Star-Forest Decompositions of Certain Geometric Graphs

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Abstract

We deal with the problem of decomposing a complete geometric graph into plane star-forests. In particular, we disprove a recent conjecture by Pach, Saghafian and Schnider by constructing an infinite family of complete geometric graphs on n vertices which can be decomposed into $\frac{2n}{3}$ plane star-forests. We also describe a method which can be potentially used to construct such infinite families of geometric graphs decomposable into cn plane star-forests given only a single such graph, for any given $c \in (\frac{1}{2}, 1)$.

1 Introduction

A classic question asked in graph theory is the following: "Given a graph G, what is the minimal number of subgraphs with property P that G can be partitioned into?" Historically, this question was asked for abstract graphs and property P was replaced with forests, trees, complete bipartite graphs and many more [2, 7, 10]. Similar questions can be asked about graphs drawn in the plane or on any other surface. Here we want to decompose our complete graph into plane/k-planar/k-quasiplanar subgraphs with a given property. Answering such questions is a similar, but separate research direction that has been pursued by many authors in discrete geometry and graph drawing communities.

A *geometric* graph is a graph drawn in the plane, with vertices represented by points in general position and edges as straight line segments between them.

Recently, there has been a lot of work done on decomposing geometric graphs into planar subgraphs of a special kind, such as trees, stars, double stars etc. [11, 6]. This paper will be concerned with plane star-forests. A *star* is a connected graph on k vertices with one vertex of degree $k-1$, which we call the center of the star, and $k-1$ vertices of degree one. This definition allows for a single edge to be a star, in this case we decide arbitrarily which of its endpoints is the center. A *star-forest* is a forest whose every connected component is a star. A star-forest is plane if it is drawn in the plane without crossings. It is easy to observe that a complete graph K_n can be decomposed into $n-1$ stars. A fact that is not obvious is that K_n cannot be decomposed into less than $n-1$ stars [3]. In the same paper, Akiyama and Kano proved that K_n can be decomposed into $\lceil \frac{n}{2} \rceil + 1$ star-forests.

The story is different for complete geometric graphs. Recently, Pach, Saghafian and Schnider [8] showed that a complete geometric graph whose vertices form a convex polygon

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cannot be decomposed into fewer than $n-1$ plane star-forests. In the same paper, the authors posed the following question:

▶ Question 1.1. What is the minimal number of plane star-forests that a complete geometric graph can be decomposed into?

Based on their findings they made the following conjecture:

▶ **Conjecture 1.2** ([8]). *Let $n \geq 1$. There is no complete geometric graph with n vertices that can be decomposed into fewer than $\lceil 3n/4 \rceil$ plane star-forests.*

The authors give a special configuration of $n = 4k$ points and construct a simple decomposition into $3n/4$ star-forests. Motivated by this example, we find configurations of n points that define a complete geometric graph which can be decomposed into $\lceil 2n/3 \rceil$ plane star-forests, disproving the conjecture.

Note on new results After submission of the paper to EuroCG we managed to obtain some better results. Among other things, we answered Conejcture 4.1 positively, thus proving that the bound $\lceil \frac{n}{2} \rceil + 1$ is indeed tight. The current version of the paper is available on arXiv [5].

2 The Construction

Firstly, we will give the most general possible construction and then present the concrete counterexample. We will write GP for a complete geometric graph whose underlying pointset is $P \subset \mathbb{R}^2$.

▶ **Theorem 2.1.** *Let $c \in (1/2, 1)$ be a constant. If there is a complete geometric graph on n_0 points which can be partitioned into cn_0 star-forests, in such a way that each vertex is a center of at least one star, then for each integer $k \geq 1$, there exists a complete geometric graph on kn_0 points that can be partitioned into ckn_0 star-forests.*

Proof. Let S be the underlying point set of the original complete geometric graph and let $k > 1$ be an integer. Label the points in S by $a_1, \ldots, a_{n_0}$. Now, replace each a_i by a set $A_i = \{a_i^1, ..., a_i^k\}$ of k points in general position in such a way that if we choose $b_1, \ldots, b_{n_0}$ where $b_i \in A_i$, we obtain a point set of the same order type as S. Call the new point set S^k. Now if $F_1, \ldots, F_{cn_0}$ is the decomposition of GS into star-forests, from this, we will obtain the decomposition of GS^k into $c(kn_0)$ star-forests. Let a_j be the center of a star in F_i. We will construct k new stars with centers in $a_j^1, \ldots, a_j^k$. Start with a_j^1, add to it all of the edges of the form $\{a_j^1, a_j^l\}$ that were not already used (in the case of a_j^1, none were used). Now for each edge of the form $\{a_j, a_{j'}\}$ in F_i, add all of the edges from a_j^1 to the vertices in $A_{j'}$. Continue doing this for each vertex a_j^l, where $l \in \{1, 2, \ldots, k\}$. We do this for each star in F_i and for each forest in the original decomposition. The result of this process is cn_0 families of star-forests, each of size k. And the planarity of the star-forests follows from the definition of the point set S^k. To see this, assume that a tree in the new decomposition has an intersection. Then the intersection is between edges whose 4 vertices are in different A_i's. But if this was the case, then a choice of transversal that includes this 4 vertices would induce a crossing inside the original decomposition of GS. ◀

We note that the assumption that each point is a center of at least one forest is crucial as otherwise the star-forests constructed in the proof do not cover all of the edges. For example see Figure 1. The vertex v is not a center of any star and thus none of the edges between vertices in A_v are covered by the star-forests on the right.

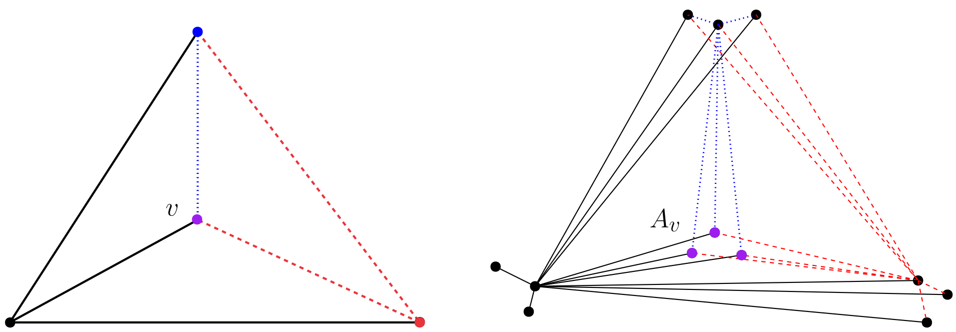

Figure 1 A complete geometric graph on 4 vertices decomposed into three star-forests and the corresponding graph on 12 vertices with the wrong "decomposition" into 9 star-forests. (only 4 are drawn for readability).

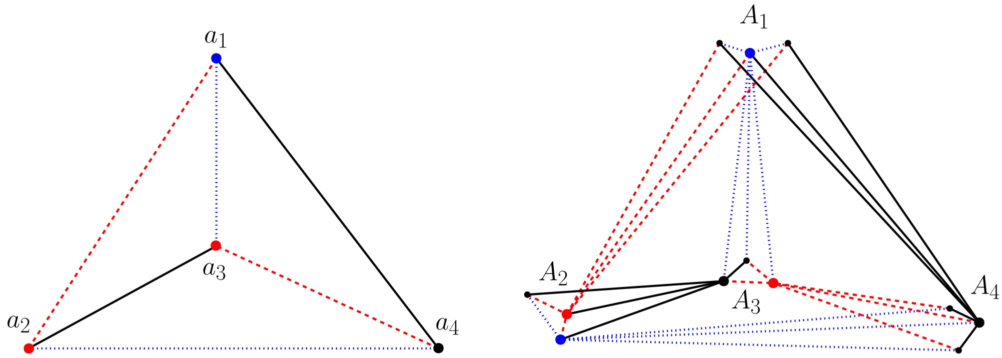

Figure 2 A complete geometric graph on 4 vertices decomposed into three plane star-forests and the corresponding graph on 12 vertices with the decomposition into 9 star-forests (only 4 are drawn for readability). Each vertex of the point set has been used as a center of some star and colored accordingly.

While Theorem 2.1 gives us a nice way of constructing infinitely many complete geometric graphs that can be partitioned into few plane star-forests, we still need concrete small examples to be able to produce the infinitudes. One example was given by the authors in [8] and can be found in Figure 2. This example motivated Conjecture 1.2. We proceed in a similar fashion.

▶ **Lemma 2.2.** *There exists a configuration of* 6 *points in the plane which can be partitioned into* 4 *plane star-forests in such a way that each point is a center of at least one star.*

Proof. We consider a configuration of 6 points which is crossing-minimal according to [9]. We decompose the graph into 4 star-forests as in the Figure 3. The graph has thus been decomposed into three 2-component star-forests colored in blue, red and black and one 3-component forest colored in purple. ◀

Now, using the pointset on $n_0 = 6$ elements from the above lemma, which can be decomposed into $2n_0/3 = 4$ star-forests, we obtain as an easy corollary a family of pointsets on $n = 6k$ points which can be decomposed into $2n/3$ star-forests, thus disproving Conjecture 1.2. We state this formally below.

▶ **Corollary 2.3.** *For every $k \in \mathbf{N}$, there exists a geometric graph on $n = 6k$ vertices which can be decomposed into $2n/3$ plane star-forests.*

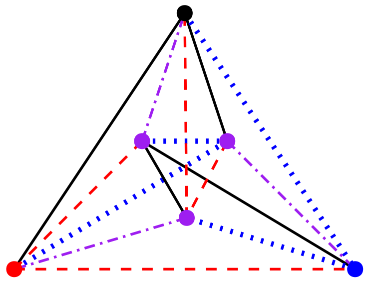

Figure 3 A complete geometric graph on 6 vertices decomposed into four star-forests, vertices are colored same as trees whose centers they are.

3 Computing Plane Star-Forest Decompositions on Pointsets with 6 Points

Using a simple computer search, we managed to find all pointsets on 6 points that can be decomposed into 4 plane star-forests. Out of the 16 order types which can be found on [1], we have found decompositions which satisfy the requirements from Theorem 2.1 for 6 of them. Those pointsets and corresponding partitions can be seen in Figure 4. The code is available at [4]. We plan to continue improving the code to be able to perform the search on bigger pointsets. Currently, the generation of appropriate decompositions is very slow, and since Stirling numbers grow very fast, we are not able to do the checks for bigger pointsets.

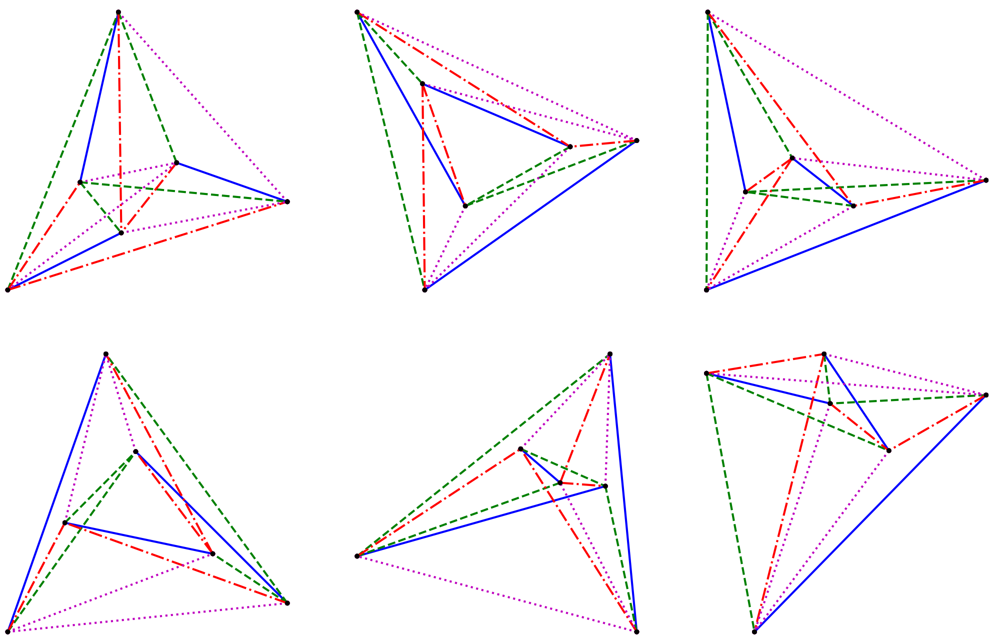

Figure 4 Star-forest decompostions of the pointsets that admit them.

4 Further Research and Open Questions

It is still unclear to us whether the number $2n/3$ is optimal, and we would be very surprised if it is. Thus we make the following conjecture:

▶ **Conjecture 4.1.** *For each $c \in (1/2, 1)$, there exists an $n \in \mathbb{N}$ and a complete geometric graph on n vertices which can be decomposed into $\lceil cn \rceil$ plane star-forests.*

If our conjecture is true, that would mean that the bound of $\lfloor n/2 \rfloor + 1$ is almost tight.

We also note that there is an interesting variation of this problem that we have not explored yet but where our approach can also be used. We define a *k-star-forest* to be a star-forest with at most k components. Authors in [8] proposed the following conjecture:

▶ **Conjecture 4.2.** *The number of plane k-star-forests needed to decompose a complete geometric graph is at least $\frac{(k+1)n}{2k}$.*

Our example does not show anything regarding Conjecture 4.2. But, it is not hard to see that the construction from Theorem 2.1 preserves the maximal number of components among all forests. Thus, we believe a similar approach could be used to attack this conjecture.

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