On exact covering with unit disks

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Abstract

We study the problem of covering a given point set in the plane by unit disks so that each point is covered exactly once. We prove that 17 points can always be exactly covered. On the other hand, we construct a set of 657 points where an exact cover is not possible.

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1 Introduction

In 2008, Inaba [10] gave the following puzzle about covering sets of points in the plane:

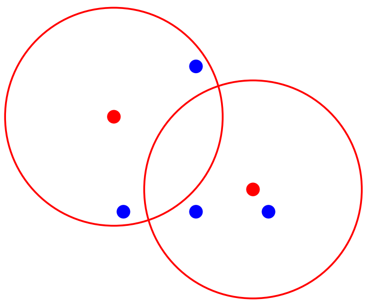
Show that any set of 10 points in $\mathbb{R}^2$ can be covered by nonoverlapping unit disks.

Inaba solved this puzzle [11, 17] with an elegant probabilistic argument (a deterministic proof is also possible). In this article, we study a relaxed version of this covering problem. Given a point set $X \subset \mathbb{R}^2$, can we find a family $\mathcal{D}$ of not necessarily disjoint unit disks so that each point $\mathbf{x} \in X$ is contained in exactly one disk $D \in \mathcal{D}$? We call such a family an *exact cover* of X. For example, in Figure 1a, the two red disks form an exact cover of the four blue points.

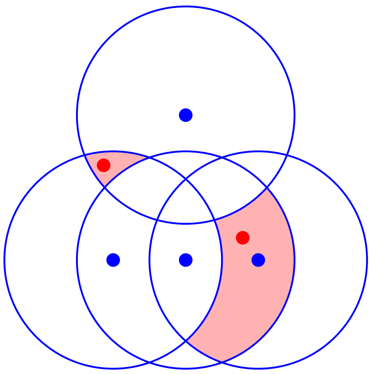
Let $B^2 \coloneqq \{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| < 1\}$, where $\| \cdot \|$ denotes the Euclidean norm. We define an (open) *disk* with *center* $\mathbf{c} \in \mathbb{R}^2$ and *radius* $r > 0$ as the set $D_{\mathbf{c},r} \coloneqq \mathbf{c} + rB^2$; if $r = 1$ we call it the *unit disk* and write $D_{\mathbf{c}}$.

▶ **Definition 1.1.** Let σ_2 be the largest $n \in \mathbb{N}$ such that any set of n points in the plane can be covered by disjoint unit disks. Let $\widehat{\sigma}_2 \in \mathbb{N}$ be the corresponding number for the relaxed problem involving exact covers.

As a covering using disjoint disks is also an exact covering, we have the basic relationship $\widehat{\sigma}_2 \geq \sigma_2$. The current best known bounds for σ_2 are $12 \leq \sigma_2 \leq 44$ [1]. Aloupis, Hearn, Iwasawa, and Uehara (2012) [1] improved Inaba's lower bound to $\sigma_2 \geq 12$ through a careful



(a)



(b)

Figure 1 Left: primal solution (exact covering). Right: dual solution (exact hitting set).

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

analysis of the probabilistic method on one-dimensional slices of the plane. In the other direction, σ_2 is finite: Intuitively, with a dense enough arrangement of points, this problem becomes similar to the problem of covering the entire set $\operatorname{conv} X$, which is impossible using disjoint disks. Specific upper bounds were reduced in rapid succession from $\sigma_2 < 60$ by Winkler (2010) [17] to $\sigma_2 < 55$ by Elser (2011) [7] and $\sigma_2 < 53$ by Okayama, Kiyomi, and Uehara (2012) [16]. Most recently, Aloupis, Hearn, Iwasawa, and Uehara (2012) [1] proved $\sigma_2 < 50$ "by hand" and demonstrated $\sigma_2 < 45$ using computer calculations.

1.1 Results

In Section 2, we build on some of the mentioned works on lower bounds to establish the following lower bound on $\widehat{\sigma}_2$:

▶ **Theorem 1.2.** *We have* $\widehat{\sigma}_2 \geq 17$.

The finiteness of $\widehat{\sigma}_2$ can be deduced by a similar argument as the finiteness of of σ_2. In Section 3 we construct a close arrangement of points that cannot be exactly covered, leading to the following (rather weak) upper bound on $\widehat{\sigma}_2$:

▶ **Theorem 1.3.** *We have* $\widehat{\sigma}_2 < 657$.

For the full proofs of Theorem 1.2 and Theorem 1.3 we refer to the appendix; nevertheless, we provide sketches of the proofs below.

1.2 Relation between exact covering and exact hitting

We denote by $\mathcal{X}$ the collection of all finite point sets in $\mathbb{R}^2$. A point $\mathbf{x} \in \mathbb{R}^2$ is contained in a unit disk $D_{\mathbf{c}}$ centered at $\mathbf{c} \in \mathbb{R}^2$ if and only if $\mathbf{c}$ is contained in the unit disk $D_{\mathbf{x}}$ centered at $\mathbf{x}$. By this simple observation, the problem of exactly covering some given $X \in \mathcal{X}$ by unit disks (*primal problem*) becomes equivalent to the following *dual problem*: Let $\mathcal{D}_X := \{D_{\mathbf{x}} \mid \mathbf{x} \in X\}$; find a $P \in \mathcal{X}$ such that each disk $D \in \mathcal{D}_X$ contains exactly one point $\mathbf{p} \in P$. See Figure 1b for an example of the dual problem. In the literature, such a set P is also called an *exact hitting set*. A dual solution P yields the solution $\mathcal{D} := \{D_{\mathbf{p}} \mid \mathbf{p} \in P\}$ to the exact covering problem. Vice versa, a solution $\mathcal{D}$ to the exact covering problem gives a solution to the dual problem by taking the disk centers.

In the dual perspective, the boundary circles of the disks $\mathcal{D}_X$ decompose the plane into *cells*. Observe that all points in a given cell are contained in the same set of disks, so the exact position of a dual solution point $\mathbf{p} \in P$ is irrelevant. Hence, it is sufficient to select a set of cells so that each $D \in \mathcal{D}_X$ contains exactly one selected cell. In the example of Figure 1b, the two red shaded cells form a solution. This observation shows that the solution space for the dual problem and for the exact covering problem is in fact discrete, and methods such as Knuth's Algorithm X (see [13] or Section 7.2.2.1 in [14]), integer programming, or SAT solvers (see [12]) can be used.

2 A lower bound

We exclude the following trivial case from our proofs in this section: If X lies on a line then X can be covered by disjoint disks. Denote by $\mathcal{X}'$ the subset of $\mathcal{X}$ that excludes every point set on a line. To prove Theorem 1.2, we have to show that all $X \in \mathcal{X}'$ with $|X| \leq 17$ can be exactly covered. We combine three separate components on top of Inaba's original probabilistic proof. In Subsection 2.1 we show that $\widehat{\sigma}_2 \geq \sigma_2 + 4$. In Subsection 2.2 we

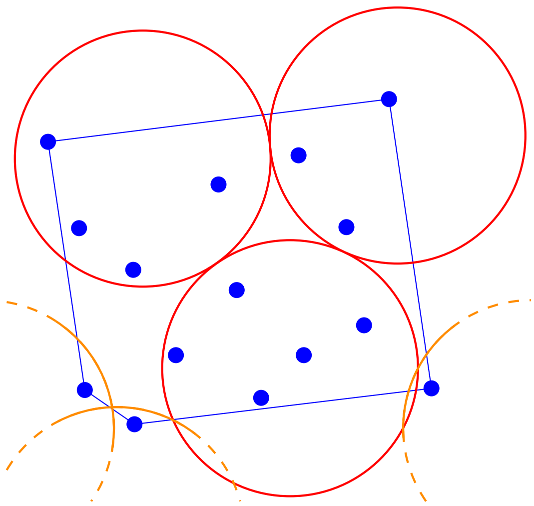

Figure 2 Extending the red disjoint disk covering of the non-boundary points by adding a new orange disk at each uncovered boundary point.

obtain $\widehat{\sigma}_2 \geq 16$ using a covering version of Betke, Henk, and Wills's parametric density [3] and $\widehat{\sigma}_2 \geq 17$ by showing that in some cases, a disk D that overlaps with $\operatorname{conv} X$ can be removed from an exact cover $\mathcal{D}$ of X so that $\mathcal{D} \setminus \{D\}$ is still an exact cover of X.

2.1 Boundary points

▶ **Definition 2.1.** Let $X \in \mathcal{X}$ and $\mathbf{v} \in X$. The point $\mathbf{v}$ is a *boundary point of* X if $\mathbf{v}$ is on the boundary of $\operatorname{conv} X$.

Let $X \in \mathcal{X}$ and $\mathbf{v}^1, \ldots, \mathbf{v}^k$ be the boundary points of X. László Kozma (private communication) observed that a covering $\mathcal{D}'$ of the non-boundary points $X \setminus \{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ by *disjoint* disks can always be extended to an exact cover of X. A boundary point $\mathbf{v}^i$ is covered by at most one disk in $\mathcal{D}'$ because the disks are disjoint. If $\mathbf{v}^i$ is not already covered by $\mathcal{D}'$, then it can be covered by a new disk which contains $\mathbf{v}^i$ but no other point of X. The resulting disk configuration yields an exact cover $\mathcal{D}$ of X (Figure 2). In particular, if $|X| \leq \sigma_2 + k$ then X can be exactly covered. We refer to this strategy as the *Extension Argument*:

▶ **Lemma 2.2** (Extension Argument). *Let $X \in \mathcal{X}$ and k be the number of boundary points of* $\operatorname{conv} X$*.*

1. *If $|X| \leq \sigma_2 + k$ then X can be exactly covered.*
2. *If $k \leq 2$ then X can be exactly covered regardless of $|X|$.*
3. *We have $\widehat{\sigma}_2 \geq \sigma_2 + 3$.*

As we assume that X does not lie on a line, we have $k \geq 3$, and the Extension Argument improves the basic inequality $\widehat{\sigma}_2 \geq \sigma_2$ to $\widehat{\sigma}_2 \geq \sigma_2 + 3$. This lower bound is limited by the case where $\operatorname{conv} X$ is a triangle, since otherwise X has at least four boundary points. Therefore, we wish to relax Definition 2.1 so that *every* $X \in \mathcal{X}'$ has at least four "generalized boundary points" that behave like boundary points.

▶ **Definition 2.3.** Let $X \in \mathcal{X}$ and $\mathbf{b} \in X$. The point $\mathbf{b}$ is a *generalized boundary point* of X if there exists a $\mathbf{c} \in \mathbb{R}^2$ such that $X \cap D_{\mathbf{c}} = \{\mathbf{b}\}$.

All vertices and boundary points of X are generalized boundary points of X. In the full version of this paper we prove the following generalization of the Extension Argument.

▶ **Lemma 2.4** (Generalized Extension Argument). *Let $X \in \mathcal{X}'$ and k be the number of generalized boundary points of* $\operatorname{conv} X$.
1. *If $|X| \leq \sigma_2 + k$ then X can be exactly covered. (That is, $\widehat{\sigma}_2(k) \geq \sigma_2 + k$.)*
2. *If $k \leq 3$ then X can be exactly covered regardless of $|X|$.*
3. *We have $\widehat{\sigma}_2 \geq \sigma_2 + 4$.*

We show that any $X \in \mathcal{X}'$ with a triangular convex hull contains at least four generalized boundary points or can be exactly covered regardless of the number of points. The fourth point is often, but not always, the closest non-vertex of X to the longest edge of $\operatorname{conv} X$.

Lemma 2.4 combined with Aloupis, Hearn, Iwasawa, and Uehara's [1] lower bound of $\sigma_2 \geq 12$ implies $\widehat{\sigma}_2 \geq 16$.

2.2 A parameterized version of Inaba's proof

Betke, Henk, and Wills (1994) [3] introduced the parametric density, a form of packing density for finitely many disks which are allowed to overlap, during their work on a packing problem called the Sausage Conjecture [8]. See [3, 5, 9] for further details on these topics. Let

$$A_2 := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x_1 = \sqrt{3}\mu, \\ x_2 = 2\lambda + \mu, \end{matrix} \quad \lambda, \mu \in \mathbb{Z} \right\}$$

be the hexagonal lattice and $\mathcal{A}_2^\rho := \left\{ \mathbf{c} + \rho B^2 \;\middle|\; \mathbf{c} \in A_2 \right\}$ be the collection of disks of radius $\rho \geq 1$ that are centered at the points of A_2. This radius ρ is called the *parameter*; the case $\rho = 1$ reduces to the usual hexagonal packing in Inaba's proof. We call the subset of $\mathbb{R}^2$ covered by exactly one disk $D_{\mathbf{c}} \in \mathcal{A}_2^\rho$ the "good" region of $\mathcal{A}_2^\rho$ and its complement the "bad" region. An exact cover of X requires each point in X to avoid the "bad" region. If $\rho > 1$, then neighboring disks of $\mathcal{A}_2^\rho$ overlap (Figure 3), so the "bad" region includes any part of the plane covered by multiple disks. The critical value for ρ minimizes the total area of the "bad" region and so maximizes the lower bound for $\widehat{\sigma}_2$ (over all coverings of the form $\mathcal{A}_2^\rho$).

We use the same argument as Inaba but with the parameterized family $\mathcal{A}_2^\rho$ and combine it with the Extension Argument for another proof of $\widehat{\sigma}_2 \geq 16$. However, $\mathcal{A}_2^\rho$ has another advantage over $\mathcal{A}_2$. Removing one disk from $\mathcal{A}_2$ strictly expands the "bad" region, so is never beneficial for exact covering. However, removing one disk $D_{\mathbf{c}}$ from $\mathcal{A}_2^\rho$ changes the subsets of $D_{\mathbf{c}}$ which are covered by another disk in $\mathcal{A}_2^\rho$ from "bad" to "good." In the next subsection, we use this feature to raise our lower bound for $\widehat{\sigma}_2$.

2.3 A redundant disk

Suppose that $X \in \mathcal{X}$ has a triangular or quadrilateral convex hull, $\mathbf{v}^1 \in X$ is a boundary point that is covered by two disks of $\mathcal{A}_2^\rho$, and $\mathcal{A}_2^\rho$ is an exact cover of $X \setminus \left\{\mathbf{v}^1\right\}$. Under certain conditions, we can remove one of the disks $D_{\mathbf{c}}$ that covers $\mathbf{v}^1$ without breaking the exact cover. In other words, although $\mathcal{A}_2^\rho$ is not an exact cover of X, we show that $\mathcal{A}_2^\rho \setminus \{D_{\mathbf{c}}\}$ is an exact cover of X. This "redundant disk" method offers a slight benefit:

▶ **Lemma 2.5.** *Let $X \in \mathcal{X}'$ with $|X| \leq 17$. If* $\operatorname{conv} X$ *is a triangle or a quadrilateral, then X can be exactly covered.*

We present the technical details and proofs in the full version of our paper. Note that unlike the Extension Argument and parameterized family, which do not depend on the underlying disk configuration, the redundant disk method uses specific properties of A_2.

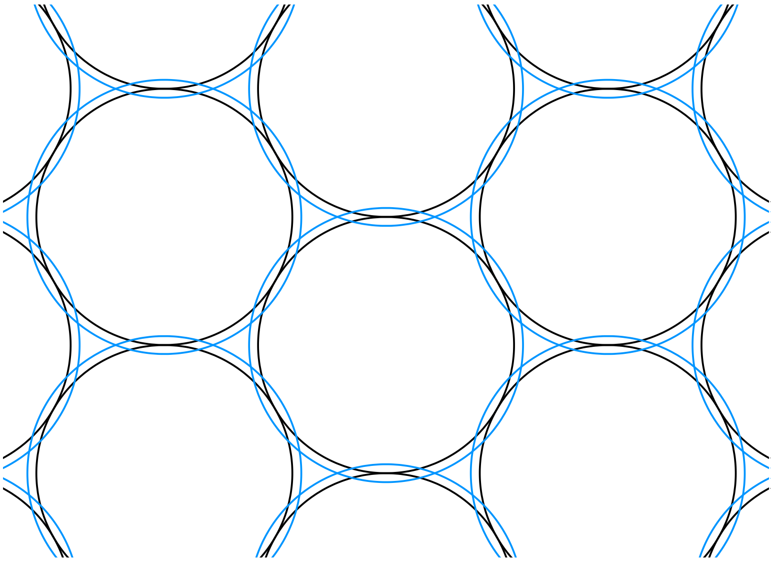

Figure 3 The disks of $\mathcal{A}_2^\rho$ for $\rho = 1$ (black) and $\rho = 1.07$ (blue).

Proof of Theorem 1.2. Let $X \in \mathcal{X}'$ with $|X| \leq 17$. If $\operatorname{conv} X$ has three or four sides, then X can be exactly covered by Lemma 2.5. If $\operatorname{conv} X$ has five or more sides, then X has at least five generalized boundary points, so X can be exactly covered by the Generalized Extension Argument 2.4 with Aloupis, Hearn, Iwasawa, and Uehara's [1] lower bound of $\widehat{\sigma}_2 \geq 12$. ◀

3 An upper bound

▶ **Definition 3.1.** Let $X \subset \mathbb{R}^2$ be a non-empty set. The *distance of a point* $\mathbf{y} \in \mathbb{R}^2$ *to* X is defined by

$$\operatorname{dist}(\mathbf{y}, X) \coloneqq \inf\{\|\mathbf{y} - \mathbf{x}\| \mid \mathbf{x} \in X\} \tag{1}$$

and the ε*-extension* of X (also called the ε*-thickening* of X) is given by

$$X_\varepsilon \coloneqq \left\{\mathbf{y} \in \mathbb{R}^2 \mid \operatorname{dist}(\mathbf{y}, X) \leq \varepsilon\right\}. \tag{2}$$

We say that X is an ε*-net* of $M \subset \mathbb{R}^2$ if $M \subset X_\varepsilon$.

▶ **Definition 3.2.** Let $M \subset \mathbb{R}^2$ and $\varepsilon > 0$. We say that M is an ε*-blocker* if every ε-net $X \in \mathcal{X}$ of M does not have an exact cover.

We recall that the covering number $N(M, \varepsilon)$ of a set $M \subset \mathbb{R}^2$ is the minimal cardinality of an ε-net of M. The following statement is a direct consequence of Definition 3.2.

▶ Proposition 3.3. Let $M \subset \mathbb{R}^2$ be an ε-blocker. Then $\widehat{\sigma}_2 < N(M, \varepsilon)$. ◀

Our upper bound on $\widehat{\sigma}_2$ follows from the following result, which asserts that every open disk of radius $R > 1$ is an ε-blocker for a suitably chosen $\varepsilon > 0$.

▶ Proposition 3.4. Let $\varepsilon \in \left(0, 7 - \sqrt{48} \approx 0.0718\right]$ and

$$R \geq \frac{3}{2}(1 + \varepsilon) - \frac{1}{2}\sqrt{1 - 14\varepsilon + \varepsilon^2}. \tag{3}$$

Then $D_{\mathbf{0},R} = \mathbf{0} + R\,B^2$ is an ε-blocker.

We now obtain Theorem 1.3 as a corollary to Proposition 3.3 by setting $\varepsilon \coloneqq 7 - \sqrt{48}$ and $R \coloneqq \frac{3}{2}(1 + \varepsilon) \approx 1.608$.

4 Conclusion

Our main result (Theorems 1.2 and 1.3) is $17 \leq \widehat{\sigma}_2 \leq 656$. An approach for improving the upper bound could be to search for small ε-nets of ε-blockers and to use Proposition 3.3.

The problem of finding $\widehat{\sigma}_2$ admits generalizations to all dimensions $d \geq 1$ and convex bodies $K \subset \mathbb{R}^d$. Let $\sigma(K)$ and $\widehat{\sigma}(K)$ be the largest n such that any n-point set in $\mathbb{R}^d$ can be covered by disjoint translates of K or exactly covered by translates of K, respectively (and write σ_d and $\widehat{\sigma}_d$ if $K = B^d$). Some of our methods, such as the Extension Argument and the parameterized family, have counterparts for other bodies K, but our other methods do not necessarily generalize.

Sphere packings are mostly empty space in high dimensions. Blichfeldt's upper bound of $\frac{d+2}{2} \cdot 2^{-\frac{1}{2}d}$ for the maximum packing density ([4], or see Section 6.1 of [18]) drops to less than or equal to 0.5 for $d \geq 6$. The density of the densest known packing in $d = 5$ is also below 0.5 (see Table 1.2 in Chapter 1 of [6], or [15]). Therefore, we cannot hope to cover many points by translating a dense packing of unit balls as in Inaba's proof [11, 17]. One possible strategy for "medium" dimensions around 5–10 is to choose one of several packings based on the arrangement of X.

With regard to lines of further research, we mention the computational complexity of disk covering. Considering the algorithmic issues that were discussed in Subsection 1.2, it is natural to ask the following question: Given $X \in \mathcal{X}$, is it NP-hard to decide whether X has an exact cover? Ashok, Basu Roy, and Govindarajan (2020) [2] showed that it is NP-hard to decide the following problem: Given a finite set $\mathcal{R}$ of unit squares and given an $X \in \mathcal{X}$, is there a subset $\mathcal{R}' \subset \mathcal{R}$ that exactly covers X? Their proof can be easily adopted for a given family $\mathcal{R}$ of unit disks. It might also be interesting to study the computational complexity if the number of disks in the exact cover is specified.

Acknowledgments

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