Non-degenerate monochromatic triangles in the max-norm plane*

Alexander Natalchenko[1] and Arsenii Sagdeev[2]

1 **MIPT, Moscow, Russia**
natalchenko.ae@gmail.com
2 **Alfréd Rényi Institute of Mathematics, Budapest, Hungary**
sagdeevarsenii@gmail.com

Abstract

For all non-degenerate triangles T, we determine the minimum number of colors needed to color the plane such that no max-norm isometric copy of T is monochromatic.

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1 Introduction

Modern *combinatorial geometry* is both deep and wide with powerful tools galore. Nevertheless, some of its questions, that may look quite simple at first glance, repel all the attempts to answer them for several decades. Perhaps the most famous problem of this sort is due to Nelson who asked in 1950 to find the *chromatic number* $\chi(\mathbb{R}^2)$ of the Euclidean plane defined as the minimum number of colors needed to color the plane $\mathbb{R}^2$ such that no two points at unit Euclidean distance apart are of the same color. Despite the long history of research, it is only known that $5 \le \chi(\mathbb{R}^2) \le 7$, where the lower bound was obtained less than five years ago, see [3, 7]. For multidimensional versions of this problem, see [2].

In their celebrated trilogy [4, 5, 6], Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus laid the foundation of *Euclidean Ramsey theory* which deals with questions of the similar flavor but with more complex configurations forbidden to be monochromatic, see [9]. After a pair of points, the second simplest configuration is a (vertex set of a) triangle. We denote by $\chi(\mathbb{R}^2, T)$ the minimum number of colors needed to color the plane such that no *isometric* (i.e., translated and rotated) copy of a triangle T is monochromatic. Erdős et al. conjectured in [6, Conjecture 3] that $\chi(\mathbb{R}^2, T) \ge 3$ for all triangles T except for an equilateral one[1], i.e., that two colors are never enough. Despite the efforts of various researchers, this conjecture was verified only for a few special families of triangles, see [6, 15, 16]. From the other direction, it is easy to see that $\chi(\mathbb{R}^2, T) \le \chi(\mathbb{R}^2)$ and thus $\chi(\mathbb{R}^2, T) \le 7$ for all triangles T. Perhaps surprisingly, no better general upper bound is known, though Graham conjectured, see [9, Conjecture 11.1.3] and [17], that $\chi(\mathbb{R}^2, T) \le 3$ for all triangles T, which was confirmed for 'not very flat' triangles in [1]. Let us also mention that currently the best bounds for multidimensional variant of this problem were recently obtained in [14].

In this paper, we continue the line of research from [8, 11, 12, 13] and consider a max-norm counterpart of the aforementioned problem. To give a formal definition, let us recall some basic notions and facts first. The ℓ_∞*-distance* between $\mathbf{z}_1 = (x_1, y_1)$, $\mathbf{z}_2 = (x_2, y_2) \in \mathbb{R}^2$ is given by $\|\mathbf{z}_1 - \mathbf{z}_2\|_\infty = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. In contrast to the Euclidean case, it is easy

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[1] For an equilateral triangle $\triangle$, the same group of authors observed that $\chi(\mathbb{R}^2, \triangle) = 2$ and conjectured in [6, Conjecture 1] that the corresponding two-coloring is unique, which was later disproved in [10].

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to find the exact value of $\chi(\mathbb{R}^2_\infty)$ defined as the minimum number of colors needed to color the plane such that no two points at unit ℓ_∞-distance apart are of the same color: see the folklore proof that $\chi(\mathbb{R}^2_\infty) = 4$ in the left-hand side of Figure 1 or e.g. in [12, Section 2.1] for more details. A subset $T' \subset \mathbb{R}^2$ is called an *ℓ_∞-isometric copy of $T \subset \mathbb{R}^2$*, if there exists a bijection $f : T \to T'$ such that $\|\mathbf{z}_1 - \mathbf{z}_2\|_\infty = \|f(\mathbf{z}_1) - f(\mathbf{z}_2)\|_\infty$ for all $\mathbf{z}_1, \mathbf{z}_2 \in T$. Finally, we denote by $\chi(\mathbb{R}^2_\infty, T)$ the minimum number of colors needed to color the plane such that no ℓ_∞-isometric copy of T is monochromatic. As earlier, it is easy to see that $\chi(\mathbb{R}^2_\infty, T) \le \chi(\mathbb{R}^2_\infty)$ and thus $\chi(\mathbb{R}^2_\infty, T)$ is equal to either 2, or 3, or 4 for every triangle T. We will show that all three of these options indeed take place.

Note that the value of $\chi(\mathbb{R}^2_\infty, T)$ depends only on the side lengths of T and is independent of the particular position of T in the plane. For all positive $a \le b \le c$ satisfying the triangle inequality $c \le a + b$, let $T(a, b, c)$ be an arbitrary triple of points in the plane with pairwise ℓ_∞-distances between them of a, b, c. This triangle is *degenerate* if $c = a + b$, otherwise it is *non-degenerate*. Observe that if both fractions $\frac{a}{c}$ and $\frac{b}{c}$ are rational, then after a proper scaling, we can assume without loss of generality that a, b, c are coprime integers. For all such non-degenerate triangles T, our next result gives the exact value of $\chi(\mathbb{R}^2_\infty, T)$.

▶ **Theorem 1.1.** *Let $a \le b \le c$ be positive integers such that $c < a + b$ and $\gcd(a, b, c) = 1$. Put $T = T(a, b, c)$. If (1) $a + b + c$ is odd, or (2) a and b are odd, $c \ge a + b - \gcd(a, b)$, then $\chi(\mathbb{R}^2_\infty, T) = 2$. Otherwise, $\chi(\mathbb{R}^2_\infty, T) = 3$.*

Our next result covers the remaining case of 'irrational' non-degenerate triangles.

▶ **Theorem 1.2.** *Let $a \le b \le c$ be positive reals such that $c < a + b$ and $\frac{a}{c}$ or $\frac{b}{c}$ is irrational. Put $T = T(a, b, c)$. If $a = q_1\xi$, $b = q_2\eta$, $c = p_1\xi + p_2\eta$ for some odd integers p_1, p_2, q_1, q_2 and reals ξ, η such that $\frac{\xi}{\eta}$ is irrational, then $\chi(\mathbb{R}^2_\infty, T) = 3$. Otherwise, $\chi(\mathbb{R}^2_\infty, T) = 2$.*

For a degenerate triangle $T = T(a, b, a + b)$, several results follow from [8], where much more general problems were studied. First, observe that every five-point subset of a nine-element set $\{0, a, a + b\}^2 \subset \mathbb{R}^2$ contains an ℓ_∞-isometric copy of T and thus $\chi(\mathbb{R}^2_\infty, T) \ge 3$. If $\frac{a}{b}$ is irrational, the axiom of choice allows one to construct the corresponding three-coloring of the plane showing that this bound is tight, see [8, Section 5]. Otherwise, after a proper scaling, we can assume without loss of generality that a and b are coprime integers. In case $a \equiv b \pmod 3$, it is again not hard to construct a coloring of the plane matching the aforementioned lower bound, see the right-hand side of Figure 1 for an illustration and [8, Section 4] for a formal proof. In the remaining case $a \not\equiv b \pmod 3$, we conjecture that three colors are not enough, which we verified for $a + b \le 7$ by computer search.

▶ **Conjecture 1.3.** *If $a, b \in \mathbb{N}$ are such that $a \not\equiv b \pmod 3$, then $\chi\big(\mathbb{R}^2_\infty, T(a, b, a + b)\big) = 4$.*

In what follows, we refer to *ℓ_∞-distances* and *ℓ_∞-isometric copies* simply as *distances* and *copies*, respectively. Whenever we consider a two-coloring of the plane, we call these colors red and blue or 0 and 1 for clarity. We also assume that side lengths a, b, c of a triangle $T = T(a, b, c)$ are reals satisfying $a \le b \le c$ and $c < a + b$, i.e., that T is non-degenerate.

We structure the reminder of the paper as follows. In Section 2, we give a necessary condition for a triple of points to form a copy of T. In Section 3, we find some properties satisfied by every red-blue coloring of the plane containing no monochromatic copies of T, for one of which, namely for Lemma 3.3, we do not provide a proof in this version of the article. Though this proof is similar to others and based on an absolutely elementary idea (if two vertices of T are red, then the third one must be blue, and vice versa), its exact implementation is not that simple, and we have to omit it in order to meet the condition on manuscript length. Finally, in Section 4, we show that these properties are mutually exclusive if T satisfies neither (1) nor (2), which would complete the proof of Theorem 1.1. Note that Theorem 1.2 follows from these properties in a very similar manner, but we have to omit this proof too.

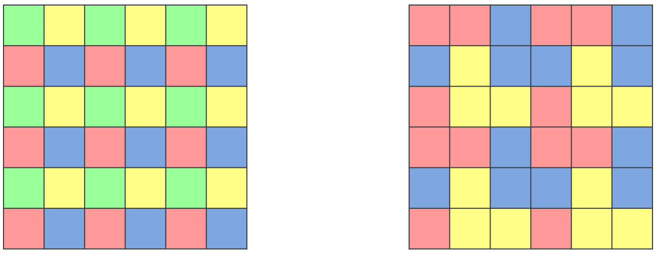

Figure 1 A 4-coloring of the plane with no monochromatic points at unit distance apart, and a 3-coloring of the plane with no monochromatic copies of degenerate triangles $T(a, b, a+b)$ for all coprime $a, b \in \mathbb{N}$ such that $a \equiv b \pmod 3$. All squares are unit. Each colored square includes only the bottom left vertex along with the left and the bottom sides from its boundary.

2 Copies of a non-degenerate triangle

▶ **Lemma 2.1.** *Let $\mathbf{z}_1 = (x_1, y_1), \mathbf{z}_2 = (x_2, y_2), \mathbf{z}_3 = (x_3, y_3) \in \mathbb{R}^2$ be a copy of T. Then at least one of the differences $|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|$ equals either a, or b, or c. Moreover, at least one of the differences $|x_1 + y_1 - x_2 - y_2|, |x_2 + y_2 - x_3 - y_3|, |x_3 + y_3 - x_1 - y_1|$ equals either $a + b - c$, or $c + a - b$, or $b + c - a$.*

Proof. The distance between two points is determined by the absolute value of the difference of their either x- or y-coordinates. Therefore, one of the axes, say x, determines at least two of the distances $\|\mathbf{z}_1 - \mathbf{z}_2\|_\infty, \|\mathbf{z}_2 - \mathbf{z}_3\|_\infty, \|\mathbf{z}_3 - \mathbf{z}_1\|_\infty$ by the pigeonhole principle. Assume that $\|\mathbf{z}_2 - \mathbf{z}_3\|_\infty = |x_2 - x_3| = a$, $\|\mathbf{z}_3 - \mathbf{z}_1\|_\infty = |x_3 - x_1| = b$. Observe that x_3 cannot lie between x_1 and x_2, since in that case we would get that $c = \|\mathbf{z}_1 - \mathbf{z}_2\|_\infty \geq |x_1 - x_2| = a + b$, a contradiction. Hence, let us assume that $x_1 = x_3 + b$, $x_2 = x_3 + a$. This implies that $|x_1 - x_2| = b - a < c = \|\mathbf{z}_1 - \mathbf{z}_2\|_\infty$ and thus $|y_1 - y_2| = c$ as desired. To prove the second half of the statement, note that if $y_1 = y_2 + c$, then $|x_1 + y_1 - x_2 - y_2| = b + c - a$, while if $y_1 = y_2 - c$, then $|x_1 + y_1 - x_2 - y_2| = c + a - b$. The same reasoning works in all the remaining cases, corresponding to possible permutations of axes, indexes, and side lengths. ◀

This simple statement immediately gives the following sufficient condition for a *horizontal coloring*, which is constant on every horizontal line $y = y_0$, or for a *diagonal coloring*, which is constant on every diagonal line $x + y = y_0$, to contain no monochromatic copies of T.

▶ **Corollary 2.2.** *The following two statements are valid:*

1. *Let $\bar{C}(\cdot)$ be a coloring of the line such that no two points at distance a, b, or c apart are monochromatic. Then the corresponding horizontal coloring of the plane, defined by the equation $C(x, y) = \bar{C}(y)$, contains no monochromatic copies of T.*
2. *Let $C'(\cdot)$ be a coloring of the line such that no two points at distance $a + b - c$, $c + a - b$, or $b + c - a$ apart are monochromatic. Then the corresponding diagonal coloring of the plane, defined by the equation $C(x, y) = C'(x + y)$, contains no monochromatic copy of T.*

3 Patterns in an arbitrary two-coloring

For this section, let us fix a red-blue coloring of the plane such that no copy of T is monochromatic. We call a vector (x_0, y_0) its *period* (resp. *anti-period*) if for all $x, y \in \mathbb{R}$, the colors of two points (x, y) and $(x + x_0, y + y_0)$ are the same (resp. distinct). It is clear that the addition of these vectors resembles the multiplication of signs: the sum of two periods or two anti-periods is a period, while the sum of an anti-period and a period is an anti-period.

▶ **Lemma 3.1.** *If one of the four vectors $(\pm b, a - c)$, $(a - c, \pm b)$ is not an anti-period, then there exists a monochromatic axis-parallel segment of length $c + a - b$. Similar statements are valid for other permutations of the side length.*

Proof. If $(a - c, b)$ is not an anti-period, then there exist $x_1, y_1 \in \mathbb{R}$ such that the two points (x_1, y_1) and $(x_1 + c - a, y_1 - b)$ are of the same color, say, both are red. It is easy to check that this pair together with an arbitrary point from the segment $\{(x_1+c, y) : y_1-c \leq y \leq y_1+a-b\}$ form a copy of T. Since there are no red copies of T, we conclude that this vertical segment of length $c+a-b$ is entirely blue, as desired. Similar arguments work for the other cases. ◀

▶ **Corollary 3.2.** *If no axis-parallel segment of length $c + a - b$ is monochromatic, then all eight vectors $(\pm a, c - b)$, $(c - b, \pm a)$, $(\pm b, a - c)$, $(a - c, \pm b)$ are anti-periods, and all six vectors $(2a, 0)$, $(2b, 0)$, $(2c, 0)$, $(0, 2a)$, $(0, 2b)$, $(0, 2c)$ are periods. Moreover, if there also exist $n, m, k \in \mathbb{Z}$ such that $0 < 2an + 2bm + 2ck \leq a + b - c$, then four vectors $(\pm c, b - a)$, $(b - a, \pm c)$ are anti-periods as well.*

▶ **Lemma 3.3.** *If $a < b$ and there exists a monochromatic axis-parallel segment of length $c + a - b$, then there also exists a monochromatic axis-parallel line.*

▶ **Lemma 3.4.** *If the horizontal line $y = 0$ is red, then both lines $y = a$ and $y = b$ are blue. Moreover, if there also exist $n, m \in \mathbb{Z}$ such that $n + m$ is even and $c - b \leq an + bm \leq a$, then the line $y = c$ is blue as well.*

Proof. Each point (x_0, a) on the line $y = a$ forms a copy of T together with two red points $(x_0 - c, 0)$ and $(x_0 + b - c, 0)$. Thus the line $y = a$ is entirely blue. Similarly, each point (x_0, b) on the line $y = b$ forms a copy of T together with two red points $(x_0 - c, 0)$ and $(x_0 + a - c, 0)$. Thus the line $y = a$ is also entirely blue.

To prove the second half of the statement, observe that the first half implies that the color of the horizontal line $y = an + bm$ is determined by the parity of $n + m$. In particular, if $n + m$ is even, then this line is red. Now it is easy to see that each point (x_0, c) on the line $y = c$ forms a copy of T together with two red points $(x_0 - b, an + bm)$ and $(x_0 + a - b, 0)$. Hence, the line $y = c$ is also entirely blue, as desired. ◀

4 Proof of Theorem 1.1

First of all, note that the upper bound $\chi(\mathbb{R}^2_\infty, T) \leq 3$ is immediate from the first half of Corollary 2.2 and the following special case of [18, Corrolary 2.1] due to Zhu.

▶ **Theorem 4.1.** *Let $a \leq b \leq c$ be positive integers such that $c < a + b$ and $\gcd(a, b, c) = 1$. Then there exists a three-coloring of the line such that no two points at distance a, b, or c apart are monochromatic.*

Therefore, to complete the proof, we only need to show that there exists a two-coloring of the plane such that no copy of T is monochromatic if and only if either (1) or (2) holds. We begin by showing the sufficiency of these conditions using the following two explicit colorings.

▶ **Lemma 4.2.** *If* (1) *holds, then no copy of* T *is monochromatic under a diagonal two-coloring of the plane defined by* $C(x, y) = \lfloor x + y \rfloor \bmod 2$.

Proof. It is clear that $\lfloor x_1 \rfloor \not\equiv \lfloor x_2 \rfloor \pmod 2$ whenever $x_1, x_2 \in \mathbb{R}$ are at odd distance apart. Since all three values $a + b - c$, $c + a - b$, and $b + c - a$ are odd by (1), the second half of Corollary 2.2 completes the proof. ◀

▶ **Lemma 4.3.** *If* (2) *holds, then no copy of* T *is monochromatic under a horizontal two-coloring of the plane defined by* $C(x, y) = \lfloor y/d \rfloor \bmod 2$*, where* $d = \gcd(a, b)$.

Proof. Assume the contrary, namely that for some $\mathbf{z}_1 = (x_1, y_1), \mathbf{z}_2 = (x_2, y_2), \mathbf{z}_3 = (x_3, y_3) \in \mathbb{R}^2$ that form a copy of T, the values $\lfloor y_1/d \rfloor, \lfloor y_2/d \rfloor, \lfloor y_3/d \rfloor$ are of the same parity, say all three are even. By Lemma 2.1, we can assume without loss of generality that $y_1 - y_2$ equals either a, or b, or c. Note that the former two cases immediately yield a contradiction since both fractions a/d and b/d are odd by (2). So in what follows we suppose that $y_1 - y_2 = c$. In particular, this implies that $\|\mathbf{z}_1 - \mathbf{z}_2\|_\infty = c$, and thus one of the distances $\|\mathbf{z}_2 - \mathbf{z}_3\|_\infty$, $\|\mathbf{z}_3 - \mathbf{z}_1\|_\infty$ equals a, while the other one equals b.

It is easy to check that if $\|\mathbf{z}_2 - \mathbf{z}_3\|_\infty = a$, $\|\mathbf{z}_3 - \mathbf{z}_1\|_\infty = b$, then $y_1 - b \leq y_3 \leq y_2 + a$. Observe that both $\lfloor (y_1 - b)/d \rfloor = \lfloor y_1/d \rfloor - b/d$ and $\lfloor (y_2 + a)/d \rfloor = \lfloor y_2/d \rfloor + a/d$ are odd. Moreover, the length of this segment is equal to $a + b - c$ which does not exceed d by (2). Therefore, this segment is too short for the parity of $\lfloor y/d \rfloor$ to change from odd to even and back again as y ranges between the endpoints. Hence, $\lfloor y_3/d \rfloor$ is also odd, and we see the contradiction. In the remaining case when $\|\mathbf{z}_2 - \mathbf{z}_3\|_\infty = b$, $\|\mathbf{z}_3 - \mathbf{z}_1\|_\infty = a$, we have $y_1 - a \leq y_3 \leq y_2 + b$, and the similar argument completes the proof. ◀

To prove the second half of the theorem, observe that if neither (1) nor (2) holds, then either
(3) a and b are of different parity, c is odd, or
(4) a and b are odd, c is even, and $c < a + b - \gcd(a, b)$.
So it remains only to show that in each of these two cases, there are no two-colorings of the plane with no monochromatic copies of T. Let us assume the contrary and fix an arbitrary such red-blue coloring.

First, we suppose that no axis-parallel segment of length $c + a - b$ is monochromatic. On the one hand, the first half of Corollary 3.2 implies that all six vectors $(2a, 0)$, $(2b, 0)$, $(2c, 0)$, $(0, 2a)$, $(0, 2b)$, $(0, 2c)$ are periods, and so are all their linear combinations. Since $\gcd(2a, 2b, 2c) = 2$, we conclude that both $(2, 0)$ and $(0, 2)$ are periods. Hence, every vector such that both its coordinates are even integers is also a period. On the other hand, note that $a + b - c$ is a positive even integer, and thus $a + b - c \geq 2 = \gcd(2a, 2b, 2c)$. Therefore, we can also apply the second half of Corollary 3.2 in our case to find twelve anti-periods in total including $(a, c - b)$, $(b, a - c)$ and $(c, b - a)$. However, both coordinates of one of these three vectors are even integers, and so this vector should be a period instead, a contradiction.

Second, suppose that there exists a monochromatic axis-parallel segment of length $c + a - b$. Besides, note that each of the conditions (3) and (4) yields $a < b$. So we can apply Lemma 3.3 to find a monochromatic axis-parallel line. Without loss of generality, assume that this line is given by the equation $y = 0$ and that it is entirely red. Now it is easy to deduce from the first half of Lemma 3.4 that for all $i, j \in \mathbb{Z}$ such that $i + j$ is odd, the horizontal line $y = ai + bj$ is blue. If (3) holds, then we obtain a contradiction by taking $i = b$, $j = -a$.

If (4) holds, we use a slightly more complex argument. Observe that there exist $n, m \in \mathbb{Z}$ such that $an + bm = a - \gcd(a, b)$. Since both a and b are odd, we conclude that $n + m$ is even. Moreover, the inequality $c - b \leq a - \gcd(a, b) = an + bm$ is immediate from (4). Therefore, we can aslo apply the second half of Lemma 3.4 in our case to deduce that the horizontal line $y = ai + cj$ is blue for all $i, j \in \mathbb{Z}$ such that $i + j$ is odd. Finally, we obtain the desired contradiction by taking $i = c$, $j = -a$.

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