On Maximal 3-Planar Graphs

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Abstract

A graph is 3-*planar* if it admits a drawing in the plane such that every edge is crossed at most three times. A 3-planar graph is *maximal* 3-planar if addition of any edge results in a graph that is not 3-planar. A 3-planar graph on n vertices has at most $5.5n - 11$ edges, and a 3-planar graph that has exactly $5.5n - 11$ edges is an *optimal* 3-planar graph. In contrast to planar graphs where maximal and optimal graphs coincide, a maximal 3-planar graph may have fewer edges than an optimal 3-planar graph. In this paper, we study properties of maximal 3-planar graphs. First, we characterize the graphs on nine vertices that are (maximal) 3-planar. Second, we show that—in contrast to maximal 1– and 2-planar graphs—maximal 3-planar graphs may contain cut vertices. Third, we give a first upper bound on the minimal edge density by constructing maximal 3-planar graphs on n vertices with only $2.375n + O(1)$ edges.

1 Introduction

Planar graphs are graphs that can be drawn on the plane without crossings. If the addition of any edge to a planar graph makes it impossible for the resulting graph to admit a plane drawing, the planar graph is said to be a *maximal* planar graph. Maximal planar graphs are well-studied and have many interesting properties. For example, the number of edges in maximal planar graphs is solely dependent on the number of vertices. Specifically, every maximal planar graph on $n \geq 3$ vertices has $3n - 6$ edges. It is natural to further explore the edge density for various families of beyond-planar graphs, which have been extensively studied over the past decade [5, 8].

In this paper, we focus on 3-planar graphs. A graph is k-planar if it admits a drawing where each edge has at most k crossings. Pach and Tóth [11] gave upper bounds on the number of edges in a k-planar graph, which they used to improve the Crossing Lemma. Unlike planar graphs, the class of k-planar graphs is not closed under edge contractions. Thus, many useful properties of minor closed classes do not apply to k-planar graphs.

As k gets larger, the density of a k-planar graph on n vertices clearly increases, but the exact correlation is still unknown. Pach and Tóth [11] showed a general upper bound of $4.108\sqrt{k}n$ edges for k-planar graphs. For small values of k, i.e., $k = 1$ and $k = 2$, they also gave tight upper bounds. The class of 1-planar graphs was introduced by Ringel [12] in the context of planar graph colorings. A 1-planar graph has at most $4n - 8$ edges, and there are infinitely many *optimal* 1-planar graphs that achieve the bound [13]. A 2-planar graph on n vertices has at most $5n - 10$ edges, and there are infinitely many *optimal* 2-planar graphs that achieve this bound [11].

A lot is left to be explored for *maximal* k-planar graphs, where a graph is *maximal k-planar* if there is no edge that can be added such that the resulting graph is still k-planar. In contrast to planar graphs, maximal k-planar graphs are not necessarily optimal k-planar graphs. Indeed the gap between maximal and optimal can be very large for k-planar graphs. Hudák et al. [9] showed an infinite family of maximal 1-planar graphs with $\frac{8}{3}n + O(1) \approx 2.667n$ edges, and a sparser construction by Brandenburg et al. [4] only has $\frac{45}{17}n + O(1) \approx 2.647n$ edges. Brandenburg et al. [4] also proved that every maximal 1-planar graph has at least

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$\frac{28}{13}n - O(1) \approx 2.153n$ edges. This lower bound was further improved by Barát and Tóth [3] to $\frac{20}{9}n \approx 2.22n$.

Interestingly, the density even decreases for maximal k-planar graphs when k increases from 1 to 2. Hoffmann and M. Reddy [6] showed that every maximal 2-planar graph on $n \geq 5$ vertices has at least $2n$ edges, and they also described an infinite family of maximal 2-planar graphs on n vertices with only $2n + O(1)$ edges. So, even though they allow more crossings, some maximal 2-planar graphs have a lower edge density than any maximal 1-planar graph.

Results. First, in Section 3 we characterize the graphs on nine vertices that are (maximal) 3-planar. Next, in Section 4 we exhibit maximal 3-planar graphs that contain vertices for which all incident edges are crossed, in every simple 3-plane drawing of the graph. As a consequence, maximal 3-planar graphs are not necessarily 2-connected and may contain vertices of degree one. In contrast, all maximal 1-and 2-planar graphs are 2-connected. Finally, in Section 5, we construct maximal 3-planar graphs on n vertices with only $2.375n + O(1)$ edges.

2 Preliminaries

A drawing is *simple* if any pair of edges has at most one common point, including endpoints. To analyze k-plane drawings of a graph, one typical restriction is to consider a drawing that minimizes the total number of crossings among all k-plane drawings of the graph, which is called a *crossing-minimal* k-plane drawing. The benefit of such a restriction is that for $k \leq 3$, a crossing-minimal k-plane drawing is always simple [10]. Consequently, 3-planar graphs always admit a simple 3-plane drawing.

▶ **Lemma 1.** *If a 3-planar graph G is not maximal 3-planar, then there exists a simple 3-plane drawing of G that is not maximal 3-plane.*

Proof. If G is not maximal 3-planar, then there exists a pair u, v of nonadjacent vertices such that $G' = G \cup e$ is 3-planar where $e = (u, v)$. Take any simple 3-plane drawing of G' and remove e to obtain a simple 3-plane drawing of G that is not maximal. ◀

In all figures of this paper, the edges are colored to indicate their number of crossings. Uncrossed edges are shown green, singly crossed edges are shown purple, doubly crossed edges are shown orange, and triply crossed edges are shown blue. Edges with an undetermined number of crossings are shown black.

3 Characterization of (Maximal) 3-planar Graphs on 9 Vertices

Angelini et al. [2] showed that K_8 is 3-planar, while K_9 is 4-planar (but not 3-planar). This motivated us to study the set of 3-planar graphs on nine vertices. With the help of a computer program, we can enumerate all possible simple 3-plane drawings of a given graph; see Section 6.

The basic idea is to check all graph structures on nine vertices in decreasing order based on the number of edges. Specifically, starting from $K_9 \setminus K_2$, which is generated by removing a single edge from a K_9, we check if the given graph admits a 3-plane drawing. Further, if we want to remove two edges from a clique, we can either remove two independent edges or remove a path P_3 of length two. If we restrict to a graph with nine vertices, that is equivalent to saying that any graph with nine vertices and thirty-four edges will be isomorphic to either $K_9 \setminus (K_2 + K_2)$ or $K_9 \setminus P_3$. We have a similar argument if we remove three edges.

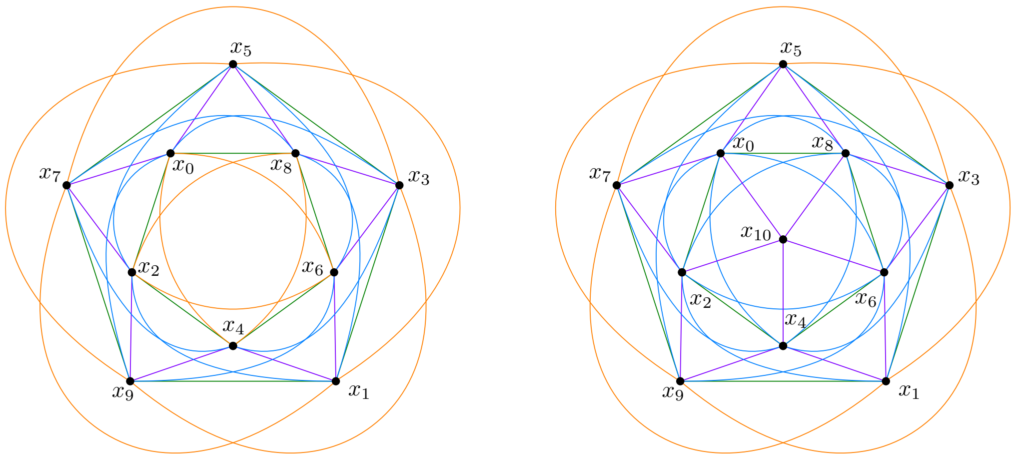

Figure 1 (Left) Drawing D_1 of graph G_1 by removing five independent edges from K_{10}. (Right) Drawing D_1' of graph G_1' by inserting a new vertex x_{10} to G_1 and adding five edges incident to x_{10}.

Using our computer program, we were able to verify that all graphs on nine vertices that have at least 34 edges do not admit a simple 3-plane drawing, while all the remaining graphs on nine vertices are 3-planar. The result can be verified with the code in our repository [7].

▶ **Theorem 2.** *A graph on nine vertices is* 3*-planar if and only if it has at most* 33 *edges, and it is maximal* 3*-planar if and only if it has exactly* 33 *edges.*

We therefore notice that maximal 3-planar graphs and maximum 3-planar graphs on n vertices coincide for $n \leq 9$.

4 Uncrossed Edge in Maximal 3-planar Graphs

Though we allow crossings in beyond-planar graphs, it does not necessarily mean that every edge will have a crossing. In a more general sense, crossings are not equally distributed over the edges. In maximal k-planar graphs where $k \leq 2$, it has been shown that every vertex must be incident to an uncrossed edge in every crossing-minimal k-plane drawing [6]. But when $k = 3$, the situation is different and we have the following

▶ **Lemma 3.** *There exist infinitely many maximal* 3*-planar graphs that each contains a vertex v such that all edges incident to v are crossed in every simple* 3*-plane drawing of the graph.*

Proof. The proof of Lemma 3 is based on a graph G_1' that has the required properties. Then we can use G_1' to create larger graphs. To construct a graph G_1', we start from a base graph G_1 that is isomorphic to K_{10} minus five independent edges. Specifically, we take the vertex set as $\{x_0, x_1, ..., x_9\}$, and include all edges of the induced complete graph except the five edges $\{x_0, x_1\}$, $\{x_2, x_3\}$, $\{x_4, x_5\}$, $\{x_6, x_7\}$ and $\{x_8, x_9\}$. We enumerated all simple 3-plane drawings of G_1 using our program and found that this graph has exactly one simple 3-plane drawing up to automorphism. This drawing is illustrated in Fig. 1, and let this drawing be D_1. We can observe that it is impossible to add an edge to D_1 while maintaining 3-planarity, thus it proves G_1 is a maximal 3-planar graph from Lemma 1.

We can further obtain a new graph G_1' by adding a new vertex x_{10} to G_1 and connecting it to the five vertices x_0, x_2, x_4, x_6, x_8. We claim that the drawing D_1' illustrated in Fig. 1(right)

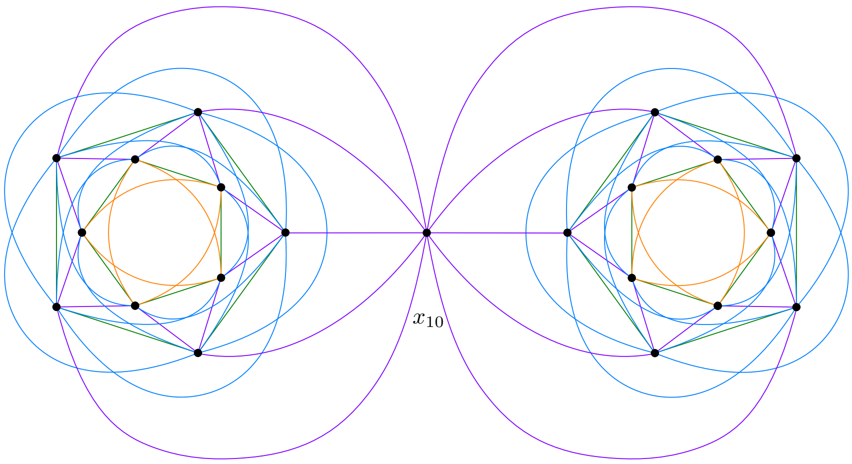

Figure 2 Graph G_2 by gluing two copies of G_1 on a merged vertex.

is the unique simple 3-plane drawing of G_1' (up to automorphisms). To see this, consider every face of D_1. It turns out that there exists exactly one face where the vertex x_{10} can be placed such that the five edges to vertices x_0, x_2, x_4, x_6, x_8 can be added while maintaining 3-planarity, and the resulting drawing is D_1'. This concludes that the graph G_1' is maximal 3-planar. We further note that x_{10} is not incident to any uncrossed edge in any 3-plane drawing of G_1'.

Further we can take two copies of the graph G_1' and merge the two vertices with degree five from each of the copies into a single vertex with degree ten to obtain a new graph G_2. It can be interpreted as gluing two copies of G_1' together at the vertex x_{10}. Refer to Fig. 2 for one possible 3-plane drawing of G_2.

We claim that G_2 is a maximal 3-planar graph. Consider an arbitrary simple 3-plane drawing of G_2. The subdrawings corresponding to the two copies of G_1 that are still simple drawings should be the same as D_1. It can also be viewed as adding a copy of D_1 into an existing drawing D_1. Again, considering all the faces of D_1 we can observe that the drawing shown in Fig. 2 is the only possible 3-plane drawing of G_2 up to automorphism.

This replication can be repeated infinitely many times to obtain larger graphs, and in each such graph the vertex x_{10} is not incident to any uncrossed edge in every simple 3-plane drawing, concluding the proof. ◀

Clearly, x_{10} is a cut vertex in G_2 and every graph obtained by following the replicating procedure. Thus, we have the following

▶ **Theorem 4.** *There exist infinitely many maximal* 3*-planar graphs that are not* 2*-connected.*

5 Number of Edges in Sparse Maximal 3-planar Graphs

We describe a construction of sparse maximal 3-planar graphs based on G_1 in this section.

▶ **Theorem 5.** *There exist an infinite family of maximal* 3*-planar graphs on* n *vertices with at most* $2.375n + O(1)$ *edges.*

Proof. We construct a nested graph with arbitrarily many layers where each layer is a variation of G_1 as shown in Fig. 3. Specifically, in each layer, we add a *hermit* vertex

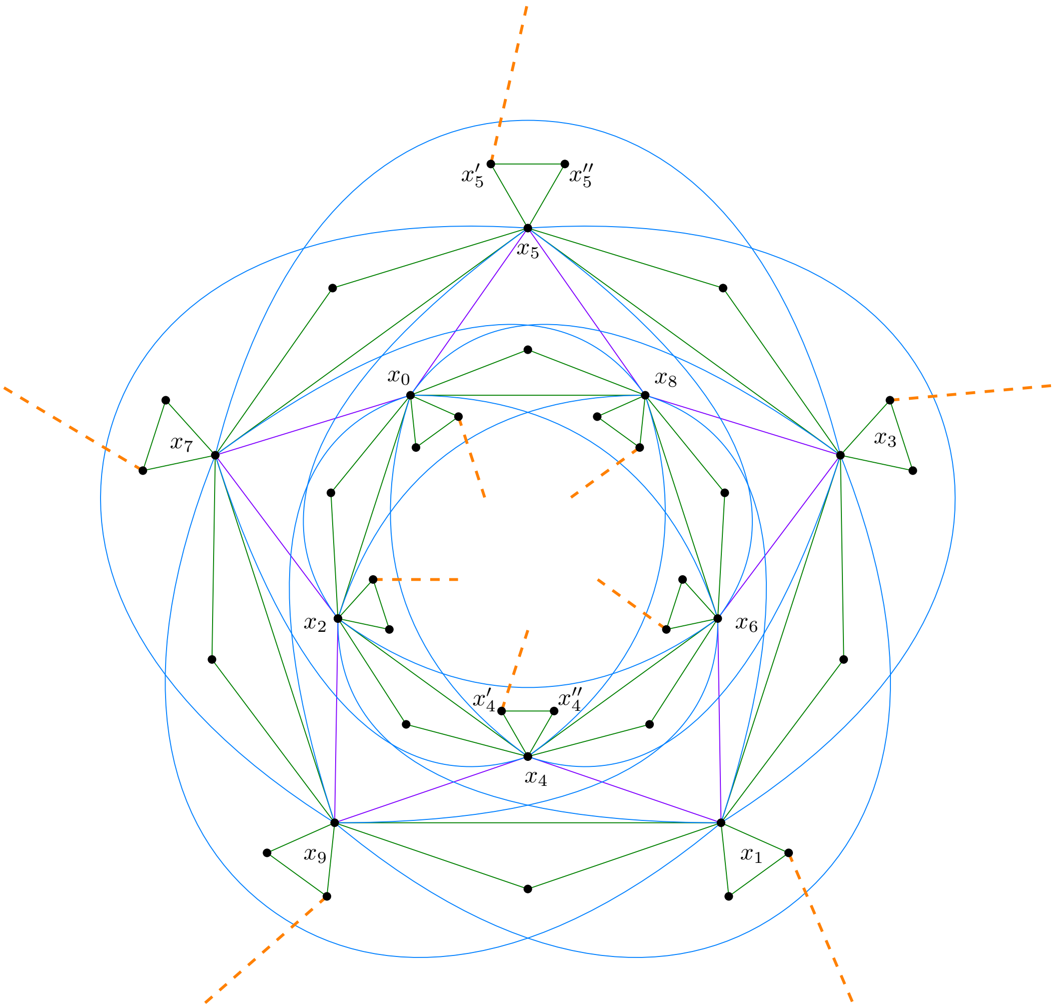

Figure 3 A single layer in the nested graph structure. Some labels are omitted for simplicity.

connecting to two endpoints of each planar edge, and a triangle to each original vertex, where a hermit is a degree-two vertex.

Further, as shown in Fig. 3, consecutive layers are connected with each other with orange dashed lines, which represent half edges. Specifically, suppose vertices from layer i are indexed as x_j^i for $0 \leq j \leq 9$, and corresponding triangle vertices are labelled as $x_j^i\,'$ and $x_j^i\,''$. To connect layers i and $i+1$, an edge is added between $x_j^i\,'$ and $x_{j+1}^{i+1}\,'$ for even j. To close the innermost and outermost layers, we simply use two 5-stars respectively to complete the graph.

We argue that the constructed graph still admits few simple 3-plane drawings. We start from each layer based on the unique simple drawing D_1. For those inserted triangles, since the drawing is symmetric considering the innermost face and the outermost face, it is enough to argue about the outer triangles. We note that outer vertices x_j^i of layer i for odd j should be reachable from a single face because all vertices from other layers have to be drawn in a single face of the drawing for layer i. Thus the only eligible face is the outermost face. Observe that $x_j^i\,''$ connects to x_j^i and $x_j^i\,'$, so the drawing of triangles has to be the same as shown in Fig. 3. We then note that each planar edge is enclosed by blue curves. Thus every hermit vertex can only be drawn in faces on two sides of the planar edge. For the closing 5-star, the argument is similar as stated for graph G_1'. Then consider all possible simple 3-plane drawing of such a nested graph, it is impossible to add any edge in any of the drawing, and this concludes that the graph is maximal 3-planar.

For each layer, we have 40 vertices and 90 edges. And between two layers, there are 5 edges connecting them. We can charge these 5 edges to one layer, and extend to arbitrarily many layers. In total we will have $40k + 2$ vertices and $95k + 5$ edges for a graph with k layers. And it gives an edge density of $\frac{95k+5}{40k+2} \cdot n = 2.375n + O(1)$ where n is the number of vertices. This concludes the proof. ◀

It is possible that the construction for 2-planar graphs with roughly $2n$ edges [6] can be extended to the 3-planar case; see Fig. 4. But new ideas are needed to prove maximality.

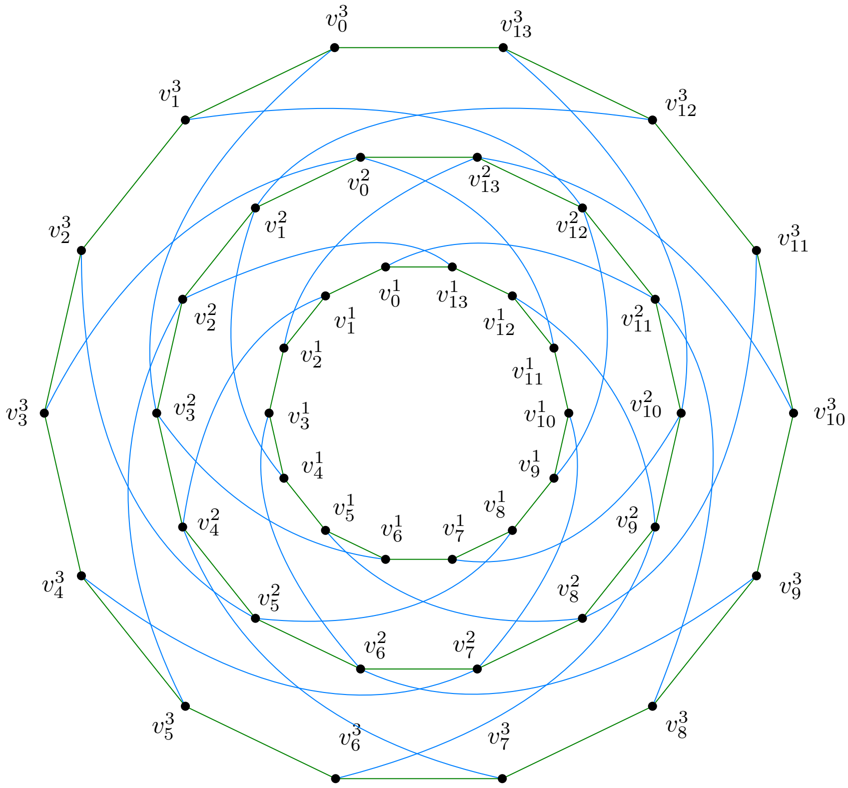

Figure 4 Nested C_{14} is a possible maximal 3-planar graph.

6 Enumeration of 3-plane drawings

In this section, we sketch the program we used to enumerate all possible simple 3-plane drawings of small graphs. This program was used to show certain properties of 3-planar graphs, like maximality. The basic idea is similar to previous work [1, 6]. We adapted the code for 2-planar graphs from [6] and extended it to 3-planar graphs. It is available in our repository [7].

To enumerate all simple k-plane drawings of a graph up to strong isomorphism (that is, up to a homeomorphism of the plane), we enumerate combinations of all possible drawings of each edge. We fix a labeling of the vertices and an ordering of the edges to then use depth first search to explore all possible simple 3-plane drawings. The restriction to simple drawings is without loss of generality by Lemma 1.

We add edges one by one to the current drawing, and try to complete the given graph. Whenever we add an edge, we take every valid drawing of the edge into consideration. After we run out of different ways to draw the current edge, we backtrack and try to draw the previous edge differently. Whenever we successfully added all edges into the drawing, we found a simple 3-plane drawing of the given graph. We record the drawing and continue.

The time complexity of such a search is exponential in the number of edges, and thus it can be used for small graphs only. In practice, it takes roughly one hour to enumerate drawings for graphs on 9 vertices, and 30 hours for graphs on 10 vertices. For larger graphs it seems challenging to enumerate all simple 3-plane drawings within reasonable time limits.

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