The Number of Non-overlapping Edge Unfoldings in Convex Regular-faced Polyhedra

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Abstract

An edge unfolding of a polyhedron is a flat polygon obtained by cutting along the polyhedron's edges and unfolding the polygon onto a plane. It is known that the number of edge unfoldings is equal to the number of spanning trees formed by the cutting edges of the polyhedron. However, some edge unfoldings overlap, i.e., two distinct faces in the edge unfoldings overlap, so they cannot be embedded in the plane. We do not know the percentage of the overlapping edge unfoldings for almost all polyhedra. In particular, there exists an interesting and well-known open problem of whether or not all convex polyhedrons have a non-overlapping edge unfolding. Horiyama et al. proposed an enumeration algorithm for edge unfoldings using zero-suppressed binary decision diagrams (ZDDs), which are compact data structures for families of sets. The ZDDs have attractive family algebraic operations; for example, we can extract sets satisfying some constraints from the family of sets over ZDDs efficiently. In this study, we propose an enumeration algorithm for non-overlapping edge unfoldings in a polyhedron using ZDDs and their operations. The algorithm first enumerates the minimal overlapping partial edge unfoldings (MOPEs) obtained through the "rotational unfolding" by Shiota and Saitoh. Then, we subtract the overlapping edge unfoldings containing the MOPEs from all edge unfoldings over ZDDs. We apply the algorithm to convex regular-faced polyhedra (including three types of Archimedean solids, twenty types of Johnson solids, nineteen types of Archimedean prisms, and twenty-one types of Archimedean antiprisms) and show the number of non-overlapping edge unfoldings for each type of polyhedron.

1 Introduction

An edge unfolding of a polyhedron is a flat polygon obtained by cutting along the polyhedron's edges and unfolding the polygon onto a plane. The origin of edge unfoldings is recognized as the illustrations found in Albrecht Dürer's "Underweysung der messung mit dem zirckel un richt scheyt" [5] published in 1525 [3]. However, the edge unfoldings can sometimes result in overlapping polygons, i.e., two distinct faces overlap, or their boundaries are in contact (Figure 1). In Dürer's book, all the polyhedra are drawn as edge unfoldings

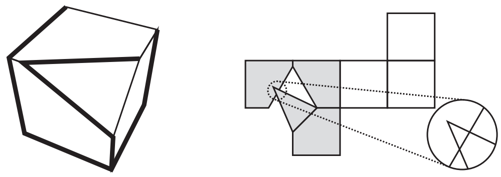

Figure 1 A cube with cut-off corners and its overlapping edge unfolding. The faces shown in gray are a MOPE.

40th European Workshop on Computational Geometry, Ioannina, Greece, March 13–15, 2024.
This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

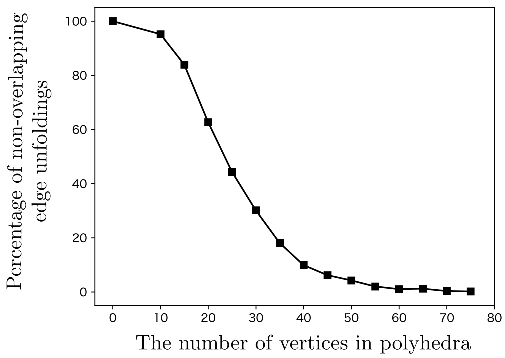

Figure 2 The line graph represents the percentage of non-overlapping edge unfoldings among 1000 randomly selected edge unfoldings. Each point on the graph represents the average values for five randomly generated convex polyhedra [15].

Table 1 Overlapping edge unfoldings for convex regular-faced polyhedra

Convex regular-faced polyhedra	Is there an overlapping edge unfolding?
Platonic solids (Total 5 types)	No [8]
Archimedean solids (Total 13 types)	No (7 types) / Yes (6 types) [2, 8, 6, 18]
Johnson solids (Total 92 types)	No (48 types) / Yes (44 types) [17]
n-gonal Archimedean prisms ($n \geq 3$)	No ($3 \leq n \leq 23$) / Yes ($n \geq 24$) [18]
m-gonal Archimedean antiprisms ($m \geq 3$)	No ($3 \leq m \leq 11$) / Yes ($m \geq 12$) [18]

without overlaps. Focusing on this point, Shephard proposed the following conjecture.

▶ **Conjecture 1.1** ([16]). *For any convex polyhedron, at least one non-overlapping edge unfolding exists.*

This conjecture is still unsolved, and some studies to solve it are ongoing. One of the studies is Schevon's experiment on randomly generated convex polyhedra [15]. She showed that the percentage of non-overlapping edge unfoldings decreases as the number of vertices increases (Figure 2). Some studies have reported the existence of an overlapping edge unfolding in a given polyhedron. Shiota and Saitoh presented an algorithm "rotational unfolding" that can quickly find an overlapping edge unfolding of a polyhedron, and they showed the existence of overlapping edge unfoldings for convex regular-faced polyhedra (Table 1).

It is known that the number of edge unfoldings is equal to the number of spanning trees formed by the cutting edges of the polyhedron. We can count the number of spanning trees using Kirchhoff's theorem [13] or a data structure called binary decision diagrams (BDD) [1] / zero-suppressed binary decision diagram (ZDD) [14]. The BDDs/ZDDs represent compact data structures for families of sets and have family algebraic operations (i.e., union, intersection, and set difference). In addition, BDDs/ZDDs allow for the counting, enumerating, and extracting of optimal families of sets. BDDs/ZDDs have been applied to enumerate specific structures on graphs [12]. Horiyama et al. enumerated spanning trees using BDDs/ZDDs and counted the number of convex regular-faced polyhedra [9, 7]. Horiyama and Shoji proposed a method for counting the number of non-overlapping edge unfolding for Platonic solids by extracting each spanning tree one by one from BDDs [8]. However, this method only applies to the polyhedra with few edge unfoldings. For example, the truncated icosahedron (Figure 3) has $375,291,866,372,898,816,000$ (approximately 3.75×10^{20}) edge unfoldings [9], so checking each unfolding individually, it would take over ten thousand years with current computers.

In this study, we propose an enumeration algorithm for non-overlapping edge unfoldings in a polyhedron using ZDDs and their operations. The algorithm first enumerates the minimal overlapping partial edge unfoldings (MOPEs), which are the minimal units of edge unfoldings with overlaps obtained through the rotational unfolding [18]. Then, we subtract the overlapping edge unfoldings containing the MOPEs from all edge unfoldings over ZDDs.

We apply this counting method to convex regular-faced polyhedra, including three types of Archimedean solids, twenty types of Johnson solids, nineteen types of Archimedean prisms, and twenty-one types of Archimedean antiprisms, and show the number of non-overlapping edge unfoldings for each type of polyhedron.

2 Preliminaries

A *polyhedron* is a three-dimensional object consisting of at least four polygons, called *faces*, joined at their edges. A *convex polyhedron* is a polyhedron with the interior angles between any two adjacent faces less than π. A *convex regular-faced polyhedron* is a convex polyhedron with all faces being regular polygons. A *Platonic solid* is a convex regular-faced polyhedron with all faces composed of congruent regular polygons. An n *prism*, where $n \geq 3$, is a polyhedron composed of two identical n-sided polygons, called *bases*, facing each other, and n parallelograms, called *side faces*, connecting the corresponding edges of the two bases. An m *antiprism*, where $m \geq 3$, is a polyhedron composed of two congruent m-sided polygonal bases and $2m$ triangular side faces alternating around the bases. An *n-gonal (anti)prism* is an n (anti)prism if the bases are n-sided regular polygons. An *n-gonal Archimedean (anti)prism* is an n-gonal (anti)prism if it is a convex regular-faced polyhedron (i.e., the side faces are regular squares (or triangles)). An *Archimedean solid* is a convex regular-faced polyhedron composed of regular polygons with the same type and order of regular polygons gathered at the vertices, except for Platonic solids and Archimedean (anti)prisms. A *Johnson solid* is a convex regular-faced polyhedron, except Platonic solids, Archimedean solids, and Archimedean (anti)prisms. There are 92 Johnson solids [11].

Let Q be a polyhedron. Two faces in Q are *neighbors* if they contain a common edge. An *unfolding* (also called a net, a development, or a general unfolding) of the polyhedron Q is a flat polygon formed by cutting Q's edges or faces and unfolding it into a plane. An *edge unfolding* of Q is an unfolding formed by cutting only Q's edges. Q can be viewed as a graph $G_Q = (V_Q, E_Q)$, where V_Q is a set of faces in the polyhedron, and E_Q is a set of edges such that two vertices are adjacent if and only if the corresponding two faces are neighbors. The following lemma applies to an edge unfolding of Q.

▶ **Lemma 2.1** (see e.g., [19], Theorem 2.2.1 and its proof). *The set of non-cutting edges for*

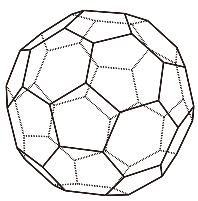

Figure 3 A truncated icosahedron (a type of Archimedean solid)

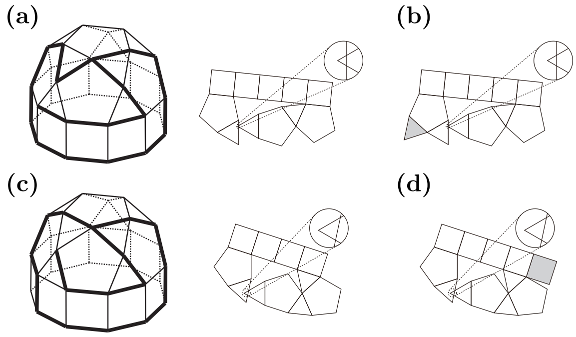

Figure 4 (a),(c) MOPEs in J21 (a type of Johnson solid). Removing any face from each MOPE results in non-connected structures, contradicting the definition of partial edge unfoldings. (b),(d) Partial edge unfoldings in J21 that are not MOPE. Removing the gray faces results in MOPEs.

an edge unfolding of Q forms a spanning tree of G_Q.

This lemma implies that counting the spanning trees of G_Q is equal to counting the edge unfoldings of Q. A *partial edge unfolding* is a flat polygon formed from a set of faces corresponding to a connected induced subgraph of G_Q.

Two distinct polygons *overlap* if there exists a point p contained in two polygons. Note that any point on a boundary is included in the polygons in this paper; the polygons overlap if they touch at the boundaries. An unfolding is *overlapping* if a pair of distinct faces exists such that the faces overlap. Herein, neighbor faces are not overlapping. The algorithm *rotational unfolding* has been developed to efficiently determine overlaps in a given polyhedron based on [4, 6] ideas [18]. Rotational unfolding can enumerate *minimal overlapping partial edge unfoldings (MOPEs)*, a partial edge unfolding with the minimal number of faces required to connect two overlapping faces. Figure 4 shows the example of MOPEs and non-MOPEs.

One method for counting spanning trees in a graph is using a *Zero-suppressed Decision Diagram (ZDD)*. A ZDD is a data structure representing families of sets compactly as a directed acyclic graph. In a ZDD, there are two types of nodes: *terminal nodes* with the out-degree zero $\top$, $\bot$, and *branching nodes*. Branching nodes are labeled by elements of the set, and each has two outgoing edges: a 1-*edge* and a 0-*edge*. The 1-edge indicates the inclusion of the labeled element, while the 0-edge indicates the exclusion of the element. In a ZDD, there is a *root node* with no incoming edges. For example, the ZDD, which represents a spanning tree as shown in Figure 5, and a path from the root node (labeled e_0) following a 1-edge, a 1-edge, a 0-edge, and a 1-edge leading to $\top$ means that the set $\{e_0, e_1, e_3\}$ forms a spanning tree. ZDDs have some operations, such as computing the union or intersection of two ZDDs [14].

3 Counting algorithm for non-overlapping edge unfoldings

In this section, we describe an algorithm counting the number of non-overlapping edge unfoldings for any polyhedron Q. Let P be a (partial) edge unfolding of Q, and let $G(P) = (V(P), E(P))$ be the graph corresponding to P. The following lemma holds.

▶ **Lemma 3.1.** *For any overlapping edge unfolding U, there exists a MOPE C such that $E(C) \subseteq E(U)$.*

Let C_i ($1 \leq i \leq k$) be MOPEs of a polyhedron, where k is the number of MOPEs. From Lemma 3.1, the following claim holds.

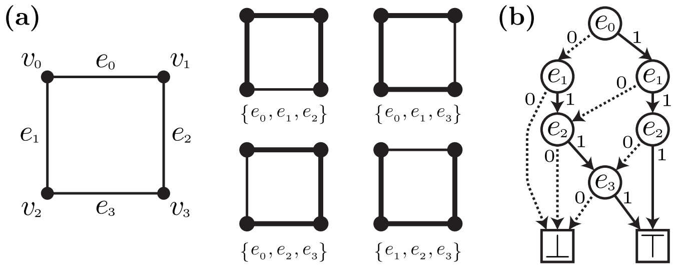

Figure 5 (a) An example of the graph C_4 and its spanning trees. (b) A ZDD representing the spanning trees of C_4. Circles represent branching nodes, labels are inside the circles, solid lines represent 1-edges, and dashed lines represent 0-edges.

▶ Claim 3.2. Let U be a non-overlapping edge unfolding. For any MOPE C_i, $E(C_i) \nsubseteq E(U)$.

The number of edge unfoldings can be counted by constructing ZDD $\mathcal{Z}_S$ [12]. However, it contains overlapping edge unfoldings. To exclude these overlapping unfoldings, we employ the *subsetting method*, an operation over ZDDs [10]. For a ZDD $\mathcal{Z}$, the subsetting method generates a new ZDD $\mathcal{Z}_C$ by extracting combinations satisfying a constraint C from $\mathcal{Z}$.

We can count the number of non-overlapping edge unfoldings by following three steps.

Step 1 Generate a ZDD $\mathcal{Z}_S$.

Step 2 For each MOPE C_i, generate a new ZDD $\mathcal{Z}_{C_i}$ representing combinations that do not simultaneously include all elements of $E(C_i)$. Herein, a ZDD $\mathcal{Z}_{C_i}$ serves as a filter to exclude overlapping edge unfoldings.

Step 3 Generate a new ZDD $\mathcal{Z}_C$ by extracting combinations that satisfy all $\mathcal{Z}_{C_i}$ from $\mathcal{Z}_S$ using the subsetting method.

4 The number of non-overlapping edge unfoldings in convex regular-faced polyhedra

In this section, we apply the counting algorithm described in Section 3 for non-overlapping edge unfoldings with regular convex regular-faced polyhedra (including three types of Archimedean solids, twenty types of Johnson solids, nineteen types of Archimedean prisms, and twenty-one types of Archimedean antiprisms). We use TdZdd library* for constructing ZDD $\mathcal{Z}_S$ and $\mathcal{Z}_{C_i}$. Experiments were conducted on a Mac OS Venture computer with an Apple M1 Max chip and 64GB of memory. To enumerate the MOPEs for Archimedean solids, Johnson solids, and Archimedean (anti)prism, we used rotational unfolding [18, 17].

We show the number of non-overlapping edge unfoldings for three types of Archimedean solids (Table 2), twenty types of Johnson solids, nineteen types of Archimedean prisms, and twenty-one types of Archimedean antiprisms†. Figure 6 shows the percentage of non-overlapping edge unfoldings from all edge unfoldings for Archimedean (anti)prisms.

From the results of these experiments, we observe the following: In Archimedean solids (Table 2), both the truncated dodecahedron and truncated icosahedron have the same number of vertices, edges, and faces, yet the truncated icosahedron has more MOPEs than that of truncated dodecahedron. Despite this, the results indicate that the percentage of

* `https://github.com/kunisura/TdZdd`

† See `https://shiotatakumi.github.io/MyPage/contents/240313-EuroCG-2024.html` for the number and percentage of non-overlapping edge unfoldings in Johnson solids, and Archimedean (anti)prisms.

Table 2 The number and percentage of non-overlapping edge unfoldings in Archimedean solids.

Archimedean solids	$\|V\|$	$\|E\|$	$\|F\|$	$\#(MOPE)$	#(Edge unfolding) [9]	#(Non-overlapping edge unfolding)	%
Sunb cube	24	60	38	72	89,904,012,853,248	85,967,688,920,076	95.6
Truncated dodecahedron	60	90	32	120	4,982,259,375,000,000,000	931,603,573,888,462,350	18.6
Truncated Icosahedron	60	90	32	240	375,291,866,372,898,816,000	366,359,657,802,290,909,354	97.6

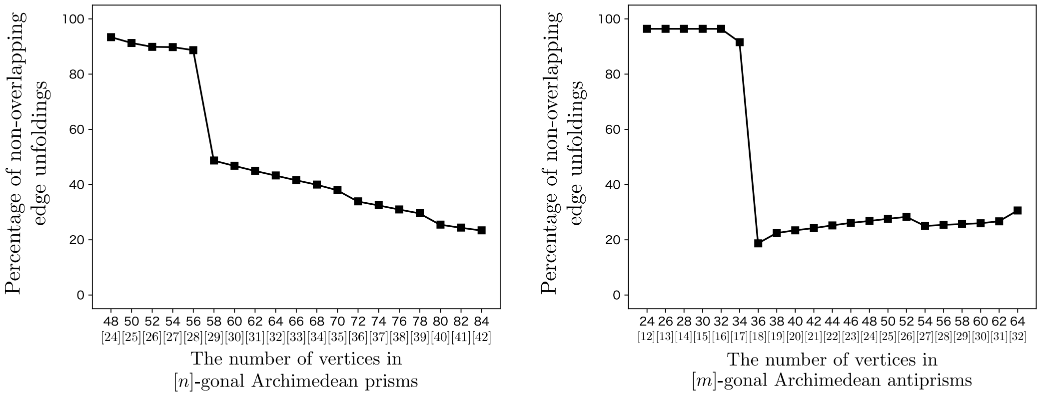

Figure 6 The relationship between the number of vertices and the percentage of non-overlapping edge unfoldings in n-gonal Archimedean prisms and m-gonal Archimedean antiprisms.

non-overlapping edge unfoldings in the truncated dodecahedron is lower than that of the truncated icosahedron. The truncated icosahedron's MOPEs consist of 8 or 9 faces (Figure 7), while the truncated dodecahedron's MOPEs have 4 faces (Figure 8). Therefore, we observe that the number of faces constituting each MOPE, rather than the number of MOPEs, influences the percentage of non-overlapping edge unfoldings. The same observation also applies to Archimedean (anti)prisms. In n-gonal Archimedean prisms, the percentage of non-overlapping edge unfoldings significantly decreases at $n = 29$, as shown in Figure 6 (left). This decrease may be attributed to the appearance of two new types of MOPEs, consisting of 4 faces for $n \geq 29$, as illustrated in Figure 10 (for $n \leq 28$, the MOPEs consist of 6, 7, or 8 faces (Figure 9)). Similarly, in m-gonal Archimedean antiprisms, the percentage of non-overlapping edge unfoldings significantly decreases at $m = 18$, as shown in Figure 6 (right). This decrease may be attributed to the appearance of three new types of MOPEs, consisting of 6 faces for $m \geq 18$, as illustrated in Figure 12 (for $m \leq 17$, the MOPEs consist of 8 faces (Figure 11)).

Acknowledgments. This work was supported in part by JSPS KAKENHI Grant Numbers JP18H04091, JP19K12098, and JP22H03549, by MEXT KAKENHI Grant Number JP20H05964, and by JST SPRING Grant Number JPMJSP2154.

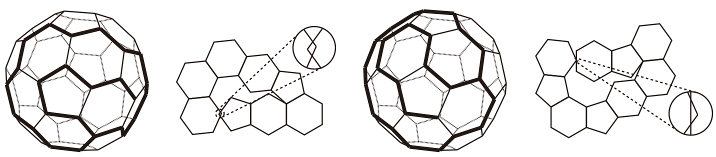

Figure 7 MOPEs in the truncated icosahedron [9, 18]

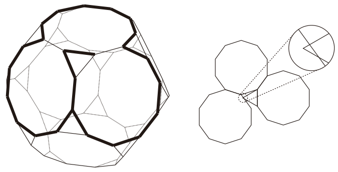

Figure 8 A MOPE in the truncated dodecahedron [9]

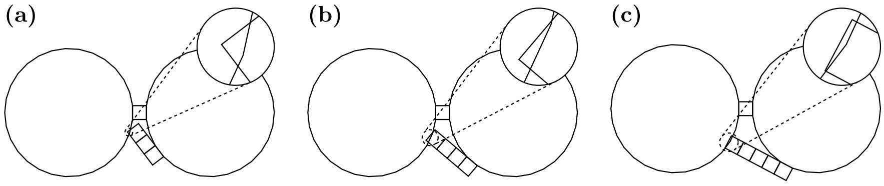

Figure 9 MOPEs in n-gonal Archimedean prisms for (a) $n \geq 24$, (b) $n \geq 26$, and (c) $n \geq 28$.

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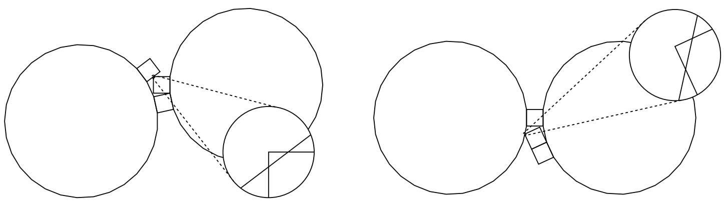

Figure 10 New MOPEs in n-gonal Archimedean prisms in $n = 29$.

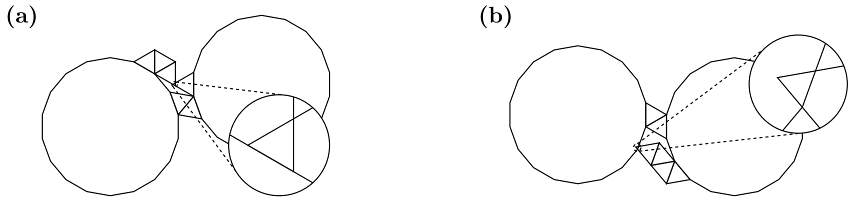

Figure 11 MOPEs in m-gonal Archimedean antiprisms for (a) $m \geq 12$, and (b) $m \geq 17$.

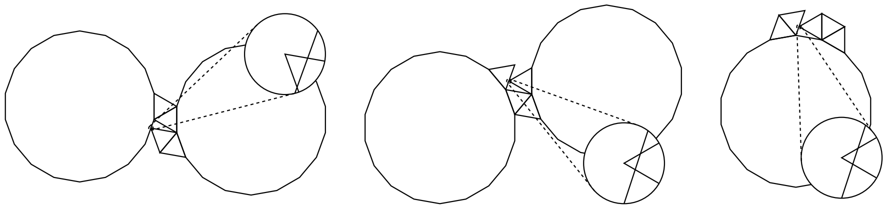

Figure 12 New MOPEs in m-gonal Archimedean antiprisms in $m = 18$.

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