Flip Graphs of Pseudo-Triangulations With Face Degree at Most Four*

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Abstract

A *pseudo-triangle* is a simple polygon with exactly three convex vertices. A *pseudo-triangulation* $\mathcal{T}$ of a point set P in $\mathbb{R}^2$ is a partitioning of the convex hull of P into pseudo-triangles, such that the union of the vertices of the pseudo-triangles is exactly P. We call a size-4 pseudo-triangle a *dart*. For a fixed $k \geq 1$, we study k-dart pseudo-triangulations (k-DPTs), that is, pseudo-triangulations in which exactly k faces are darts and all other faces are triangles. Our results are as follows. We prove that the flip graph of 1-DPTs is generally not connected, and show how to compute its connected components. Furthermore, for k-DPTs on a point configuration called the *double chain* we analyze the structure of the flip graph on a more fine-grained level.

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1 Introduction

Pseudo-triangulations were introduced in the early 1990's by Pocchiola and Vegter to study the visibility complex of disjoint convex regions [13] and by Chazelle et al. for ray shooting in polygons [6]. It was in the early 2000's that pseudo-triangulations of point sets became popular, when Streinu showed that *pointed* pseudo-triangulations of point sets, those in which every vertex is *pointed*, i.e., incident to an angle larger than π, are minimally rigid [15] and used this for a solution of the Carpenter's Rule Problem. The converse statement that every planar minimally rigid graph admits a drawing as a pointed pseudo-triangulation was proved by Haas et al. [8], later generalized to non-minimally rigid and non-pointed pseudo-triangulations by Orden et al. [12] using the notion of *combinatorial pseudo-triangulation*, an embedding of a planar graph together with a labelling of the angles mimicking the properties of angles in a geometric pseudo-triangulation.

Among the many other results on pseudo-triangulations, for which we refer to the survey by Rote et al. [14], let us highlight the notion of a *flip* [1, 11]. There are three types of flips. The first one follows the spirit of flips in triangulations, exchanging the only interior edge in the two geodesic diagonals of a pseudo-quadrilateral, as in Figure 1a-b. This includes

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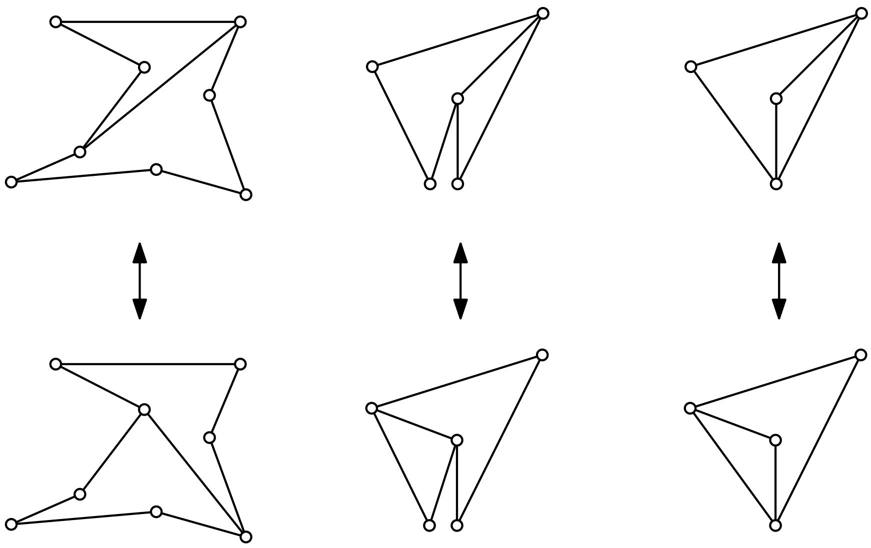

Figure 1 Some flips in pseudo-triangulations.

the case of a degenerate pseudo-quadrilateral, with two consecutive corners merged into a single one, as in Figure 1c where the two lowest vertices from Figure 1b have been merged. The two remaining types of flip insert or remove an interior edge to obtain another pseudo-triangulation, respectively increasing or decreasing by one the number of pointed vertices (this will be fixed in the present work, so such types of flip will not appear). The flip graph for pseudo-triangulations turns out to be connected and have diameter in $O(n \log n)$ [5].

The fact that a pseudo-triangle can have linear size and, therefore, the flip operation cannot be computed in constant time as for triangulations, led to the consideration of pseudo-triangulations with bounded size of the internal faces. In particular, Kettner et al. [9] showed that for every point set there is a pointed pseudo-triangulation with internal faces of size 3 or 4, called a *4-pointed pseudo-triangulation* or *4-PPT*. These 4-PPTs fulfill nice properties, like being properly 3-colorable while that question is NP-complete for general pseudo-triangulations [2]. By contrast, some properties known for general pseudo-triangulations turned out to be elusive for 4-PPTs. In particular, a long-standing open problem is whether the flip graph of 4-PPTs is connected, which has only been proved for combinatorial 4-PPTs [3]. The aim of this work is to generalize this problem and prove results on cases that can provide additional insight towards solving that open problem.

We consider flips in *4-pseudo-triangulations* or *4-PT*s, which are defined as general, not necessarily pointed, pseudo-triangulations with internal faces of size 3 or 4. We call a size-4 pseudo-triangle a *dart*, with its *tail* being the concave (also called reflex) vertex, its *tip* being the vertex not adjacent to the tail, and its two *wings* being the remaining two vertices. The segment between the tip and tail of a dart will be referred to as its *spine*, though such a segment is necessarily not an edge and therefore missing in a dart. In a 4-PT, each interior pointed vertex is the tail of a dart, and 4-PTs with k interior pointed vertices are 4-PTs with k darts or *k-dart 4-PTs*, which will be denoted as k-DPTs. See Figure 2.

For a size-n point set P with a convex hull of size $h \leq n$, the maximum number of interior pointed vertices is $n - h$, and therefore the maximum number of darts in a 4-PT is $n - h$ as well. In particular, 4-PPTs coincide with k-DPTs for $k = n - h$, since they are those 4-PTs in which every interior vertex is pointed. Thus, the aforementioned open problem in [3] asks

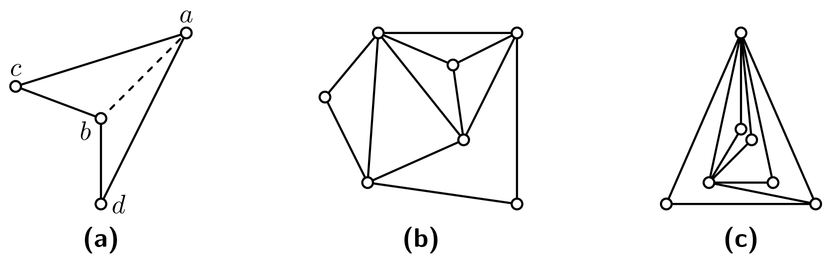

Figure 2 (a) A dart with tip a, tail b, wings c and d, and a dashed spine. **(b)** A 1-DPT on 7 points. **(c)** An $(n-h)$-DPT with $n = 7$ and $h = 3$; each vertex not on the convex hull is a dart tail.

about the connectivity of the flip graph of k-DPTs for the largest possible value of k, that is $k = n - h$. Our first goal is to look at the opposite end of the range and study the flip graph of k-DPTs for the smallest possible value of k, that is $k = 1$. This corresponds to 4-PTs with only one dart, i.e., only one interior pointed vertex. We show that the resulting flip-graph of 1-DPTs is not connected and we show how to compute its connected components. Furthermore, for k-DPTs on a frequently-studied point configuration, the double chain [4], we analyze the structure of the flip graph on a more fine-grained level.

Due to space constraints, most proofs are omitted and can be found in the full version [10].

2 Components of the Flip Graph for 1-DPTs

To compute the number of components of the flip graph in the presence of a single dart, we first partition the class of 1-DPTs on a point set P into separate classes $\mathcal{G}_p$ where $p \in P$. Each such class $\mathcal{G}_p$ consists of all those 1-DPTs that have the tail of the dart located at p. We show that all 1-DPTs in $\mathcal{G}_p$ are in the same connected component of the flip graph. The proof consists of three steps: First, we show that for any dart d on P there exists a *dart triangle*, defined as a triangle on the three corners of a dart containing no more points of P than the tail of that dart (see Figure 3a), with the property that such a dart triangle shares the tip and tail with the original dart d (but may have other wings, see Figures 3b-d).

▶ **Lemma 2.1.** *For any* 1*-DPT of a point set P containing an arbitrary dart d there is a flip sequence to a* 1*-DPT with a dart triangle that has the same tip and tail as d.*

Proof sketch. We define a flip sequence that creates a dart triangle sharing the tip and tail with d. We first flip to a triangulation where the wings of d are connected by an edge (see Figure 3a). Such a flip sequence exists, since Dyn et al. [7] proved that a triangulation can be flipped to any other triangulation, without ever flipping an edge in a set D of constrained edges. We insert the spine edge and add it to D along with all other edges of d. If there are points in the dart triangle (see Figure 3b), we flip to a triangulation on those points that can easily be flipped to an empty dart triangle (see Figures 3c-d), and remove the spine edge. ◀

Second, we show that for a 1-DPT with the tail of the dart at point $p_d \in P$, the tip can be flipped to any point $p \in P \setminus \{p_d\}$ if this allocation of tip and tail permits a 1-DPT of P.

▶ **Lemma 2.2.** *If a point $p_d \in P$ is the tail of the single dart in two* 1*-DPTs $\mathcal{T}_1$ and $\mathcal{T}_2$ on point set P, then there is a flip sequence between $\mathcal{T}_1$ and $\mathcal{T}_2$.*

Proof sketch. We apply Lemma 2.1 to $\mathcal{T}_1$ and $\mathcal{T}_2$ to ensure the respective darts are in dart triangles, and use the obtained flip sequences as prefixes/suffixes for our final sequence. We

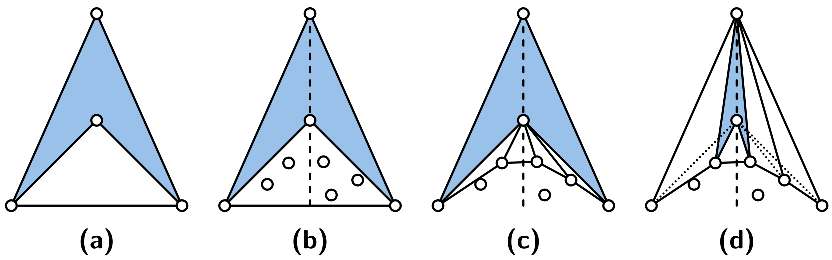

Figure 3 **(a)** A dart triangle. **(b)** A dart with an edge connecting the wings. The vertices in the bottom face are split by the extension of the (dashed) spine. **(c)** The specific triangulation between the dart and the upper envelop of the subset P' of points of P in the triangle formed by the wings and tail of d, together with the wings of d. **(d)** A number of flips linear in $|P'|$ creates a dart triangle.

remove the tail vertex from the obtained triangulations and mark the new faces as special faces. This results in two (proper) triangulations $\mathcal{T}_1'$ and $\mathcal{T}_2'$, for which we can find a flip sequence transforming one into the other. We apply this flip sequence to flip from $\mathcal{T}_1$ and $\mathcal{T}_2$ with one change: any flip involving an edge adjacent to a special face is substituted for a short flip sequence as shown in Figure 4, to preserve the dart during the flip sequence. ◀

The flip sequence of Lemma 2.2 implies that all 1-DPTs in the class $\mathcal{G}_p$ of 1-DPTs that have the tail of the dart located at p reside in the same connected component of the flip graph. As the third and final step in finding the connected components of the flip graph for 1-DPTs on P, we still have to check whether the tail of the dart can move from one vertex $p \in P$ to $q \in P$, i.e., whether the 1-DPTs in $\mathcal{G}_p$ and $\mathcal{G}_q$ all reside in the same connected component. To do so, we consider, for each pair of vertices $p, q \in P$, each triple of points in P distinct from p and q and check whether no other points are located in the faces of any of the small configurations in Figure 5a-d. Observe that these five vertices must admit two overlapping darts with different tails. If this is the case, then by Lemma 2.2 we can flip to the 1-DPTs where the respective darts have their tips in the position prescribed in the small configuration, and perform the flip in the configuration to swap the tails. To complete the argument, we can use a finite case analysis to prove that we have to check only the configurations in Figure 5a-d; no other ways to swap tails in 1-DPTs exist.

▶ **Lemma 2.3.** *There exist exactly four configurations of five points, that allow a dart to move its tail with an edge flip, as illustrated in Figure 5a-d.*

▶ **Theorem 2.4.** *For 1-DPTs on a set P of n points with h points on the convex hull, there are at most $n - h$ components in the flip graph. The exact components can be determined by checking every quintuple of points that have a triangular convex hull.*

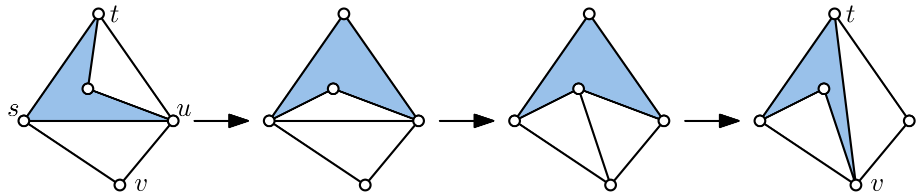

Figure 4 A flip sequence to flip an edge incident with the face containing the tail p_d of a dart.

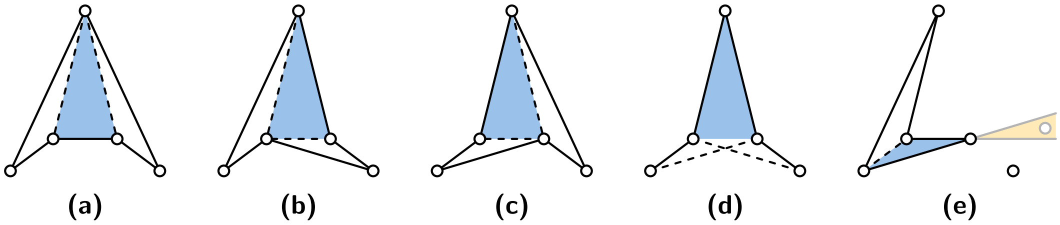

Figure 5 (a)-(d) All dart configurations of five points that allow us to move the tail of a dart using an edge flip. The darts share the blue triangle and each require one dashed edge. **(e)** Other triangles cannot use both middle vertices as tails, without a (non-existing) point in the yellow area.

3 Characterizing the Flip Graph for the Double Chain

In this section we consider the double chain, a point set consisting of a convex 4-gon being the hull of two concave chains of points next to opposite edges of the 4-gon, $P_{\asymp} = P_1 \cup P_2$, such that these concave chains do not cross the diagonals of the 4-gon, see Figure 6a. We can completely characterize the flip graph of k-DPTs on $P_{\asymp}$, for any possible k.

A k-DPT on a point set $P_{\asymp}$ admits two kinds of darts, *aligned darts* for which the spine connects two adjacent vertices of one concave chain, and *crossing darts* which have tip and tail in opposite concave chains. In this section we prove that the tail of a dart cannot swap between concave chains, and we say that a dart is *designated* to the chain where the tail is located. Additionally, we show that we can use edge flips to flip any k-DPT of an instance $P_{\asymp}$ to a *canonical* k-DPT. In such a k-DPT, all darts are aligned, and flipped as far left as possible (see Figure 6b); for darts designated to P_1 the wings on the opposite chain are at the leftmost point on P_2, while for darts designated to P_2 the analogous wings are at the rightmost tail on P_1. Furthermore, all wings inside the convex hulls of their designated chains are located at the rightmost point of the chain. By analyzing the number of canonical k-DPTs, we will analyze the number of connected components of the flip graph.

We first prove a few properties of aligned and crossing darts using geometric observations.

▶ **Lemma 3.1.** *In a k-DPT of point set $P_{\asymp}$, any aligned dart has one wing on the opposite concave chain, and for any choice of wings on the respective concave chains, a dart exists.*

▶ **Lemma 3.2.** *In a k-DPT of point set $P_{\asymp}$, the wings and tail of any crossing dart are consecutive on one concave chain; for any choice of tip on the opposite chain, a dart exists.*

▶ **Lemma 3.3.** *A k-DPT of point set $P_{\asymp}$ admits only crossing and aligned darts.*

Again using quintuples of points, we prove that tails cannot swap between concave chains.

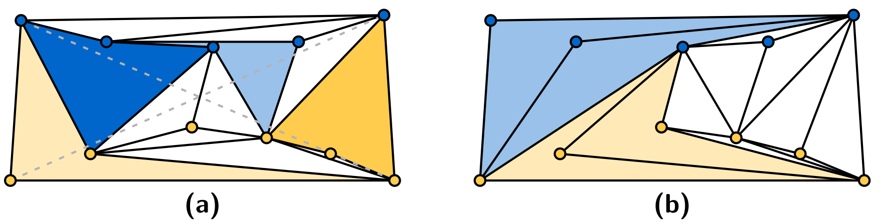

Figure 6 (a) A k-DPT for concave chains P_1 (blue) and P_2 (yellow), with $k = 4$. The aligned and crossing darts are light and dark colored, respectively. Note that P_1 and P_2 do not cross the dashed-grey diagonals. **(b)** The canonical k-DPT of **(a)**; white faces are triangulated arbitrarily.

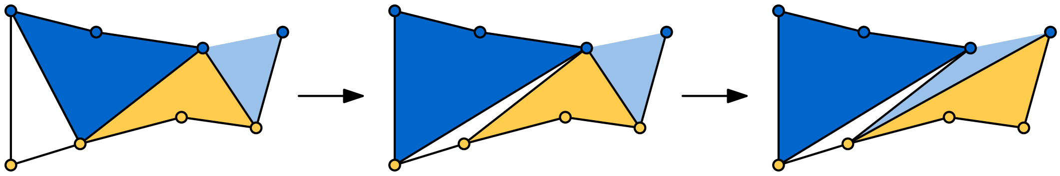

Figure 7 Moving the tip of a crossing dart, followed by moving the wing of an aligned dart. Faces (or parts thereof) inside the convex hull of either chain remain unchanged and are hence not drawn.

▶ **Lemma 3.4.** *In any k-DPT of point set $P_{\asymp} = P_1 \cup P_2$ the tail of a dart cannot swap between P_1 and P_2 through an edge flip.*

Next we show how to flip any k-DPT on an instance $P_{\asymp}$ to the canonical k-DPT for $P_{\asymp}$.

▶ **Lemma 3.5.** *Any k-DPT of point set $P_{\asymp}$ can be flipped to the canonical k-DPT.*

Proof sketch. We first consider all darts with tails on P_1 and flip them to be aligned darts on the left of the k-DPT, followed by all darts with tails on P_2. We move the vertices on the opposite chain to their (final) leftmost position, as in Figure 7. Then we move the spines in place by swapping between crossing and aligned darts with a single flip per swap, as in Figure 8. Some tails may not be located in the leftmost position, but they can move leftward when their dart is crossing, as in Figure 9. The end result can be seen in Figure 6b. ◀

Since any k-DPT on a set $P_{\asymp}$ can flip to a canonical k-DPT, we can now prove that:

▶ **Theorem 3.6.** *The number of connected components of the flip graph of k-DPTs on a point set $P_{\asymp} = P_1 \cup P_2$ is equal to the number of ways to designate k darts to P_1 or P_2.*

Proof. By Lemma 3.4 we know that the tails of darts cannot swap between convex chains. Furthermore, Lemma 3.5 tells us that all such k-DPTs can be flipped to a canonical k-DPT. Now observe that the number of darts designated to P_1 and P_2 completely determines the canonical k-DPT. Thus all k-DPTs with the same designation of darts to the concave chains are part of the same connected component of the flip graph. ◀

If P_1 and P_2 have $a + 2$ and $b + 2$ points, respectively, with $a, b \geq 0$, then there are $a + b$ points that can be a tail of a dart, and thus $k \leq a + b$. Distinguishing several cases depending on a, b, and k, we arrive at a unified formula for the number of components of the flip graph: Intuitively, we distribute k indistinguishable balls over 2 distinguishable (fixed-size) bins.

▶ **Corollary 3.7.** *The number of connected components of the flip graph of k-DPTs on a point set $P_{\asymp} = P_1 \cup P_2$, with $|P_1| = a + 2$ and $|P_2| = b + 2$, is equal to* $\min\{a, b, k, a + b - k\} + 1$ *for* $0 \leq k \leq a + b$.

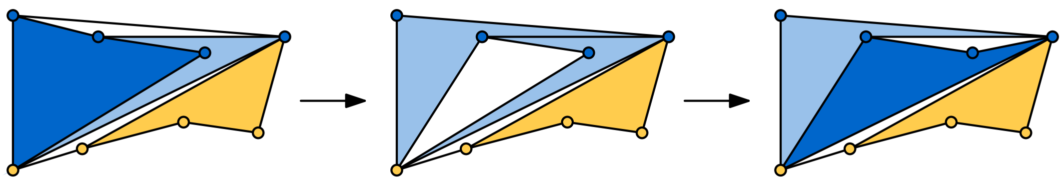

Figure 8 Flipping a crossing dart to be aligned, followed by a flip from an aligned to a crossing dart.

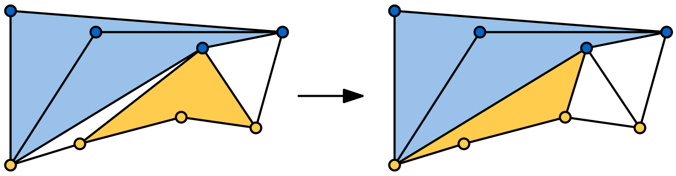

Figure 9 Moving a crossing dart designated to P_2 leftwards.

4 Conclusion

We studied the flip graph of pseudo-triangulations with faces of size 3 (triangles) and a bounded number k of size-4 faces (darts). For $k = 1$ in any point configuration, and for any k in the double chain point configuration, we showed how to find the connected components of the flip graph. For general point configurations, we conjecture that a similar approach will work for slightly higher values of k, such as $k \in \{2, 3\}$. Our goal in studying these special configurations is to obtain new insights into the flip graph of pointed pseudo-triangulation with faces of size 3 or 4. However, our current findings do not seem to allow us to reach much further than low values of k, as the required number of cases for higher k becomes infeasible.

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