronoi diagram of order k of S, $OV_k(S)$, can be defined as a subdivision of $\mathbb{R}^d$ into cells such that points in the same cell have the same ordered k nearest points of S. Thus, each cell $f(\langle P_k \rangle)$ of $OV_k(S)$ is defined by an ordered subset $\langle P_k \rangle$ of size k of S, where the points are arranged in order of proximity starting from the closest to the farthest. Note that, by definition, the union of all the cells of $OV_k(S)$ corresponding to the different permutations of a fixed subset of length k of S is the cell, $f(P_k)$, associated to such subset in the (ordinary) order-k Voronoi diagram, $V_k(S)$. See Figure 1.

For the order-k Voronoi diagram of S, the region $R_k(\ell)$ of $Q_\ell \in S$ is defined as the set of cells of $V_k(S)$ that have the point Q_ℓ as one of their k nearest neighbours. See Figure 2. For $OV_k(S)$ we can define these regions in the same way. These regions are not necessarily convex but star-shaped, see [2, 4, 10, 16], and it is known that $R_1(\ell)$ is contained in the kernel of $R_k(\ell)$; see [3]. Also, these regions are related to Brillouin zones. For a given k, the region $R_k(\ell) \setminus R_{k-1}(\ell)$ is known as a Brillouin zone of Q_ℓ. Brillouin zones have been studied mainly for lattices but also for arbitrary discrete sets, see e.g. [6, 17].

Local coordinates based on Voronoi diagrams were introduced by Sibson [13]. He states that, given a set S of n points of $\mathbb{R}^d$ in general position, a point $Q_\ell \in S$ can be expressed as a convex combination of its nearest points of S. This is described next. Cells of $V_2(S)$ that intersect $f(\{Q_\ell\})$ in $V_1(S)$ are of the form $f(\{Q_\ell, Q_j\})$, i.e., cells defined by Q_ℓ and another point Q_j, that we call its natural neighbour. These intersections give ratios of volumes which are the coefficients multiplying the corresponding natural neighbours in the convex

40th European Workshop on Computational Geometry, Ioannina, Greece, March 13–15, 2024.
This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

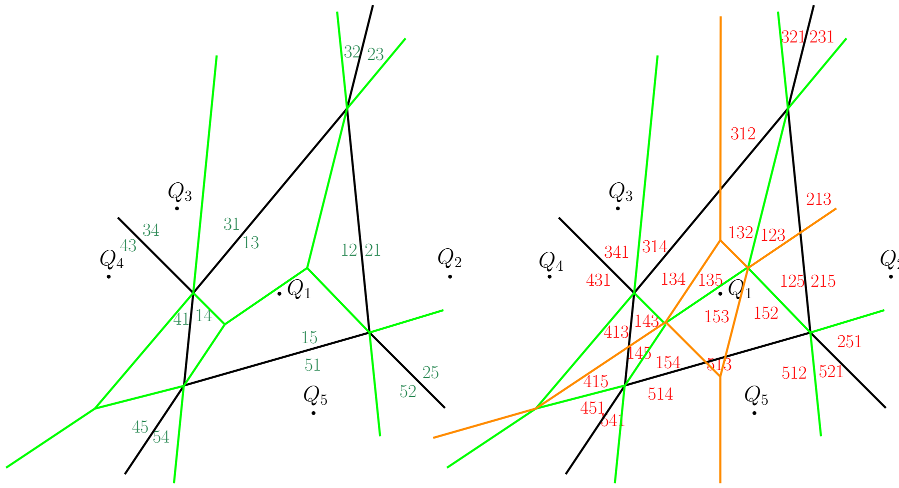

Figure 1 For a set $S = \{Q_1, \cdots, Q_5\}$ of five points in $\mathbb{R}^2$. Each cell of $OV_k(S)$ is labeled by the indices of its k nearest points of S. $V_1(S)$ is shown in black, $V_2(S)$ in green, and $V_3(S)$ in orange colour. Left: The cells of $OV_2(S)$ with the same nearest neighbour Q_i from S form the cell $f(\{Q_i\})$ in $V_1(S)$. The cells of $OV_2(S)$ with the same subset P_2 of two points of S (in any order) form the cell $f(P_2)$ of $V_2(S)$. Right: $OV_3(S)$ is shown together with $V_1(S)$, $V_2(S)$, and $V_3(S)$.

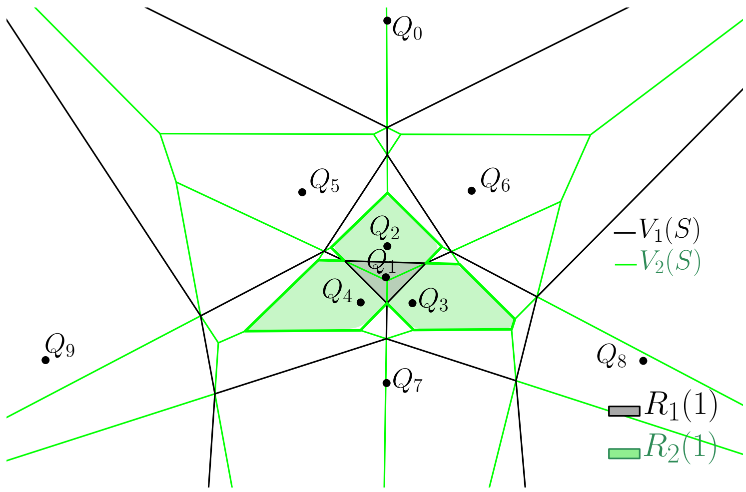

Figure 2 $R_1(1)$ is the cell $f(\{Q_1\})$ in $V_1(S)$. $R_2(1)$ is the union of cells of $V_2(S)$ that have Q_1 as one of its two nearest neighbours. $R_1(1) \subset R_2(1)$.

combination that expresses Q_ℓ. Volumes $\sigma(f(\{Q_\ell, Q_j\}) \cap f(\{Q_\ell\}))$ are equal to the volumes given by the intersection of the cells of $V_1(S \setminus \{Q_\ell\})$ and $f(Q_\ell)$ in $V_1(S)$, see Figure 3.

▶ **Theorem 1.1.** *(Local coordinates property [13]). For a bounded cell $f(\{Q_\ell\})$ of $V_1(S)$,*

$$Q_\ell = \sum_{j \neq \ell} \frac{\sigma(f(\{Q_\ell, Q_j\}) \cap f(\{Q_\ell\}))}{\sigma(f(\{Q_\ell\}))} Q_j \tag{1}$$

Sibson's formula has been used to define the natural neighbour interpolation method [14]. Given a set of points and a function, this interpolation method provides a smooth approximation of new points to the function. Sibson's algorithm uses the closest subset of the input set $S \setminus \{Q_\ell\}$ to interpolate a query point, Q_ℓ, and applies weights based on the ratios of volumes provided by Theorem 1.1. Local coordinates and the natural neighbour interpolation method have been studied e.g. in [5, 11, 15], and they have many applications such as reconstruction of a surface from unstructured data or interpolation of rainfall data, see [9, 15].

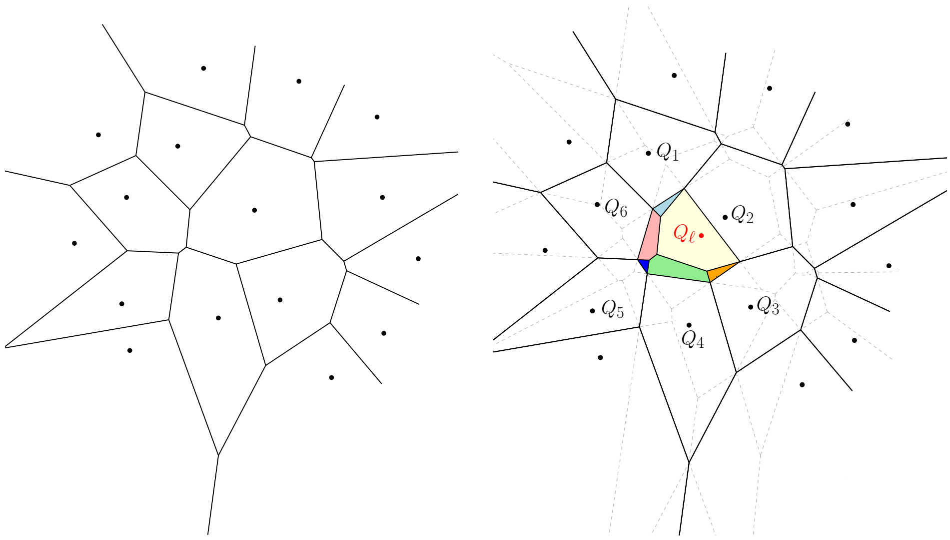

Figure 3 In $\mathbb{R}^2$. Left: The initial Voronoi diagram $V_1(S \setminus \{Q_\ell\})$ without query point Q_ℓ. Right: Colored areas given by the intersections of $f(\{Q_\ell\})$ and the cells of $V_1(S \setminus \{Q_\ell\})$, are the same as the ones given by the intersections of the cells of $V_2(S)$ (shown in dashed) with the cell $f(\{Q_\ell\})$.

Aurenhammer gave a generalization of Sibson's result to Voronoi diagrams of higher order, and more generally to power diagrams, see [1]. Aurenhammer's formula allows to write a point Q_ℓ of S as a linear combination of other points of S. We state this in Theorem 2.1 and Corollary 2.2 below. The formula in Theorem 2.1 is defined in terms of $OV_{k+1}(S)$. It is restated in Corollary 2.2 in terms of intersections of cells of $V_{k-1}(S)$ and $V_{k+1}(S)$ with a cell of $V_k(S)$. This formula works for a bounded cell of $V_k(S)$.

Our main contribution is another generalization of Sibson's result, stated in Theorem 2.3. In this theorem, we express a point $Q_\ell \in S$ as a convex combination of its neighbours of S using ratios of volumes in the region $R_k(\ell)$. Similar to Sibson's formula that required the cell of the point Q_ℓ to be bounded, our formula requires its region $R_k(\ell)$ to be bounded. For the case $k = 1$, Theorem 2.3 coincides with Theorem 1.1.

This paper is organized as follows. Section 2 details the generalization of Sibson's formula. In Section 3 we give a geometric interpretation of the formulas presented in Section 2 for point sets in the plane. Finally, Section 4 is on how the generalization of Sibson's formula could be used for interpolation. Proofs are omitted in this abstract.

2 Coordinates based on Voronoi diagrams

In this section we present a generalization of Sibson's formula that expresses a point using its neighbours of the Voronoi diagram of any given order. For this, we recall results from Aurenhammer [1] in Theorem 2.1 and Corollary 2.2, using a different notation.

Let $F_{k+1}(P_k)$, to simplify $F(P_k)$, be the set of cells of $OV_{k+1}(S)$, $1 \le k \le n-2$, whose k nearest neighbours are the points of $P_k \subset S$ in any order, and the $(k+1)$-th nearest neighbour is another point of S not in P_k. Let $f_{i,j}$ denote the union of cells of $OV_{k+1}(S)$ whose k-th nearest neighbour is Q_i and whose $(k+1)$-th nearest neighbour is Q_j.

▶ **Theorem 2.1.** *([1]) If all cells in $F(P_k)$ are bounded in $OV_{k+1}(S)$, then*

$$\sum_{\substack{j \\ f_{i,j} \in F(P_k)}} \sigma(f_{i,j}) Q_i = \sum_{\substack{i \\ f_{i,j} \in F(P_k)}} \sigma(f_{i,j}) Q_j$$

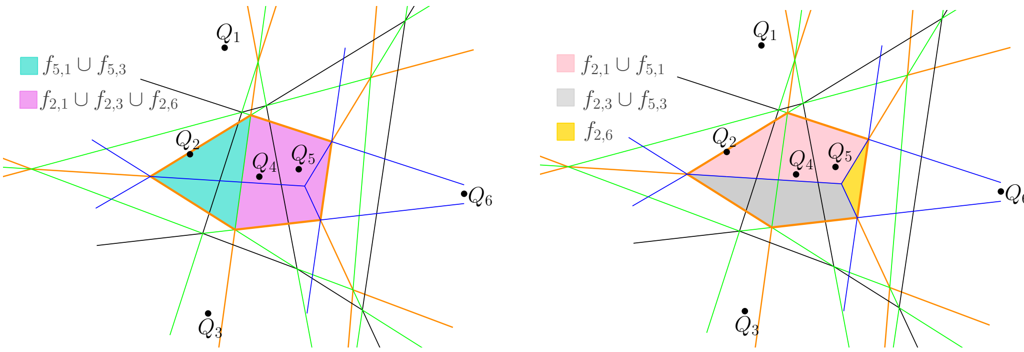

Figure 4 Illustrating Theorem 2.1 for $F(\{Q_2, Q_5\})$ in $OV_3(S)$, where S is a set of six points in $\mathbb{R}^2$. In this case the equation reduces to $\sigma(f_{5,1} \cup f_{5,3})Q_5 + \sigma(f_{2,1} \cup f_{2,3} \cup f_{2,6})Q_2 = \sigma(f_{2,1} \cup f_{5,1})Q_1 + \sigma(f_{2,3} \cup f_{5,3})Q_3 + \sigma(f_{2,6})Q_6$. Left: cells grouped according to its k-nearest neighbour. Right: cells grouped according to its $(k+1)$-nearest neighbour.

Note that, the subdivisions induced by $V_{k-1}(S)$ at the interior of $f(P_k)$ correspond to grouping the cells of $F(P_k)$ in $OV_{k+1}(S)$ that have the same k-nearest neighbour. Also, the subdivisions induced by $V_{k+1}(S)$ at the interior of $f(P_k)$ correspond to grouping the cells of $F(P_k)$ in $OV_{k+1}(S)$ that have the same $(k+1)$-nearest neighbour. See Figure 4.

By these observations, Theorem 2.1 can be stated as follows.

▶ **Corollary 2.2.** *([1]) Let $2 \le k \le n-2$ and let $f(P_k)$ be a bounded cell of $V_k(S)$. Then,*

$$\sum_{\substack{f(P_{k-1}) \in V_{k-1}(S) \\ Q_i \in P_k \setminus P_{k-1}}} \sigma(f(P_{k-1}) \cap f(P_k)) Q_i = \sum_{\substack{f(P_{k+1}) \in V_{k+1}(S) \\ Q_j \in P_{k+1} \setminus P_k}} \sigma(f(P_{k+1}) \cap f(P_k)) Q_j$$

Note that, by the relation between the Voronoi diagrams and the ordered Voronoi diagrams, $R_k(\ell)$ is the set of cells of $OV_{k+1}(S)$ that have Q_ℓ as one of their k nearest neighbours from S, i.e., $R_k(\ell) = \cup_{Q_\ell \in P_k} F(P_k)$. Based on this observation and Theorem 2.1 we can prove the following result.

▶ **Theorem 2.3.** *If $R_k(\ell)$ is a bounded region, then*

$$Q_\ell = \sum_{\substack{i \\ f_{i,j} \in R_k(\ell)}} \frac{\sigma(f_{i,j})}{\sigma(R_k(\ell))} Q_j.$$

▶ **Corollary 2.4.** *Let $1 \leq k \leq n-2$ and let $R_k(\ell)$ be a bounded region. Then,*

$$Q_\ell = \sum_{f(P_k)\in R_k(\ell)} \sum_{\substack{f(P_{k+1})\in V_{k+1}(S)\\ Q_j\in P_{k+1}\setminus P_k}} \frac{\sigma(f(P_{k+1})\cap f(P_k))}{\sigma(R_k(\ell))} Q_j$$

3 A geometric interpretation

In the following we examine the generalization of Sibson's theorem to higher order Voronoi diagrams from Corollary 2.2 in more detail for cells $f(P_k)$ of $V_k(S)$, when S is a point set in $\mathbb{R}^2$. Divide both sides of the equation given in Corollary 2.2 by $\sigma(f(P_k))$; then, each side of the equation describes a point H that is a convex combination of points from S. We have

$$H = \sum_{\substack{f(P_{k-1})\in V_{k-1}(S)\\ Q_i\in P_k\setminus P_{k-1}}} \frac{\sigma(f(P_{k-1})\cap f(P_k))}{\sigma(f(P_k))} Q_i = \sum_{\substack{f(P_{k+1})\in V_{k+1}(S)\\ Q_j\in P_{k+1}\setminus P_k}} \frac{\sigma(f(P_{k+1})\cap f(P_k))}{\sigma(f(P_k))} Q_j \tag{2}$$

What can we say about this point H?

Let $f(P_k)$ be an r-gon. Then S contains r points $Q_1,\dots,Q_r$, such that each edge of the r-gon lies on a perpendicular bisector between two of these r points, and each vertex, $C_{ij\ell}$, of $f(P_k)$ is the center of a circle passing through three of them, Q_i, Q_j, and Q_ℓ; see e.g. [3].

We denote with $\Delta(ABC)$ the triangle with vertices A, B, and C, and with $\square(ABCD)$ the quadrilateral with vertices A, B, C and D, in cyclic order.

Let us consider the case when $f(P_k)$ is a quadrilateral cell of $V_k(S)$ with vertices C_{123}, C_{124}, C_{134}, and C_{234}, in cyclic order along the boundary of the quadrilateral cell $f(P_k) = \square(C_{123}C_{124}C_{134}C_{234})$. One of the diagonals $C_{123}C_{134}$ and $C_{124}C_{234}$ is an edge of $V_{k-1}(S)$ and the other one of $V_{k+1}(S)$. Figure 5 shows an example. We refer to [3, 7] for a more detailed discussion on the structure of cells of $V_k(S)$. Corollary 2.2 states in this case that

$$\begin{aligned} H &= Q1\cdot\frac{\sigma(\Delta(C_{123}C_{134}C_{234}))}{\sigma(\square(C_{123}C_{124}C_{134}C_{234}))} + Q3\cdot\frac{\sigma(\Delta(C_{123}C_{124}C_{134}))}{\sigma(\square(C_{123}C_{124}C_{134}C_{234}))}\\ &= Q2\cdot\frac{\sigma(\Delta(C_{124}C_{134}C_{234}))}{\sigma(\square(C_{123}C_{124}C_{134}C_{234}))} + Q4\cdot\frac{\sigma(\Delta(C_{124}C_{234}C_{123}))}{\sigma(\square(C_{123}C_{124}C_{134}C_{234}))} \end{aligned} \tag{3}$$

It follows that H is the intersection point of diagonals Q_1Q_3 and Q_2Q_4 of $\square(Q_1Q_2Q_3Q_4)$. This implies that given a quadrilateral cell $\square(C_{123}C_{124}C_{134}C_{234})$ of $V_k(S)$, the four corresponding points of S also form a convex quadrilateral, $\square(Q_1Q_2Q_3Q_4)$. Moreover, we can show that areas of triangles with vertices in $\square(C_{123}C_{124}C_{134}C_{234})$ are proportional to areas of triangles with vertices in $\square(Q_1Q_2Q_3Q_4)$, also see [8, 12].

Let us then consider the case when $f(P_k)$ is a cell of $V_k(S)$ with more than four sides. Equation (2) gives a point H that can be expressed in two ways as convex combination of points of S. Let us look at a pentagonal cell $f(P_k) = \pentagon(C_{123}C_{134}C_{145}C_{245}C_{125})$ of $V_k(S)$; See Figure 6. For $r > 5$ the situation is similar. Corollary 2.2 here gives

$$\begin{aligned} H &= Q_1\cdot\frac{\sigma(\square(C_{123}C_{125}C_{145}C_{134}))}{\sigma(\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125}))} + Q_5\cdot\frac{\sigma(\Delta(C_{125}C_{245}C_{145}))}{\sigma(\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125}))}\\ &= Q_2\cdot\frac{\sigma(\square(C_{245}C_{125}C_{123}C_{234}))}{\sigma(\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125}))} + Q_4\cdot\frac{\sigma(\Delta(C_{245}C_{234}C_{1345}C_{145}))}{\sigma(\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125}))}\\ &+ Q_3\cdot\frac{\sigma(\Delta(C_{123}C_{234}C_{134}))}{\sigma(\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125}))} \end{aligned}$$

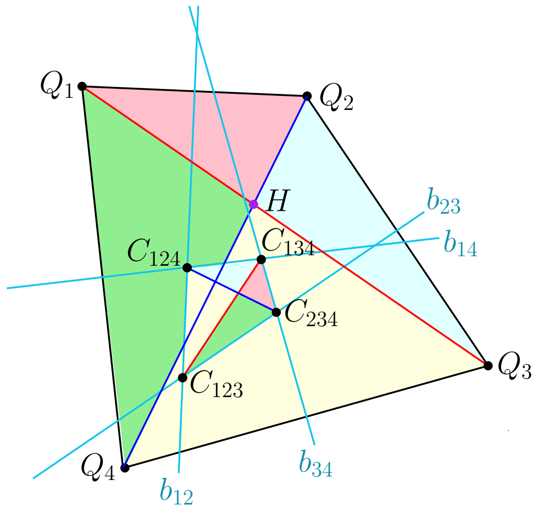

Figure 5 The quadrilateral cell $f(P_k) = \square(C_{123}C_{124}C_{134}C_{234})$ of $V_k(S)$ is obtained by perpendicular bisector construction from $\{Q_1, Q_2, Q_3, Q_4\} \subset S$. Point H given by Equation (2) is the intersection point of diagonals Q_1Q_3 and Q_2Q_4. Triangles with same color have proportional area.

We get that H lies on the segment Q_1Q_5 and inside the triangle $\Delta(Q_2Q_3Q_4)$. Furthermore, H divides the segment Q_1Q_5 in the same proportion as the edge $C_{125}C_{145}$ divides the pentagon $\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125})$ into the the quadrilateral $\square(C_{125}C_{145}C_{134}C_{123})$ and the triangle $\Delta(C_{125}C_{145}C_{245})$. And H divides triangle $\Delta(Q_1Q_2Q_3)$ in the same proportion into triangles $\Delta(Q_3HQ_4)$, $\Delta(Q_2HQ_3)$, and $\Delta(Q_2HQ_4)$ as C_{234} divides $\pentagon(C_{123}C_{134}C_{145}C_{245}C_{125})$ into $\square(C_{245}C_{125}C_{123}C_{234})$, $\square(C_{245}C_{234}C_{134}C_{145})$ and $\Delta(C_{134}C_{234}C_{123})$.

4 Towards higher order natural neighbour interpolation

Sibson's theorem (Theorem 1.1) gave rise to the natural neighbour interpolation method. Given a set of points S and known function values $G(Q_j)$ for $Q_j \in S \setminus \{Q_\ell\}$, the function value $G(Q_\ell)$ of a point Q_ℓ is interpolated by $G(Q_\ell) = \sum_j c_j G(Q_j)$, where the sum is over the natural neighbours Q_j of Q_ℓ in $V_1(S)$. The local coordinates c_j are given by Theorem 1.1. Note that they satisfy $\sum_j c_j = 1$ and $c_j \geq 0$ for all j. Then, Sibson's natural neighour interpolation is given by

$$G(Q_\ell) = \sum_{j \neq \ell} \frac{\sigma(f(\{Q_\ell, Q_j\}) \cap f(\{Q_\ell\}))}{\sigma(f(\{Q_\ell\}))} G(Q_j). \tag{4}$$

The generalization of Sibson's formula given in Theorem 2.3 suggests to approximate the function value $G(Q_i)$ by using the natural neighbours of higher order Voronoi diagrams. By using the region $R_k(\ell)$ for $k > 1$, we can estimate the function value of a point Q_ℓ as

$$G(Q_\ell) = \sum_{\substack{i \\ f_{i,j} \in R_k(\ell)}} \frac{\sigma(f_{i,j})}{\sigma(R_k(\ell))} G(Q_j). \tag{5}$$

Note that $R_1(\ell) = f(\{Q_\ell\})$ in $V_1(S)$, and for $k = 1$ Equations (4) and (5) coincide.

A better estimation can be obtained by using Theorem 2.3 in a combination of different values of k. We explore this for the 1-dimensional case.

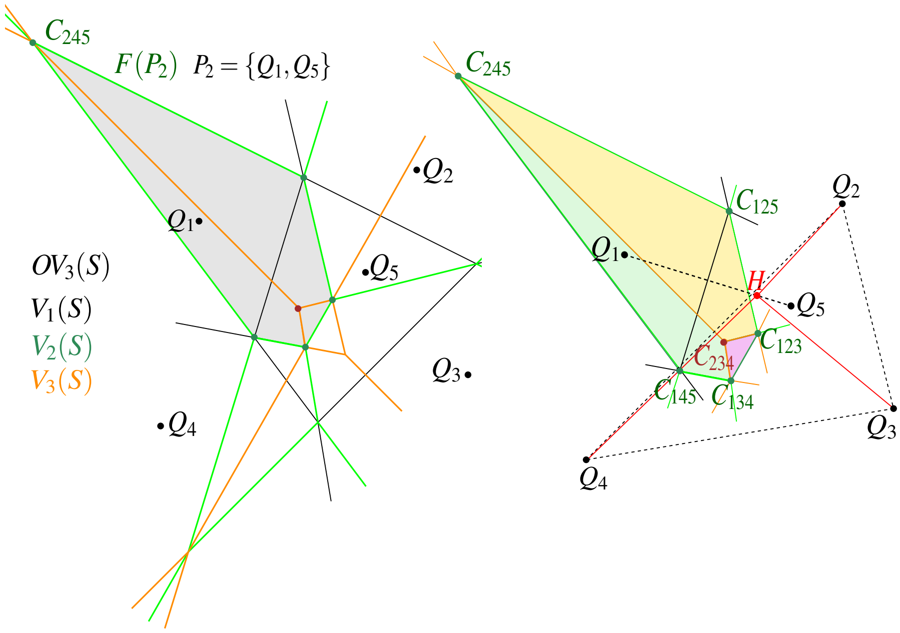

Figure 6 (Left) $OV_3(S)$ for a set of five points $S = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$. For $P_2 = \{Q_1, Q_5\}$, the grey region $F(P_2)$ of $OV_3(S)$ is the pentagonal cell $f(P_2)$ of $V_2(S)$. $f(P_2)$ is divided by an edge of $V_1(S)$ and is also divided by three edges of $V_3(S)$. (Right) The point H lies on the segment Q_1Q_5 and inside the triangle $\Delta(Q_2, Q_3, Q_4)$. Triangle areas of $\Delta(Q_2HQ_3), \Delta(Q_3HQ_4)$ and $\Delta(Q_2HQ_4)$ are proportional to the areas of the three colored regions inside $f(P_2)$, green, yellow, and pink, respectively. The lengths of segments HQ_1 and HQ_5 are proportional to the areas $\sigma(f(P_2) \cap f(\{Q_1\}))$ and $\sigma(f(P_2) \cap f(\{Q_5\}))$, respectively.

Theorem 2.3, respectively Corollary 2.4, for dimension 1 reduces to the following statement.

▶ **Property 4.1.** *Let $S = \{x_0, x_1, \ldots x_{2\ell}\}$ with $x_0 < x_1 < \ldots < x_{2\ell}$ be real numbers. Then,*

$$x_\ell = \frac{1}{x_{2\ell} - x_0} \left(\left(\sum_{i=0}^{\ell-1} x_i (x_{\ell+1+i} - x_{\ell+i}) \right) + \left(\sum_{i=\ell+1}^{2\ell} x_i (x_{i-\ell} - x_{i-\ell-1}) \right) \right). \tag{6}$$

▶ Remark. Property 4.1 has actually a more general statement. The assumption $x_0 < x_1 < \ldots < x_{2\ell}$ is not needed.

We denote points Q_i of S as x_i and their function values $G(Q_i)$ as y_i. When $k = 1$ we have Sibson's classical nearest neighbour interpolation, which for dimension $d = 1$ is piecewise linear interpolation. Let $x_0, x_1, \ldots, x_5$ be six points on the real line in that order. And let $x_2 < x < x_3$ be a query point whose function value $G(x)$ we want to interpolate. To avoid degenerate cases where bisectors between points coincide, we also assume that all midpoints $(x_i + x_j)/2$ with $x_i, x_j \in \{S \cup \{x\}\}$ are different. Sibson's classical formula, Equation (4), uses the two neighbours x_2 and x_3 of x, and gives the interpolation

$$G_1(x) = \frac{1}{x_3 - x_2} \left(y_2(x_3 - x) + y_3(x - x_2) \right), \tag{7}$$

i.e. point $(x, G_1(x))$ lies on the line segment connecting points (x_2, y_2) and (x_3, y_3). This can also be deduced from Property 4.1. Combining Equation (5) for $k = 1$ and $k = 2$, we obtain

$$G_2(x) = \frac{1}{x_4 - x_1 + x_3 - x_2}(y_1(x_3 - x) + y_2(x_4 - x) + y_3(x - x_1) + y_4(x - x_2)). \quad (8)$$

In the same way, combining Equation (5) for $k = 1$, $k = 2$, and $k = 3$, we obtain

$$G_3(x) = \frac{1}{x_5 - x_0 + x_4 - x_1 + x_3 - x_2}(y_0(x_3 - x) + y_1(x_4 - x) + y_2(x_5 - x) + y_3(x - x_0) + y_4(x - x_1) + y_5(x - x_2)). \quad (9)$$

Figure 7 shows an example of the interpolation formulas given in Equations (7), (8), and (9).

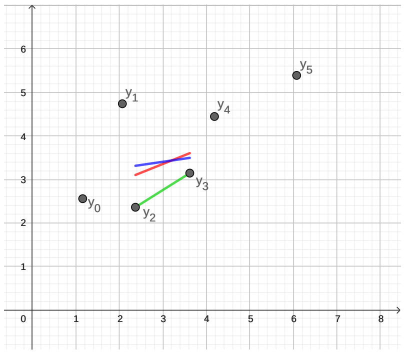

Figure 7 The generalized Sibson interpolation in $\mathbb{R}^1$. In green: Sibson's original interpolation, Equation (7), used only $R_1(x)$. The blue segment shows the interpolation using $R_1(x)$ and $R_2(x)$, given by Equation (8). Four points are used. The red segment shows the interpolation using $R_1(x)$, $R_2(x)$, and $R_3(x)$, given by Equation (9). Six points are used.

We conclude with some comments on the proposed interpolation formulas. First, they appear in a natural way from the generalization of Sibson's formula. This already makes it worth to study such generalized interpolation formulas. In Equations (8) and (9), the coefficients c_j in $G_i(x) = \sum_j c_j y_j$, $i = 2, 3$, satisfy $\sum_j c_j = 1$ and $c_j \geq 0$ for every c_j. We also mention that it can not be guaranteed that $G_i(x)$ coincides with $G_i(x_2)$ or with $G_i(x_3)$, when x coincides with one of the endpoints of the interval, x_2 or x_3, respectively. Though, we observe that in this case, the point farthest away from x on one side, drops from being used in the interpolation formula. This also holds for the classical case $k = 1$.

Finally, we expect that the generalized interpolation formulas can have applications. For instance, when the used values for the interpolation are obtained by measurements and measurement inaccuracy can not be ruled out. Then reliability might be improved by using nearest neighbours from $V_k(S)$ or by using $R_k(x)$, instead of only $V_1(S)$.

Acknowledgements

This research has been supported by projects 2021SGR00266 and PID2019-104129GB-I00/ MCIN/ AEI/ 10.13039/501100011033.

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