Revisiting the Fréchet distance between piecewise smooth curves

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Abstract

With the notable exception of an algorithm for the decision problem for planar piecewise smooth curves due to Rote (2007), research into algorithms for computing the Fréchet distance has concentrated on comparing polygonal curves. We present an algorithm for the decision problem for piecewise smooth curves that is both conceptually simple and naturally extends to the first algorithm for the problem for piecewise smooth curves in $\mathbb{R}^d$. To this end, we introduce a decomposition of the free space diagram into a controlled number of pieces that can be used to solve the decision problem using techniques similar to the polygonal case. Assuming the algorithm is given two continuous curves, each consisting of a sequence of m, resp. n, smooth pieces, where each piece belongs to a sufficiently well-behaved class of curves, such as the set of algebraic curves of bounded degree, we solve the decision problem in $O(mn)$ time. Furthermore, we study approximation algorithms for piecewise smooth curves that are also c-packed. We adapt the existing framework for $(1+\varepsilon)$-approximations and show that an approximate decision can be computed in $O(cn/\varepsilon)$ time for any $\varepsilon > 0$.

1 Introduction and motivation

The Fréchet distance is a well-studied distance measure between curves, with a long history in both applications and algorithmic research. The wealth of work surrounding the analysis of algorithms for computing the Fréchet distance is centered primarily on polygonal curves. However, more complicated curves and especially splines are natural objects that have become commonplace in industrial applications for, e.g., computer graphics, robotics and to represent motion tracking or planning data. A crucial prerequisite to using smooth curves similarly to polygonal curves in such contexts is the ability to effectively answer elementary algorithmic questions for such curves. A natural and fundamental task in computational geometry is the computation of the Fréchet distance between smooth curves such as splines. Despite this, as far as we know, there is no known approach to realizing such a computation for curves in $\mathbb{R}^d$. To tackle the case of smooth curves in the plane ($d = 2$), Rote [3] introduced an approach based on analyzing the turning angle and planar curvature of the planar curves. However, this approach does not easily generalize to higher dimensions. We revisit this problem and present a novel, simpler approach, with the additional benefit that it works for higher dimensions, with the same time complexity. Our methods are conceptually simple, but rely on a number of key technical ingredients.

Problem definition Throughout the paper, γ_1 and γ_2 will be used to denote two piecewise smooth curves in $\mathbb{R}^d$ with d fixed, that is, continuous maps $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^d$ that are comprised of m and n smooth pieces, each of class C^2. Let $A_{[0,1]}$ be the set of continuous and bijective maps $\alpha : [0,1] \to [0,1]$ that are increasing. The **Fréchet distance** between γ_1 and γ_2 is defined as $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) := \inf_{\alpha,\beta \in A_{[0,1]}} \max_{t\in[0,1]} \|\gamma_1(\alpha(t)) - \gamma_2(\beta(t))\|$. Our methods

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naturally allow any fixed ℓ_p norm with $1 < p < \infty$ for the norm $\|\cdot\|$ (the cases $p = 1, \infty$, while possible, would add a level of technicality to our treatment that distracts from its relative simplicity). We focus primarily on the **decision problem** of deciding whether the Fréchet distance between two piecewise smooth curves is at most a given $\delta > 0$.

Results Our first main contribution is that we establish an algorithm to solve the decision problem for the Fréchet distance between piecewise smooth curves. Assuming that the curves are **algebraically bounded curves**, i.e., piecewise smooth algebraic curves where the degree of the curves is bounded by a constant, we obtain a bound of $O(mn)$ for the time complexity of the decision problem, which matches the polygonal case. The running time is independent of the ambient dimension but the algebraic complexity of the operations involved in the algorithm depends on the dimension and the nature of the curves. Our algorithm for the decision problem results in an algorithm for the computation of the Fréchet distance for algebraically bounded curves in $O(mn \log(mn))$ time using parametric search, similarly to the polygonal case.

It is known [1] that the decision problem cannot be solved in strongly subquadratic time, so research has focused on investigating algorithms for restricted classes of curves. Our second contribution is that we show that we can adapt the framework from [2] for an efficient $(1 + \epsilon)$-approximation algorithm for the Fréchet distance between two c-packed, polyognal curves to the setting of c-packed piecewise smooth curves in $\mathbb{R}^d$. To this end, we introduce a simplification procedure for piecewise smooth curves and distill the necessary ingredients to obtain a linear time decision algorithm for algebraically bounded c-packed curves.

Comparison to previous work To arrive at an algorithm for the decision problem for smooth planar curves for the ℓ_2-norm for a given δ in general position, Rote uses a partitioning of the smooth curves, induced by condition on the turning angle and planar curvature, to obtain pieces for which the associated **free space diagram** FSD_δ (Section 2) is well-behaved. In contrast to this, our approach is to analyze the free space diagram directly, by studying the boundary of the **free space** $\mathcal{D}_\delta$ in FSD_δ, leading to a conceptually simpler algorithm. The free space is defined as the set of parameter value pairs at which the curves are at most a distance of δ apart. We propose a refined decomposition of each cell of FSD_δ into a controlled number (depending on the degree of the curves) of subcells, for which determining the existence of a monotone path connecting two intervals on the boundary of a subcell is easy. Here, the role of convexity of the free space in a cell for polygonal curves is replaced by monotonicity of the boundary curves of $\mathcal{D}_\delta$ within each subcell of the refined decomposition. We emphasize that our construction of the refined decomposition exclusively accesses the same values that are also required in Rote's work to process each subcell of FSD_δ.

Unlike the polygonal case, the free space within a cell of FSD_δ can be very complicated, as illustrated by a contour plot of the distance function in parameter space for two degree 3 splines in $\mathbb{R}^3$ in Figure 1 for different values of δ. Figure 3 shows another example of the kind of behavior of the free space one can expect within a cell. We note that both Rote's decision algorithm as well as ours assume values of δ for which the boundary of $\mathcal{D}_\delta$ has no singularities. We show that singularities of the boundary of $\mathcal{D}_\delta$ are confined to a small number of critical values of δ and are thus not necessary for the computation of $\mathrm{d}_\mathcal{F}$.

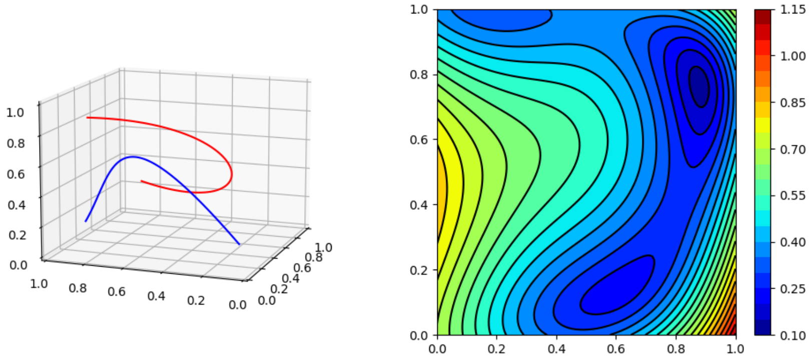

Figure 1 Two smooth curves in $\mathbb{R}^3$ and a contour plot of the associated distance function in the joint parametric space of the curves.

Computational assumptions We assume that we can compute the intersection of a curve with a sphere of a given radius centered at a point of another curve and find the parameter values in $[0,1]$ that correspond to the intersections.

2 A combinatorial description of the free space diagram

For two piecewise smooth curves $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^d$ consisting of m and n pieces, respectively, and $\delta > 0$, the **free space** $\mathcal{D}_\delta = \mathcal{D}_\delta(\gamma_1, \gamma_2)$ is defined as

$$\mathcal{D}_\delta(\gamma_1, \gamma_2) = \left\{(x, y) \in [0,1]^2 | \|\gamma_1(x) - \gamma_2(y)\| \le \delta\right\}.$$

The complement of $\mathcal{D}_\delta$ in $[0,1]^2$ is referred to as the **forbidden region**. There is a natural partition of the joint parameter space $[0,1]^2$ of both curves into $m \cdot n$ rectangular cells such that γ_1 and γ_2 are smooth when restricting to the interior of each rectangle. The resulting decomposition of $[0,1]^2$ together with the partitioning into the free space and forbidden region is known as the **free space diagram** FSD_δ. A key motivation behind the definition is the observation that $\mathrm{d}_\mathcal{F}(\gamma_1, \gamma_2) \le \delta$ iff there is a path from $(0,0)$ to $(1,1)$ through the free space in $[0,1]^2$ that is monotone in both coordinates.

Overview of the algorithm Similarly to the classical polygonal case, to solve the decision problem, we investigate the existence of a monotone (in both coordinates) path from $(0,0)$ to $(1,1)$ in the free space $\mathcal{D}_\delta$. To this end, we refine the free space diagram using the boundary B_δ of the free space. Our decision algorithm has the following high-level description.

1. Mark the minima and maxima of the boundary B_δ of the free space in FSD_δ in the x (horizontal) and y (vertical) direction.
2. Cut each cell of FSD_δ into subcells, horizontally (vertically) through each marked point if it has a vertical (horizontal) tangent. Mark each point of intersection of a cut with B_δ.
3. For each resulting subcell, pair the marked points on the boundary according to how they are connected by B_δ through monotone arcs, so that adjacent points are paired.
4. Solve the decision problem for FSD_δ using only the marked points and pairings by computing reachable intervals on the boundaries of cells, in particular
   a. process all cells in lexicographical order of their indices (row by row, from the left);
   b. for each cell, process all subcells within the cell in lexicographical order.

2.1 Refining the free space diagram

We consider the boundary B_δ of FSD_δ as a set of curves, as opposed to the boundary of a region. Let I_{sing} be the set of singularities of B_δ in the interior of the cells of FSD_δ, consisting of points where B_δ has a cusp or intersects itself. Like Rote, we assume that $I_{\mathrm{sing}} = \emptyset$. It turns out that for almost all δ, there are no singular points of B_δ in the interior of each cell in FSD_δ associated to the smooth pieces of the curves, so that $I_{\mathrm{sing}} = \emptyset$, after possibly applying a small perturbation to δ, as illustrated in Figure 2. Intuitively, the scarcity of critical values for δ can be explained by noting that each critical value corresponds to a value of δ for which there are points $(t_1, t_2) \in B_\delta$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\|\gamma_1(t_1) - \gamma_2(t_2)\|_p = 0 = \frac{\mathrm{d}}{\mathrm{d}t_2}\|\gamma_1(t_1) - \gamma_2(t_2)\|_p, \tag{1}$$

equations which themselves do not depend on δ. In contrast, we note that for the norms ℓ_1 and ℓ_∞ in the definition of the Fréchet distance, B_δ may contain cusp singularities for all values of δ in an open interval.

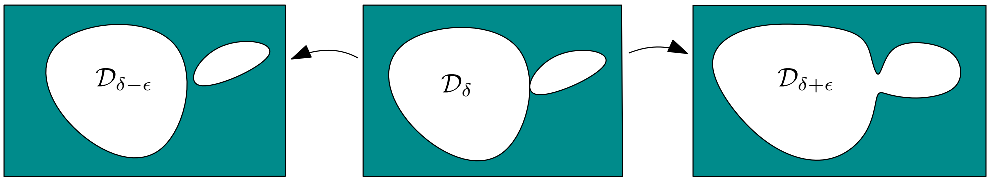

Figure 2 Illustration of singular points and changes of the boundary B_δ as δ changes.

Let $E_h \subset B_\delta$ ($E_v \subset B_\delta$) be the set of extrema of the free space in the y direction, with horizontal tangent (in the x direction, with vertical tangent). For simplicity of exposition, we assume that each of E_h and E_v is a collection of isolated points. In particular, B_δ does not have a vertical or horizontal segment, which means that there is no arc of one curve that lies at a constant distance δ from a point on the other.

For a point $z \in E_h$ (E_v), we fix the cell in FSD_δ containing z, and trace the vertical (horizontal) line incident to z inside this cell. The result is a refinement of each cell of FSD_δ into a collection of **subcells** $\{S\}$, illustrated in Figure 3 for one cell of FSD_δ.

▶ **Lemma 2.1.** *In the interior of each subcell in $\{S\}$, B_δ is a union of smooth arcs that are monotone in both coordinates of $\mathbb{R}^2$ and disjoint except possibly at the boundary of a subcell.*

We record each intersection $\mathcal{I}_\mathcal{S}$ of B_δ with the boundary of each subcell $\mathcal{S}$, which together form the set $\mathcal{I} = \bigcup_{\mathcal{S}\text{ is subcell}} \mathcal{I}_\mathcal{S}$ of all intersections of subcell walls with B_δ. Notice that B_δ can be naturally interpreted as a graph G_δ with vertex set $\mathcal{I}$, and each edge a monotone arc contained in a subcell. We partition the two sets E_h and E_v into the sets E_h^+ and E_h^-, and E_v^+ and E_v^-, respectively, according to whether the forbidden region lies locally to the right of or above the point $(-)$, or to the left of or below the point $(+)$. For the bottom and left edge of each subcell $\mathcal{S}$, we refer to the information of whether the boundary B_δ at each point in $\mathcal{I}_\mathcal{S}$ is increasing or decreasing as a function of the horizontal x-coordinate as the **slope information** of these points. In other words, the slope information at a point $z \in \mathcal{I}_\mathcal{S}$ can be thought of as an extra bit associated to z that encodes whether B_δ curves to the left or to the right at z, illustrated in Figure 3 by arrows.

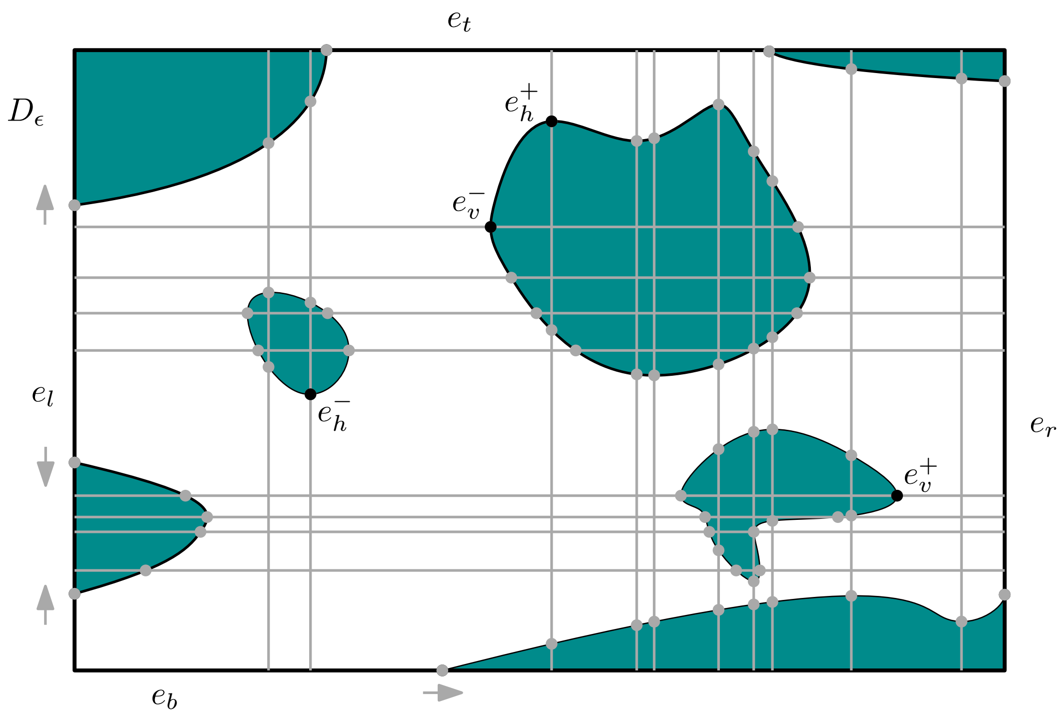

Figure 3 The decomposition of a cell of the free space diagram into subcells arising from the horizontal and vertical lines at extremities of the forbidden region in the coordinate directions.

The slope information on the bottommost and leftmost edges of the original cells of FSD_δ leads to a construction recipe for the combinatorial structure of G_δ from its vertex set.

▶ **Lemma 2.2.** *Assume δ is such that $I_{sing} = \emptyset$. There is an algorithm that reproduces the combinatorial structure of G_δ, using the sets $E_h^+, E_h^-, E_v^+, E_v^-$, and $\mathcal{I}$ along with the slope information on the bottommost and leftmost edges of* FSD_δ*, in time $O(|\mathcal{I}|)$.*

▶ Remark. Figure 4 illustrates the necessity of some knowledge of the slope information on edges of a subcell for the accurate reconstruction of B_δ inside a cell.

As illustrated in Figure 5, the slope information for points in a subcell can be gleaned by evaluating the distance between the two curve segments at certain test points.

The parts of the cell walls that are reachable by monotone paths in the free space can be computed in a structurally similar way to the polygonal case, leading to an algorithm for the

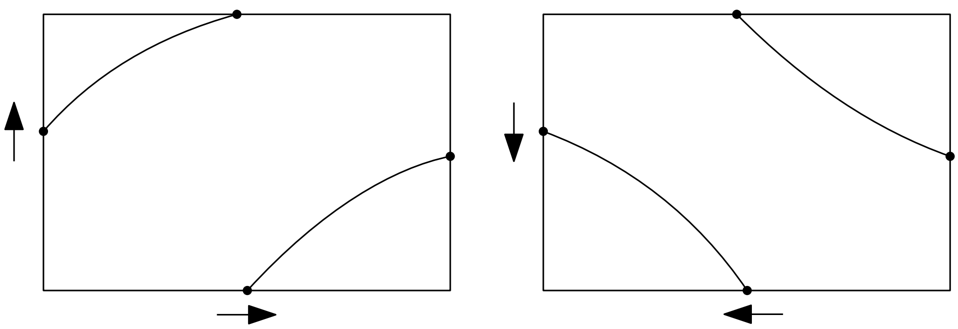

Figure 4 Two different sets of slope information and their combinatorial structures in a subcell.

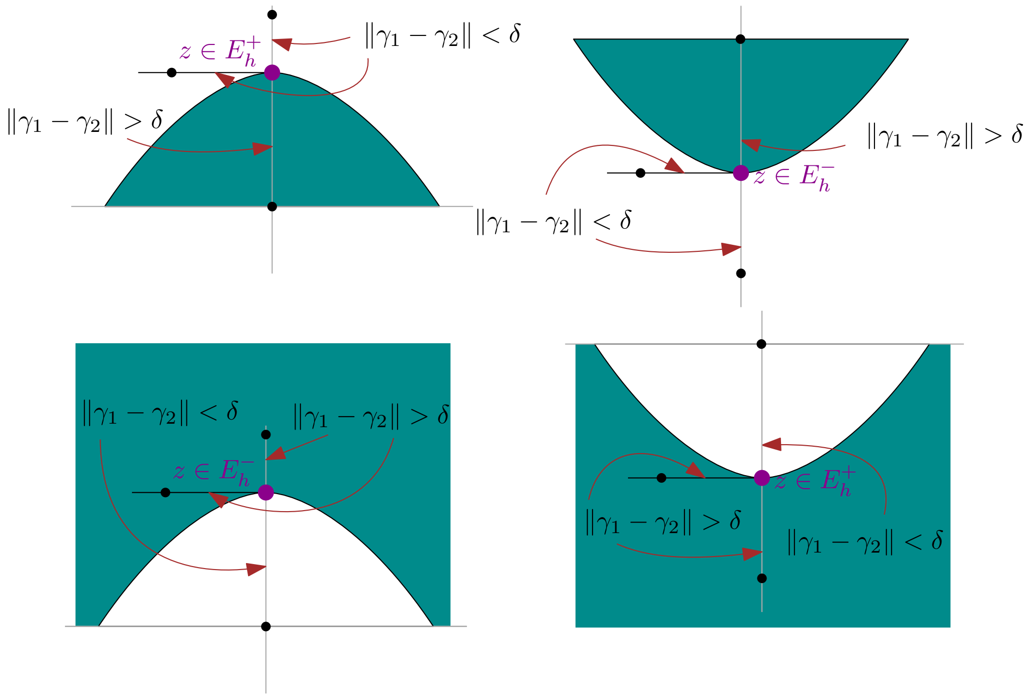

Figure 5 Finding slope information by evaluating the distance at points.

decision problem. The crucial insight is that each arc of the boundary of the free space inside each subcell is a monotone arc, which allows for to transfer the reachable intervals on the bottom and left subcell walls to neighboring cell walls in constant time. The following result is due to there only being a constant number of subcells in each original cell for algebraically bounded curves, with constant depending only on the allowed degree of the curves.

▶ Proposition 2.3. Given two algebraically bounded piecewise smooth curves γ_1, γ_2 in $\mathbb{R}^d$ comprised of m and n pieces, respectively, and a value of δ such that B_δ has no singularities, one can decide if $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) \leq \delta$. The running time is bounded by $O(mn)$.

The solution to the decision problem can be used to compute the Fréchet distance in the same way as in the case of polygonal curves, using parametric search. For this, the first step is to identify the $O(mn)$ critical values where marked points appear or disappear, components merge, appear, or start touching the boundary of cells. We then apply a binary search among these $O(mn)$ critical values to narrow down the range of values for the Fréchet distance to be a critical value corresponding to a change of the order in the x- and y-direction of the marked points in the free space diagram. Inbetween these $O(mn)$ critical values, Cole's variant of parametric search with a parallel sorting algorithm for both the x- and y-coordinates of all the marked points of B_δ yields an overall running time of $O(mn \log(mn))$ for the computation of the Fréchet distance.

▶ **Theorem 2.4.** *Let γ_1 and γ_2 be two algebraically bounded curves in $\mathbb{R}^d$ consisting of m and n pieces, respectively. Then the Fréchet distance between γ_1 and γ_2 can be computed in $O(mn)$ space and in $O(mn \log(mn))$ operations (of bounded algebraic complexity).*

3 The decision problem in linear time for c-packed curves

A curve γ is **c-packed** if the total arc length of γ inside any ball of radius r is at most cr. By utilizing a simplification procedure for piecewise smooth curves that transforms c-packed curves into c'-packed curves and guarantees a minimum arclength of each piece of the simplification, one can show that the number of grid cells that are reachable and contain free space, of simplified c-packed curves, depends linearly on n. This ultimately leads to our main result concerning approximate decision algorithms.

▶ **Corollary 3.1.** *Let γ_1 and γ_2 be two piecewise smooth algebraically bounded c-packed curves, $1 \geq \epsilon > 0$ and $\delta > 0$. There is an algorithm that correctly outputs, in $O(cn/\epsilon)$ time, either (i) a $(1+\epsilon)$-approximation to $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2)$, (ii) $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) < \delta$, or (iii) $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) > \delta$.*

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