Exact solutions to the Weighted Region Problem*

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Abstract

In this paper, we consider the Weighted Region Problem. In the Weighted Region Problem, the length of a path is defined as the sum of the weights of the subpaths within each region, where the weight of a subpath is its Euclidean length multiplied by a weight $\alpha \geq 0$ depending on the region. We study a restricted version of the problem of determining shortest paths through a single weighted rectangular region. We prove that even this very restricted version of the problem is unsolvable within the Algebraic Computation Model over the Rational Numbers (ACM$\mathbb{Q}$). On the positive side, we provide the equations for the shortest paths that are computable within the ACM$\mathbb{Q}$. Additionally, we provide equations for the bisectors between regions of the Shortest Path Map for a source point on the boundary of (or inside) the rectangular region.

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1 Introduction

The Weighted Region Problem (WRP) [10] is a well-known geometric problem that, despite having been studied extensively, is still far from being well understood. Consider a subdivision of the plane into (usually polygonal) regions. Each region R_i has a weight $\alpha_i \geq 0$, representing the cost per unit distance of traveling in that region. Thus, a segment σ, of length $|\sigma|$, between two points in the same region has weighted length $\alpha_i|\sigma|$ when traversing the interior of R_i, or $\min\{\alpha_i, \alpha_j\}|\sigma|$ if it goes along the edge between R_i and R_j. Then, the weighted length of a path $\pi(s,t)$ through a subdivision is the sum of the weighted lengths of its subpaths through each face or edge. The resulting metric is called the *Weighted Region Metric*. The WRP entails computing a shortest path between two given points s and t under this metric. We denote the weighted length of $\pi(s,t)$ by $d(s,t)$. Figure 1 shows how the shape of a shortest path changes as the weight of one region varies.

Existing algorithms for the WRP—in its general formulation—are approximate. Since the seminal work by Mitchell and Papadimitriou [10], with the first $(1+\varepsilon)$-approximation, several algorithms have been proposed, with improvements on running times, but always keeping some dependency on the vertex coordinates sizes and weight ranges. These methods are usually based on the continuous Dijkstra's algorithm (e.g., [10]), or on adding Steiner points (e.g., see [1, 2, 3, 4, 13]). However, rather recently it has been proved that computing an exact shortest path between two points using the weighted region metric, even if there are

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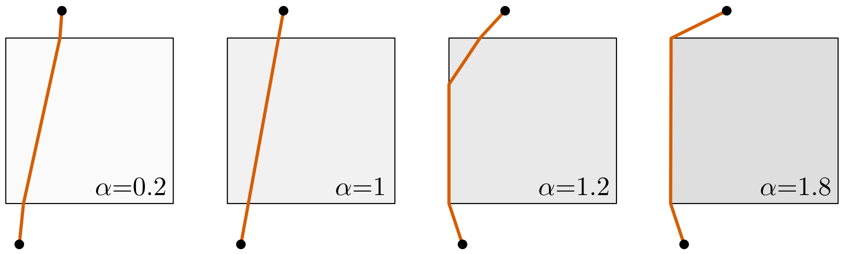

Figure 1 Examples of shortest paths between two points—shown in orange—for two weighted regions. The unbounded region has weight 1, the squares have varying weight α.

only three different weights, is an unsolvable problem in the Algebraic Computation Model over the Rational Numbers (ACM$\mathbb{Q}$) [5]. In the ACM$\mathbb{Q}$ one can compute exactly any number that can be obtained from rational numbers by applying a finite number of operations from $+, -, \times, \div$, and $\sqrt[k]{}$, for any integer $k \geq 2$. This provides a theoretical explanation for the lack of exact algorithms for the WRP, and justifies the study of approximation methods.

This also raises the question of which are the special cases for which the WRP can be solved exactly. Two natural ways to restrict the problem are by limiting the possible weights, and by restricting the shape of the regions. For example, computing a shortest path among polygonal obstacles can be seen as a variant of the WRP with weights in $\{1, \infty\}$. The case for weights in $\{0, 1, \infty\}$ can be solved in $O(n^2)$ time [6]. Other variants that can be solved exactly correspond to regions shaped as regular k-gons with weight ≥ 2 (since they can be considered as obstacles), or regions with two weights $\{1, \alpha\}$ consisting of parallel strips [11].

Our results. In light of the fact that the WRP is unsolvable in the ACM$\mathbb{Q}$ for three different weights, in this work we study the case of two arbitrary weights, that is, weights in $\{1, \alpha\}$. This case is particularly interesting, since an algorithm for weights $\{1, \alpha\}$ can be transformed into one for weights in $\{0, 1, \alpha, \infty\}$ [8]. However, the variant with weights $\{1, \alpha\}$ was conjectured to be as hard as the general WRP problem, see the first open problem in [6, Section 7]. (The results in [5] do not directly apply to weights $\{0, 1, \alpha, \infty\}$.)

This paper is organized as follows. First we present some preliminaries in Section 2. In Section 3 we consider two weights and one rectangular region R, with the source point s on its boundary or inside. For this setting, we figure out all types of possible optimal paths and give exact formulas to compute their lengths. In Section 3.3 we focus on the case where s is outside of R, and prove that in this case the WRP with weights $\{1, \alpha\}$ is already unsolvable in the ACM$\mathbb{Q}$. In Section 4 we explore the computation of the shortest path map for s. We finish with some conclusions in Section 5. Omitted proofs are in the full version of the paper.

2 Shortest paths and their properties

In this section we briefly review some key properties of shortest paths in weighted regions.

First, shortest paths in the weighted region problem will always be piecewise linear, see [10, Lemma 3.1]. Second, it is known that shortest paths must obey Snell's law of refraction. So we can think of a shortest path as a ray of light. We define the *angle of incidence* θ as the minimum angle between the incoming ray and the vector perpendicular to the region boundary. The *angle of refraction* θ' is defined as the minimum angle between the outgoing ray and the normal (see Figure 3). Snell's law states that whenever the ray goes from one region R_i to another region R_j, then $\alpha_i \sin\theta = \alpha_j \sin\theta'$. In addition, whenever

$\alpha_i > \alpha_j$, the angle θ_c at which $\frac{\alpha_i}{\alpha_j} \sin\theta_c = 1$ is called the *critical angle*. A ray that hits an edge at an angle of incidence greater than θ_c, will be totally reflected from the point at which it hits the boundary. In our problem, a shortest path will never be incident to an edge at an angle greater than θ_c. Finally, if the space only contains orthoconvex regions[1] with weight at least $\sqrt{2}$, they can be simply considered obstacles [11]. Thus, since we focus on a rectangular region R, we assume that its weight is $0 < \alpha < \sqrt{2}$. However, first we provide some general properties of shortest paths for arbitrary weighted regions that are interesting on their own.

▶ **Lemma 2.1.** *Let $\mathcal{S}$ be a polygonal subdivision for which each region has a weight in the set $\{1, \alpha\}$, with $\alpha > 0$. A shortest path $\pi(s,t)$ visits any edge of the subdivision at most once.*

Proof. Assume, for the sake of contradiction, that $\pi(s,t)$ intersects an edge e in at least two disjoint intervals I_1 and I_3. Moreover, let $p_1 \in I_1$ and $p_3 \in I_3$ be points for which the subpath $\pi(p_1,p_3) \subseteq \pi(s,t)$ does not intersect e in any points other than p_1 and p_3. Let p_2 be a point on $\pi(p_1,p_3)$ between p_1 and p_3, which thus does not lie on e. Now observe that there exists a path $\overline{p_1p_3}$ from p_1 to p_3 of length $\min\{1,\alpha\}|\overline{p_1p_3}|$. Since p_2 does not lie on $\overline{p_1p_3}$, it follows by the triangle inequality that the length of $\pi(p_1,p_3)$ is strictly larger than $\min\{1,\alpha\}|\overline{p_1p_3}|$. Hence, $\pi(s,t)$ is not a shortest path, and we obtain a contradiction. ◀

▶ **Corollary 2.2.** *Let $\mathcal{S}$ be a polygonal subdivision with n vertices, such that the weight of each region is within the set $\{1, \alpha\}$, with $\alpha > 0$. Any shortest path $\pi(s,t)$ is a polygonal chain with at most $O(n)$ vertices.*

Proof. Any shortest path is a polygonal chain whose interior vertices all lie on edges of $\mathcal{S}$, see [10, Proposition 3.8]. By Lemma 2.1, each edge contributes with at most two vertices. ◀

We observe that if the regions use only one of two weights $\{1, \alpha\}$, Corollary 2.2 implies that the time complexity of the algorithm proposed by Mitchell and Papadimitriou [10] can be improved by a quartic factor to $O(n^4 L)$, where L is the precision of the instance.

3 Computing a shortest path

We now consider the problem of computing a shortest path $\pi(s,t)$ from s to t when the region R is an axis-aligned rectangle of weight α. The exact shape of $\pi(s,t)$ depends on the position of s and t with respect to R, and on the value of α.

In Sections 3.1 and 3.2 we consider the case that s lies on the boundary or inside of R, respectively. We categorize the various types of shortest paths, and show that we can compute the shortest path of each type, and thus we can compute $\pi(s,t)$. In Section 3.3, we consider the case that s and t lie outside R. In this case $\pi(s,t)$ may have two vertices on the boundary of R. We show that the coordinates of these vertices cannot be computed exactly within the ACM$\mathbb{Q}$.

3.1 The source point s lies on the boundary of R

Throughout this section we consider the case where s is restricted to the boundary of R, a rectangle of unit height with top-left corner at $(0,0)$. Let $s = (s_x, 0)$, $s_x > 0$, be a point on the top side of R, see Figure 2. In addition, we assume that t is to the left of the line through s perpendicular to the top side of R. The other cases are symmetric.

[1] A region is orthoconvex if its intersection with every horizontal and vertical line is connected or empty [12].

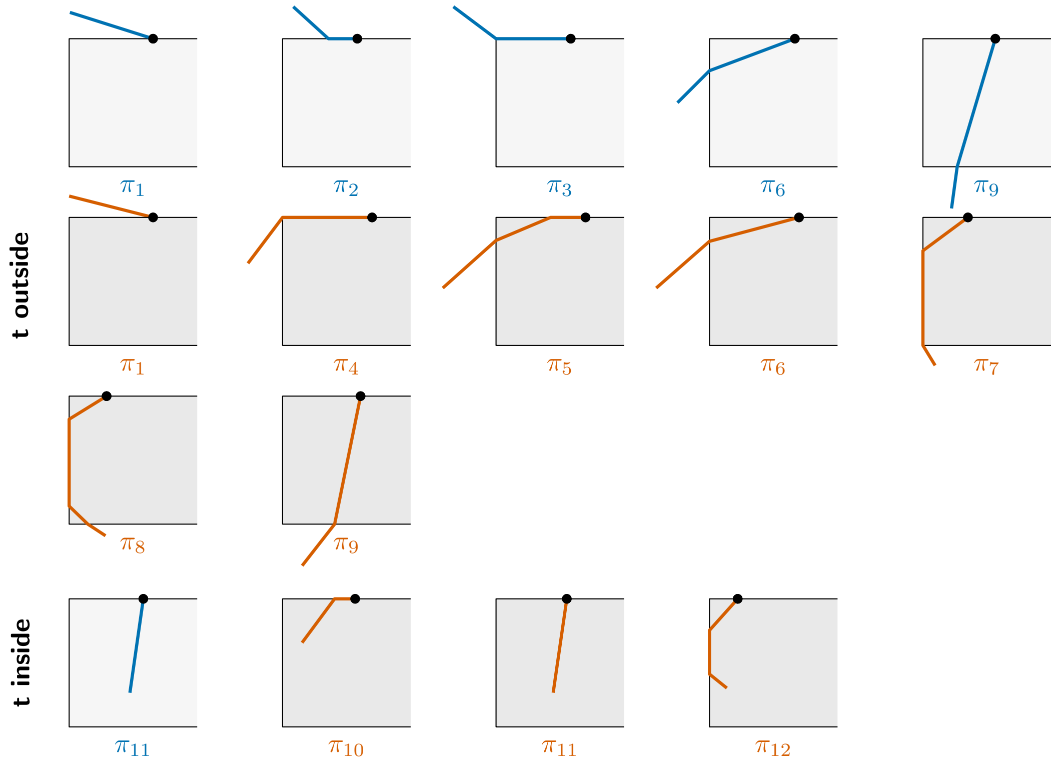

Figure 2 Path types for s on the boundary of R of weight $\alpha < 1$ (blue) and $1 < \alpha < \sqrt{2}$ (orange).

Shortest path types. Lemma 2.1 implies that in this setting, there are only $O(1)$ combinatorial types of paths that we have to consider. More precisely, we have that:

▶ **Observation 3.1.** *Let s be a point on the top boundary of a rectangle R with weight $0 < \alpha < \sqrt{2}$. There are 12 types of shortest paths $\pi_i(s,t)$, shown in Figure 2, up to symmetries.*

Length of $\pi_i(s,t)$. When s is on the boundary of R, there is at most one vertex of $\pi_i(s,t)$ without the critical angle property. This allows us to compute the exact coordinates of the vertices of $\pi_i(s,t)$. Theorem 3.2 gives the length $d_i(s,t)$ of the path $\pi_i(s,t)$. The proofs of the equations, which are based on Snell's law of refraction, are deferred to the full version.

▶ **Theorem 3.2.** *Let $s = (s_x, 0)$ be a point on the boundary of R with weight $0 < \alpha < \sqrt{2}$. A shortest path $\pi(s,t) = \pi_i(s,t)$ from s to a point $t = (t_x, t_y)$ and its length can be computed in*

$O(1)$ time in the ACM$\mathbb{Q}$. In particular, the length $d(s,t) = d_i(s,t)$ is given by

$$d_i(s,t) = \begin{cases} \sqrt{(s_x - t_x)^2 + t_y^2} & \text{if } i = 1 \\ \alpha(s_x - t_x) + \sqrt{1-\alpha^2}t_y & \text{if } i = 2 \\ \alpha s_x + \sqrt{t_x^2 + t_y^2} & \text{if } i = 3 \\ s_x + \sqrt{t_x^2 + t_y^2} & \text{if } i = 4 \\ s_x - \sqrt{2-\alpha^2}t_x - \sqrt{\alpha^2 - 1}t_y & \text{if } i = 5 \\ \alpha\sqrt{s_x^2 + y^2} + \sqrt{t_x^2 + (t_y - y)^2} & \text{if } i = 6 \\ \sqrt{\alpha^2 - 1}s_x + 1 + \sqrt{t_x^2 + (t_y + 1)^2} & \text{if } i = 7 \\ \sqrt{\alpha^2 - 1}(s_x + t_x) - \sqrt{2 - \alpha^2}(1 + t_y) + 1 & \text{if } i = 8 \\ \alpha\sqrt{(s_x - x)^2 + 1} + \sqrt{(t_x - x)^2 + (t_y + 1)^2} & \text{if } i = 9 \end{cases} \quad \left(\begin{array}{l}\textit{and thus } t \textit{ lies} \\ \textit{outside } R\end{array}\right), \textit{ and}$$

$$d_i(s,t) = \begin{cases} s_x - t_x - \sqrt{\alpha^2 - 1}t_y & \text{if } i = 10 \\ \alpha\sqrt{(s_x - t_x)^2 + t_y^2} & \text{if } i = 11 \\ \sqrt{\alpha^2 - 1}(s_x + t_x) - t_y & \text{if } i = 12 \end{cases} \quad \left(\begin{array}{l}\textit{and thus } t \textit{ lies} \\ \textit{inside } R\end{array}\right),$$

in which x is the unique real solution in the interval (t_x, s_x) to the equation

$$\begin{aligned} &\beta x^4 - 2\beta(t_x + s_x)x^3 + [\beta(s_x^2 + t_x^2 + 4s_xt_x) + \alpha^2(1 + t_y)^2 - 1]x^2 \\ &- 2[\beta(t_xs_x^2 + t_x^2s_x) + \alpha^2(1 + t_y)^2s_x - t_x]x + \beta t_x^2s_x^2 + \alpha^2(1 + t_y)^2s_x^2 - t_x^2 = 0, \end{aligned} \qquad (1)$$

where $\beta = \alpha^2 - 1$, and y is the unique real solution in the interval $(t_y, 0)$ to the equation

$$\beta y^4 - 2t_y\beta y^3 + [\alpha^2t_x^2 + \beta t_y^2 - s_x^2]y^2 + 2s_x^2t_yy - s_x^2t_y^2 = 0.$$

3.2 The source point s lies inside R

We now consider the case where s is restricted to the interior of the rectangle R.

▶ **Observation 3.3.** *Let s be a point in a rectangle R with weight $0 < \alpha < \sqrt{2}$. There are 6 types of shortest paths, up to symmetries, namely $\pi_i(s,t)$, for $i \in \{6, 7, 8, 9, 11, 12\}$.*

The types of shortest paths are similar to the ones defined in Observation 3.1, see the paths in Figure 2 where the top side of R or the region above R is not intersected. As in Theorem 3.2, we can thus compute the (length of) a shortest path (of each type) exactly, albeit that the expressions for the length are dependent on the location of s in R. Theorem 3.2 gives exact lengths for all path types when R has height > 1 and s is at distance exactly 1 from the bottom boundary of R.

3.3 The source point s lies outside of R

When both the source and the target point are outside of R, the shortest path can again be of many different types. In particular, the types in Figure 2 can be generalized to this setting. There are two special cases where the shortest path bends *twice* at an angle that is not the critical angle: it can bend on two opposite sides of the rectangle, or on two incident sides. In the first case, the angles at both vertices of $\pi(s,t)$ are equal, and the shortest path can be computed exactly [11]. For the second case, we show that it is not possible to compute the coordinates of the vertices of $\pi(s,t)$ exactly in the ACM$\mathbb{Q}$. Hence, the WRP limited to two weights $\{1, \alpha\}$ is not solvable within the ACM$\mathbb{Q}$. Note that this path type can occur in an even simpler setting, where R is a single quadrant instead of a rectangle.

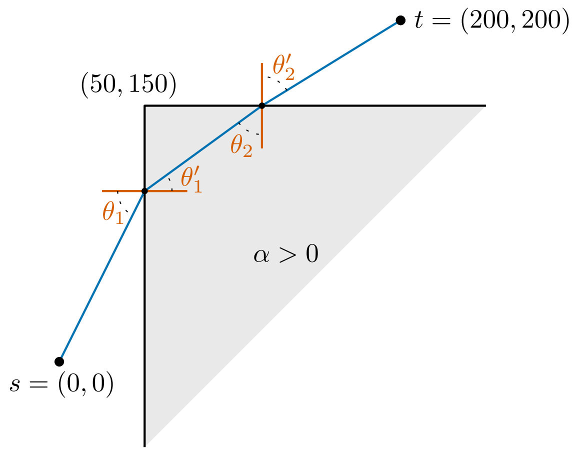

Figure 3 A shortest path from s to t that bends twice under different angles.

▶ **Theorem 3.4.** *The weighted region problem with weights in the set* $\{1, \alpha\}$*, with* $0 < \alpha < \sqrt{2}$*, cannot be solved exactly within the ACM*$\mathbb{Q}$*, even if* R *is a single quadrant.*

Proof sketch. We obtain this result by following the approach of De Carufel et al. [5] to show that the polynomial that represents a solution to the WRP in this situation is not solvable within the ACM$\mathbb{Q}$. We consider the situation in Figure 3. By applying Snell's law on both vertices of $\pi(s,t)$, and several trigonometric identities, we find the following expression for $u = \sin\theta_1$, where θ_1 is the angle at which the path leaves s, with respect to the x-axis:

$$\sqrt{\alpha^2 - u^2}\left(\frac{3}{u} - \frac{1}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-\alpha^2+u^2}}\right) = 3.$$

Note that a solution to this equation would give us the shortest path, as we can use Snell's law to find the other angles θ_1' and θ_2'. By squaring appropriately, this equation can be transformed into a degree 11 polynomial $p(u)$. Finally, we show that this polynomial is unsolvable within the ACM$\mathbb{Q}$ by applying a general lemma on the unsolvability of polynomials. ◀

4 Computing a Shortest Path Map

To find a shortest path from a source point s to *all* points at once, one can build a *Shortest Path Map* (*SPM*), see e.g., [7, 9, 10]. A *SPM* is a subdivision of the space for a given source s, where for each cell the paths $\pi(s,t)$, with t in the cell, have the same type. To compute the *SPM*, we consider computing the bisector $b_{i,j} = \{q \mid q \in \mathbb{R}^2 \wedge d_i(s,q) = d_j(s,q)\}$ for all relevant pairs of shortest path types π_i, π_j, i.e., pairs for which $b_{i,j}$ appears in the shortest path map. As before, we consider the setting where R is a rectangular region. In Section 4.1, we first consider the case when s lies on the boundary of R. In Section 4.2, we do the same for the case s lies inside R. The case that s lies outside R is not interesting, as we cannot even compute exactly a single shortest path in that case.

4.1 The source point s lies on the boundary of R

The *SPM* is given by the boundary of R and several bisector curves, expressed as points $(x, b_{i,j}(x))$. If $\alpha < 1$, these curves all lie outside R (the interior of R is a single region in the *SPM*). Bisectors involving $\pi_9(s,t)$ are of a much more complicated form, as might be expected from the implicit representation used for $d_9(s,t)$ in Theorem 3.2. Therefore, Lemma 4.1 gives the bisector curves, excluding the ones related to $\pi_9(s,t)$. The proofs are deferred to the full version.

▶ **Lemma 4.1.** *The SPM for a point $s = (s_x, 0)$ on the boundary of R is defined by:*

$$b_{i,j}(x) = \begin{cases} \frac{\sqrt{1-\alpha^2}}{\alpha}(s_x - x) & \text{if } i=1, j=2 \\ -\frac{\sqrt{1-\alpha^2}}{\alpha}x & \text{if } i=2, j=3 \\ 0 & \text{if } i=3, j=6 \end{cases} \quad (\text{ when } \alpha < 1\,), \text{ and}$$

$$b_{i,j}(x) = \begin{cases} 0 & \text{if } i=1, j=4 \\ \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x & \text{if } i=4, j=5 \\ \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x - \sqrt{\alpha^2-1}s_x & \text{if } i=5, j=6 \\ x=0 & \text{if } i=6, j=7 \\ -1 - \frac{\sqrt{2-\alpha^2}}{\sqrt{\alpha^2-1}}x & \text{if } i=7, j=8 \\ -\sqrt{\alpha^2-1}(s_x - x) & \text{if } i=10, j=11 \\ -\frac{(s_x+x)+2\alpha\sqrt{s_x x}}{\sqrt{\alpha^2-1}} & \text{if } i=11, j=12 \end{cases} \quad (\text{ when } 1 < \alpha < \sqrt{2}\,).$$

We conjecture the following on the bisectors involving $\pi_9(s,t)$.

▶ **Conjecture.** *No point on $b_{i,9} \setminus R$, $i \in \{4, \ldots, 8\}$, can be computed exactly within ACM$\mathbb{Q}$.*

We tried to prove this conjecture by taking a similar approach as in Theorem 3.4. However, the solution to Equation (1) already seems to be of high degree. We therefore did not manage to formulate a point on the bisector as a polynomial equation (not containing roots).

Note that in the more restrictive case where R is a single quadrant and s lies on the boundary, the only types of shortest paths that exist are $\pi_i(s,t)$, for $i \in \{1,2,3,4,5,6,10,11,12\}$. Thus, we can compute the *SPM* in the ACM$\mathbb{Q}$ (the bisectors are given by the equations in Lemma 4.1).

4.2 The source point s lies inside R

In this case we have shortest paths of type $\pi_i(s,t)$, for $i \in \{6,7,8,9,11,12\}$. Hence, the equations of the bisectors of the *SPM* are given by the sides of R, and bisector $b_{6,9}$ if $\alpha < 1$, and bisectors $b_{6,7}, b_{7,8}, b_{6,9}, b_{7,9}, b_{8,9}$ and $b_{11,12}$ if $1 < \alpha < \sqrt{2}$. See Lemma 4.1.

5 Conclusion

We analyzed the WRP when there is only one weighted rectangle R, and showed how to obtain the exact shortest path $\pi(s,t)$ and its length when s lies in or on R. When both s and t lie outside R the exact solution is unsolvable in ACM$\mathbb{Q}$. We obtain similar results in the case where R is a single quadrant. For future work, it would be interesting to analyze if or how we can generalize this to other convex shapes.

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