Recognition Complexity of Subgraphs of 2- and 3-Connected Planar Cubic Graphs

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Abstract

We study the recognition complexity of subgraphs of 2- and 3-connected planar cubic graphs. Recently, we presented [ESA 2022] a quadratic-time algorithm to recognize subgraphs of planar cubic bridgeless (but not necessarily connected) graphs, both in the variable and fixed embedding setting (the latter only for 2-connected inputs). Here, we extend our results in two directions: First, we present a quartic-time algorithm to recognize subgraphs of 2-connected planar cubic graphs in the fixed embedding setting, even for disconnected inputs. Second, we prove NP-hardness of recognizing subgraphs of 3-connected planar cubic graphs in the variable embedding setting.

1 Introduction

Given a planar graph G of maximum degree at most 3 (i.e., a *subcubic* planar graph), we want to augment G by adding further vertices and edges to obtain a 3-regular (i.e., *cubic*) planar graph H, which is then called a 3-*augmentation* of G. An embedding[1] of H induces an embedding $\mathcal{E}$ of G, and each face f of $\mathcal{E}$ may contain edges of $E(H) - E(G)$, called *new edges*, or even vertices of $V(H) - V(G)$, called *new vertices*, or none of these. It is easy to see that 3-augmentations always exist. However, this becomes non-trivial if we require H to be k-connected for some $k \in \{1, 2, 3\}$. See Fig. 1 for some problematic cases. Here, we study whether a subcubic planar graph G admits some k-connected 3-augmentation H, or equivalently, the recognition of subgraphs of k-connected cubic planar graphs. We consider several variants where the input graph G is given with a fixed embedding $\mathcal{E}$ and the desired 3-augmentation H must extend $\mathcal{E}$, and/or where the input graph G is already k'-connected for some $k' \in \{0, 1, 2\}$. (If G is 3-connected, then $H = G$ is the only connected 3-augmentation.)

[1] We consider combinatorial (crossing-free) embeddings with no specific choice of an outer face.

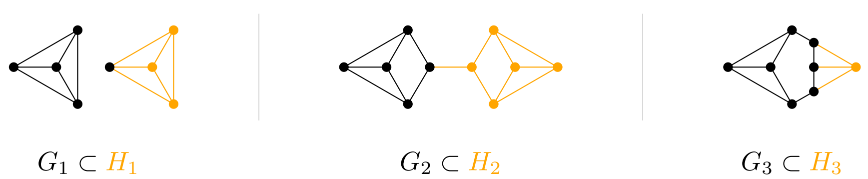

Figure 1 For $k = 1, 2, 3$, the planar subcubic graph G_k (in black) admits a $(k-1)$-connected 3-augmentation H_k (new vertices and edges in orange), but no k-connected 3-augmentation.

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

Previous Results. Motivated by the problem whether a given planar graph G is 3-edge-colorable (whose complexity is still open), we recently considered the sufficient (although not necessary) condition of whether G admits a bridgeless 3-augmentation. In fact, this can be tested in quadratic-time in the variable embedding setting, and also in the fixed embedding setting, provided that G is 2-connected [3].

Hartmann, Rollin and Rutter [5] studied, for each $k, r \in \mathbb{N}$, whether a planar graph G can be augmented by adding edges (but no vertices!), to a k-connected r-regular planar graph H. In particular, for $r = 3$, they show that the problem is NP-complete in the variable embedding setting for all $k \in \{0, 1, 2, 3\}$, as well as in the fixed embedding setting when $k = 3$. For the remaining cases of fixed embedding and $k \in \{0, 1, 2\}$ they present a polynomial-time algorithm.

Let us also refer to [3] for more related work and other augmentation problems.

Our Results. We resolve the complexity of finding a k-connected 3-augmentation for a given subcubic planar graph G in two new cases. See Fig. 2 and the following theorem.

▶ **Theorem 1.1.** *Let G be an n-vertex planar graph with an embedding $\mathcal{E}$ and $\Delta(G) \leq 3$.*

1. *We can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H extending $\mathcal{E}$, or conclude that none exists. If G is connected, then $\mathcal{O}(n^2)$ time suffices.*
2. *It is NP-complete to decide whether G admits a 3-connected 3-augmentation.*

Note that Statements 1 and 2 concern the fixed and variable embedding setting, respectively.

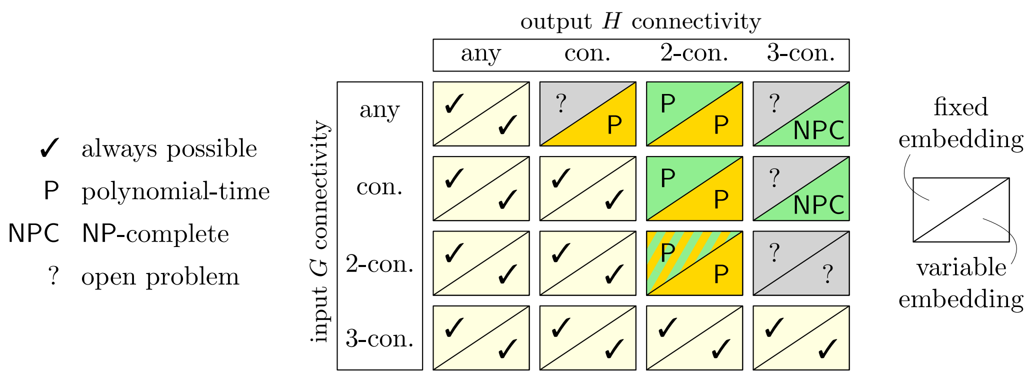

Figure 2 Complexity of finding 3-augmentations. Colors indicate results in this paper and [3].

▶ Remark. For our polynomial-time algorithms in the fixed embedding setting, we shall reduce the problem to a particular version of the Generalized Factor problem (definitions in Section 2). This approach is similar to the treatment of 2-connected input graphs in [3]. But here, for graphs containing bridges or consisting of several connected components, additional tools and a more refined analysis are needed, which is also reflected in the increased runtime.

Statements marked with $(\star)$ are proven in the full version [4].

2 Preliminaries

A graph $G = (V, E)$ is *k-connected* if $G - S$ is connected for every set $S \subseteq V$ of at most $k - 1$ vertices in G. Similarly, G is *k-edge-connected* if $G - S$ is connected for every set $S \subseteq E$ of at most $k-1$ edges in G. We denote by $\theta(G)$ the largest k for which G is k-edge-connected. If G has maximum degree at most 3, then G is k-connected if and only if $\theta(G) \geq k$.

For an integer $\ell \geq 3$, let W_ℓ be the graph obtained from $C_\ell \square P_2$ by subdividing each edge in one cycle C_ℓ exactly once. See Fig. 3 (left) for an illustration. Consider a planar graph G with an embedding $\mathcal{E}$, and a vertex $v \in V(G)$ with $\deg_G(v) = \ell \geq 3$. A *wheel-extension at* v is the graph and embedding obtained by replacing v with W_ℓ, and by attaching v's incident edges to the subdivision vertices of W_ℓ in a one-to-one non-crossing way. See Fig. 3 (right).

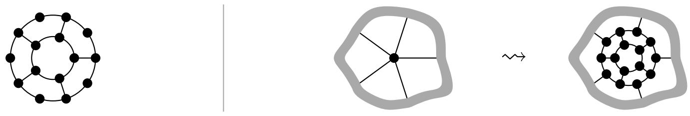

Figure 3 Left: $C_5 \square P_2$ with subdivision vertices. Right: Wheel-extension.

▶ **Observation 2.1** ($\star$). *Let G be a graph (possibly containing multi-edges, but no loops), let $v \in V(G)$ be a vertex with $\deg_G(v) \geq 3$, and let G' be obtained from G by a wheel-extension at v. Then $\theta(G') \geq \min\{\theta(G), 3\}$.*

Generalized Factors. Let H be a graph with a set $B(v) \subseteq \{0, \dots, \deg_H(v)\}$ assigned to each vertex $v \in V(H)$. Following Lovász, a spanning subgraph $G \subseteq H$ is called a *B-factor* of H if and only if $\deg_G(v) \in B(v)$ for every vertex $v \in V(H)$ [6]. Deciding whether a graph H admits a B-factor is known as the Generalized Factor problem. In general, the Generalized Factor problem is NP-complete [6]. Still, for certain well-behaved sets $B(\cdot)$, the problem becomes polynomial-time solvable. A set $B(v)$ is said to have a *gap of length* $\ell \geq 1$ if there is an integer $i \in B(v)$ such that $i+1, \dots, i+\ell \notin B(v)$, and $i+\ell+1 \in B(v)$. If all gaps of each $B(v)$ have length 1, then an algorithm by Cornuéjols can compute a B-factor in time $\mathcal{O}(|V(H)|^4)$ [1]. Moreover, if there are no two consecutive forbidden degrees $i, i+1 \in \{0, \dots, \deg_H(v)\}$ for any v, i.e., $i, i+1 \notin B(v)$, then a B-factor can be computed in time $\mathcal{O}(|V(H)| \cdot |E(H)|)$ by a result of Sebő [7]. (The latter condition is slightly stronger than requiring gaps of length at most 1, explaining the better runtime.)

3 2-Connected 3-Augmentations with a Fixed Embedding

We consider the 3-augmentation problem for arbitrary input graphs G and 2-connected output graphs H, corresponding to the third column of the table in Fig. 2. For the variable embedding setting, a quadratic-time algorithm is given in [3, Theorem 2]. For the fixed embedding setting here, we present a quartic-time algorithm. We start with a reduction to graphs G with $\delta(G) \geq 2$.

▶ **Lemma 3.1** ($\star$). *Let G be a planar graph with embedding $\mathcal{E}$. There is a planar supergraph $G' \supseteq G$ with $\delta(G') \geq 2$ whose embedding $\mathcal{E}'$ extends $\mathcal{E}$, such that G has a 2-connected 3-augmentation extending $\mathcal{E}$ if and only if G' has one extending $\mathcal{E}'$.*

The proof is simple and just requires the following preprocessing of G: Replace all vertices $v \in V(G)$ with $\deg_G(v) \leq 1$ by copies of K_3 as shown in Fig. 4 (left/middle).

▶ **Lemma 3.2.** *Let G be a planar n-vertex graph with an embedding $\mathcal{E}$, $\delta(G) \geq 2$, and $\Delta(G) \leq 3$. Then we can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H of G extending $\mathcal{E}$, or conclude that none exists. If G is connected, then time $\mathcal{O}(n^2)$ suffices.*

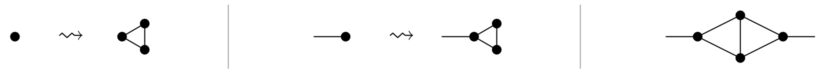

Figure 4 Left/Middle: Replacement rules. Right: Gadget to avoid parallel edges.

Proof. The proof is by a linear-time reduction to an equivalent instance A of the GENERALIZED FACTOR problem, such that A fulfills the necessary condition to apply an $\mathcal{O}(n^4)$-time algorithm by Cornuéjols [1, Section 3], or even an $\mathcal{O}(n^2)$-time algorithm by Sebő [7, Section 3].

We construct the 2-connected 3-augmentation H of G by adding new edges and vertices into the faces of $\mathcal{E}$. Therefore, the obtained embedding of H extends $\mathcal{E}$.

Some faces of $\mathcal{E}$ stand out, as these *must* contain new edges (and possibly vertices) to reach 2-connectedness. We call these the *connecting faces* F_c. Obviously, all faces incident to at least two connected components are connecting faces. Further, for each bridge e of G, the unique face f incident to both sides of e is a connecting face because the only way to add new connections between the components separated by e is through f. Recall that a 3-regular graph is 2-connected if and only if it is connected and bridgeless, so these are the only two types of connecting faces. All other faces are considered to be *normal* faces, denoted by F_n.

For a connecting face $f \in F_c$, let G_f be the subgraph of G on the vertices and edges incident to f, let B_f be its blocks (i.e., maximal 2-connected components or bridges), and let T_f be its block-cut-forest. We partition B_f into $S_f \cup I_f \cup L_f$:

$$S_f \coloneqq \{b \in B_f \mid b \text{ forms a trivial (i.e., single-vertex) tree in } T_f\} \qquad (\textit{singleton} \text{ blocks})$$
$$I_f \coloneqq \{b \in B_f \mid b \text{ is an inner vertex of a non-trivial tree in } T_r\} \qquad (\textit{inner} \text{ blocks})$$
$$L_f \coloneqq \{b \in B_f \mid b \text{ is a leaf in a non-trivial tree in } T_f\} \qquad (\textit{leaf} \text{ blocks})$$

The GENERALIZED FACTOR instance A is a bipartite graph with bipartition classes $\mathcal{V}$ and $\mathcal{F}$. Here, $\mathcal{V} \coloneqq \{v \in V \mid \deg_G(v) = 2\}$ contains all vertices of G not yet having degree 3. Similarly, vertices in $\mathcal{F}$ represent the faces of $\mathcal{E}$. Edges of a B-factor of A will determine the faces of $\mathcal{E}$ containing the new edges. In particular, $\mathcal{F}$ contains one vertex corresponding to each normal face in F_n of $\mathcal{E}$. Additional vertices in $\mathcal{F}$ are needed to handle the connecting faces. For each connecting face $f \in F_c$, we add all blocks in B_f as vertices to $\mathcal{F}$. (If there are two faces f, g in $\mathcal{E}$ such that B_f and B_g contain blocks corresponding to the same subgraph of G, then $\mathcal{F}$ contains two such vertices: one corresponding to the block in B_f, and another to the block in B_g.)

In A, each $x \in \mathcal{F}$ is incident to exactly the following $v \in \mathcal{V}$: If x is a normal face $f_n \in F_n$, then x is connected to all $v \in \mathcal{V}$ that are incident to f_n in $\mathcal{E}$. Otherwise, if x is a block $b \in B_f$ for some connecting face $f_c \in F_c$, then x is connected to all $v \in \mathcal{V}$ that are contained in b. See Fig. 5 for an example.

Lastly, we need to assign a set $B(x) \subseteq \{0, 1, \ldots, \deg_A(x)\}$ of possible degrees to each vertex $x \in (\mathcal{V} \cup \mathcal{F})$:

$$B(x) \coloneqq \begin{cases} \{1\}, & x \in \mathcal{V} \\ \{0, 2, 3, \ldots, \deg_A(x)\}, & x \in F_n \\ \{0, 1, 2, \ldots, \deg_A(x)\}, & x \in I_f \text{ for some connecting face } f \in F_c \\ \{1, 2, 3, \ldots, \deg_A(x)\}, & x \in L_f \text{ for some connecting face } f \in F_c \\ \{2, 3, 4, \ldots, \deg_A(x)\}, & x \in S_f \text{ for some connecting face } f \in F_c \end{cases}$$

The order and size of A are linear in n as every vertex $v \in V(G)$ is contained in at most

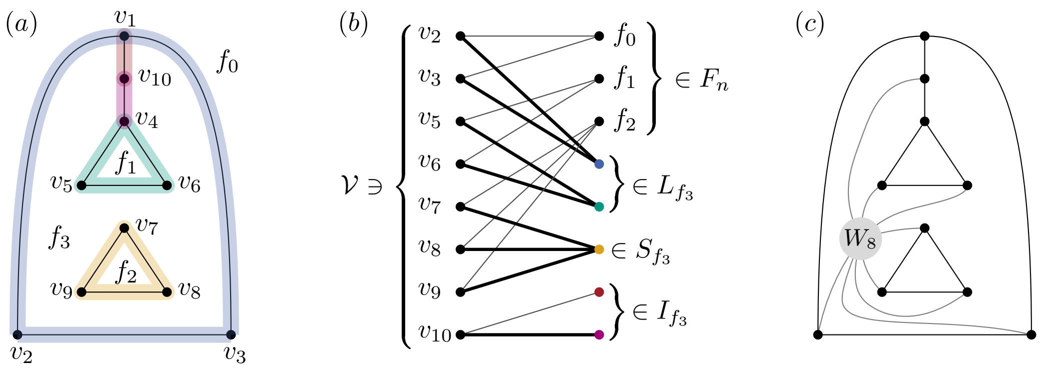

Figure 5 (a) A planar subcubic graph G. (b) Its corresponding Generalized Factor instance. Thick edges denote a possible solution. (c) A 2-connected 3-augmentation of G (with an indicated wheel-extension).

three distinct faces, and at most three blocks of G_f for any face f of $\mathcal{E}$. Moreover, A can be computed in linear time; see the full version [4, Claim 3.3] for a proof.

Also in the full version [4, Claims 3.4 and 3.5], we formally prove that A admits a B-factor if and only if G has a 2-connected 3-augmentation H. To convey an intuition, consider a B-factor A' of A. Every vertex $v \in \mathcal{V}$ is incident to exactly one edge $vx \in E(A')$. Here, x is either a face f, or there is a face f such that x is a block in B_f. In order to obtain H, we add a new half-edge incident to v into f. Now, for each face f of $\mathcal{E}$, we connect all half-edges within f to a new vertex v_f. Applying a wheel-extension (Obs. 2.1) to every vertex v_f of degree larger than 3, and replacing each vertex v_f of degree 2 by the gadget in Fig. 4 (right), yields a 2-connected 3-augmentation of G. For the other direction, observe that each degree-2-vertex of G is, in H, incident to a new edge inside a face f of $\mathcal{E}$. A B-factor A' of A is obtained by adding vx to $E(A')$, where $x \in \mathcal{F}$ is either f or a block in B_f (containing v).

It remains to argue that we can compute a B-factor of A efficiently. By inspecting the sets $B(x)$ for all $x \in (\mathcal{V} \cup \mathcal{F})$, we can see that none of them contains a gap of size 2 or greater. Therefore, we are in a special case of the Generalized Factor problem that can be solved, in $\mathcal{O}(n^4)$ time, by Cornuéjols' algorithm [1].

A closer inspection yields that only for $x \in S_f$ the sets $B(x)$ contain two forbidden degrees. (Note that $\deg_A(v) \leq 2$ for all $v \in \mathcal{V}$: If there is a face f such that v is contained in two blocks of G_f, then both edges incident to v are bridges; thus v is incident to no other face. Otherwise, this follows from $\deg_G(v) \leq 2$, i.e., v being incident to at most two faces.) Therefore, if $S_f = \emptyset$ for all connecting faces $f \in F_C$, then we can even apply the algorithm by Sebő, taking only $\mathcal{O}(n^2)$ time [7]. In particular, this is the case if G is connected. ◀

4 NP-Hardness for 3-Connected 3-Augmentations

In this section, we shall prove that deciding whether a given planar graph G admits a 3-connected 3-augmentation is NP-complete. In particular, we show that the problem remains NP-complete when restricted to connected graphs G. This implies the NP-completeness results represented in the fourth column of the table in Fig. 2.

Recall that an embedding of any 3-connected 3-augmentation H induces an embedding $\mathcal{E}$ of G, and for convenience, let us call the pair $(H, \mathcal{E})$ a *solution* for G. Let us also define a (≤ 2)-*subdivision* of a graph R to be the result of subdividing each edge in R with up to two vertices. Note that, if R is 2-connected, then so is every (≤ 2)-subdivision of R.

▶ **Lemma 4.1** (⋆). *Let G be a graph obtained from a (≤ 2)-subdivision R_2 of a 3-connected planar graph R by attaching a degree-1 vertex to each subdivision vertex. Then G admits a solution $(H, \mathcal{E})$ if and only if no face of $\mathcal{E}$ has exactly one or two incident degree-1 vertices.*

By Lem. 4.1, any graph G as described in the lemma admits a 3-connected 3-augmentation if and only if it admits an embedding $\mathcal{E}$ with no face incident to exactly one or two degree-1 vertices. Testing such graphs for such embeddings, however, turns out to be NP-complete.

▶ **Theorem 4.2** (⋆). *Deciding whether a given graph is a subgraph of a 3-regular 3-connected planar graph is* NP*-complete.*

Proof Idea. We reduce from the NP-complete problem PLANAR-MONOTONE-3SAT [2], where an instance is a monotone 3SAT-formula Ψ together with a planar embedding $\mathcal{E}_\Psi$ of its bipartite variable-clause incidence graph I_Ψ, such that **(1)** the x-axis contains each variable, **(2)** no edge crosses the x-axis, and **(3)** each clause above (respectively below) the x-axis is positive (respectively negative).

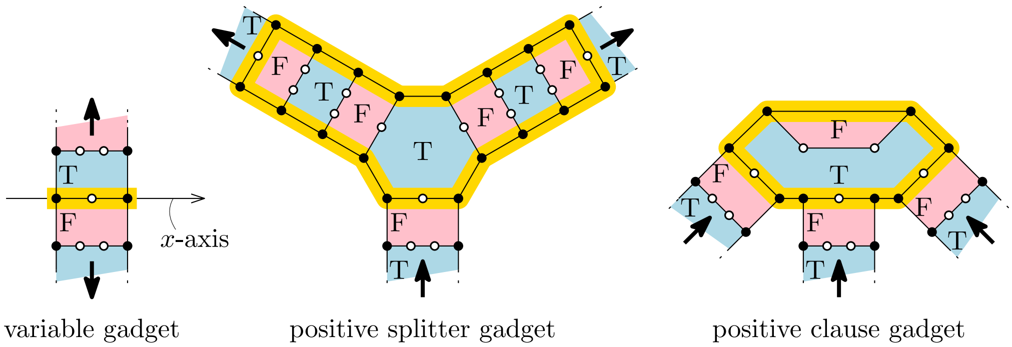

Figure 6 Gadgets used in the NP-hardness reduction.

We obtain a graph G_Ψ from the embedding $\mathcal{E}_\Psi$ using the gadgets in Fig. 6. Each variable gadget is the start of one upper and one lower corridor, each with alternating red (standing for FALSE) and blue (standing for TRUE) faces. Each positive splitter gadget splits one upper corridor into two, and each positive clause gadget is the end of one upper corridor of each appearing variable. Negative splitter and negative clause gadgets below the x-axis are symmetric, with red and blue swapped. See Fig. 7 for a full example.

With some extra vertices and edges, the result is a (≤ 2)-subdivision of a 3-connected 3-regular graph R. Attaching degree-1 vertices as in Lem. 4.1 gives G_Ψ, and we ask for an embedding $\mathcal{E}$ of G_Ψ with no face having one or two degree-1 vertices. The crux is, that except for the two highlighted faces in the clause gadgets, any pair of neighboring red and blue faces has in total at most five subdivision vertices. Thus, for each variable x, either only blue or only red faces in all splitters and corridors have degree-1 vertices. Setting x to TRUE in the former and to FALSE in the latter case, satisfies all clauses. See Fig. 8 for an example.

In fact, Ψ is satisfiable if and only if G_Ψ admits an embedding as required by Lem. 4.1. ◀

▶ Remark. The (≤ 2)-subdivision in the above reduction behaves quite similar to G_Ψ. The only problem is that a face f may have been chosen by exactly *two* incident degree-2 vertices to contain their third (new) edge without creating a 2-edge-cut; namely, with a direct edge. Thus, the above reduction also yields NP-completeness of recognizing *induced* subgraphs of 3-connected 3-regular planar graphs, even for 2-connected inputs with a unique embedding.

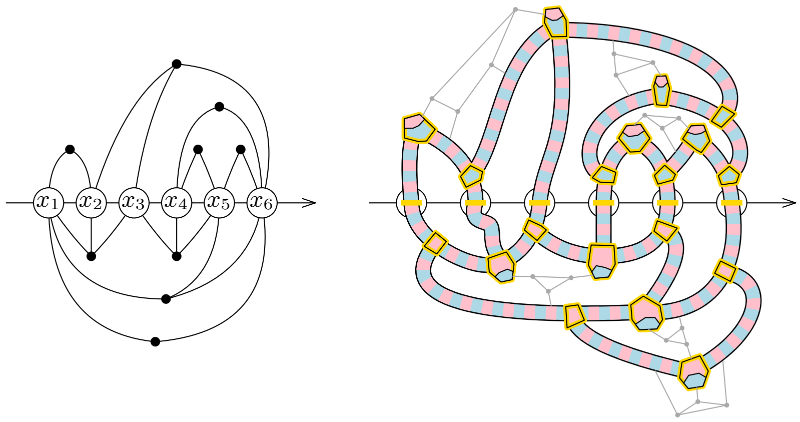

Figure 7 Illustration of a PLANAR-MONOTONE-3SAT embedding $\mathcal{E}_\Psi$ and a corresponding graph G_Ψ. Extra vertices and edges added for 3-connectivity of R are shown in gray.

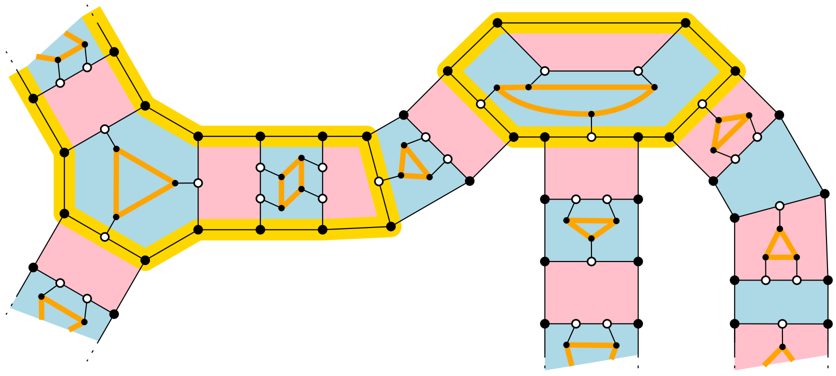

Figure 8 Part of the graph G_Ψ together with a 3-connected 3-augmentation in orange. This corresponds to a clause with two true variables (left with a splitter gadget, and middle) and one false variable (right).

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