The k-Transmitter Watchman Route Problem is NP-Hard Even in Histograms and Star-Shaped Polygons*

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Abstract

A k-transmitter g in a polygon P, with n vertices, k-sees a point $p \in P$ if the line segment $\overline{gp}$ intersects P's boundary at most k times. In the k-Transmitter Watchman Route Problem we aim to minimize the length of a k-transmitter watchman route along which every point in the polygon—or a discrete set of points in the interior of the polygon—is k-seen. We show that the k-Transmitter Watchman Route Problem for a discrete set of points is NP-hard for histograms, uni-monotone polygons, and star-shaped polygons given a fixed starting point. For histograms and uni-monotone polygons it is also NP-hard without a fixed starting point. Moreover, none of these versions can be approximated to within a factor $c \cdot \log n$, for any constant $c > 0$.

1 Introduction

k-transmitters were introduced as a generalization of the classical illumination problems [1]. Guards are replaced by *modems*, also called *k-transmitters*, who send a signal that can pass through up to k walls.

In the watchman route problem (WRP), introduced by Chin and Ntafos [3], instead of placing several stationary guards, we are given one mobile watchman who moves within the given environment, and want to compute a shortest watchman route such that all points in the environment are seen from some point on the route. This problem has shown to be solvable in polynomial time both with [3, 4, 10] and without [2, 9] a fixed starting point. Several different variations of the WRP have been considered, for example for polygons with and without holes [7], for lines and line segments [5], and using a k-transmitter as a watchman [8]. Nilsson and Schmidt proved NP-hardness for the k-transmitter watchman route problem for a discrete set of points within a simple polygon and provided a polylogarithmic approximation algorithm for that case [8]. In this paper, we show that it is also NP-hard for certain classes of simple polygons, namely histograms, uni-monotone, and star-shaped polygons, given a fixed starting point. Extending this, we also show NP-hardness without a fixed starting point for histograms and uni-monotone polygons.

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2 Notation and Preliminaries

Let P be a simple polygon having n vertices. P is called a *histogram* if it is rectilinear and has one horizontal edge (a base) that is equally long as the sum of lengths of all other horizontal edges. A *k-transmitter* is a modem that can see through up to k walls. Our goal is to see a set of points in the interior of P with one mobile k-transmitters. We say that point p k-sees point q (and p and q are *k-visible* to each other) if the segment between p and q intersects the polygon boundary at most k times. For a point $p \in P$ the k-visibility region of p is the set of all points in P that p k-sees.

k-Transmitter Watchman Route Problem with Starting Point (k-TrWRP(S, P, s)). Given a polygon P with n vertices, an integer $k \geq 2$, a starting point s in P, and a set of interior points S in P, find a minimum length watchman route that starts at s and lies within P such that all points in S are k-visible from the route.

k-Transmitter Watchman Route Problem (k-TrWRP(S, P)). Given a polygon P with n vertices, an integer $k \geq 2$ and a set of interior points S in P, find a minimum length watchman route that lies within P such that all points in S are k-visible from the route.

Since both the watchman and the points that need to be seen lie in the interior of P, it is sufficient to consider only even values for k. We therefore assume that k is even.

In Section 3, we prove that k-TrWRP(S, P, s) is NP-hard for histograms by providing a reduction from Set Cover. We then extend this reduction to uni-monotone and star-shaped polygons.

Set Cover Problem. Given a universe $\mathcal{U}$ and a family $\mathcal{R}$ of subsets of $\mathcal{U}$, find a subfamily $\mathcal{C} \subseteq \mathcal{R}$ that contains all elements of $\mathcal{U}$ and is of minimum cardinality.

Feige [6] showed that Set Cover cannot be approximated to within a factor $(1 - o(1)) \ln |\mathcal{U}|$ in polynomial time. Thus, there exists no polynomial time algorithm that approximates k-TrWRP(S, P, s) for histograms, uni-monotone polygons or star-shaped polygons, and k-TrWRP(S, P) for histograms and uni-monotone polygons, within approximation ratio of $c \log |S|$ for any $c > 0$. Nevertheless, we can apply the approximation algorithm presented in [8] to a histogram, uni-monotone or star-shaped polygon P with n vertices. This algorithm has an approximation factor of $O(\log^2(|S| \cdot n) \log \log(|S| \cdot n) \log(|S| + 1))$.

3 NP-Hardness for Histograms: Reduction from Set Cover

▶ **Theorem 3.1.** *For any $k \geq 2$, k-TrWRP(S, P, s) is NP-hard for histograms and cannot be approximated within a factor $c \log n$, for any $c > 0$.*

Proof. We provide a reduction from Set Cover. Let $(\mathcal{U}, \mathcal{R})$ be an instance of the Set Cover Problem. We construct a bipartite graph G with $V(G) = \mathcal{U} \cup \mathcal{R}$ and $E(G) = \{(u, R) \mid u \in \mathcal{U}, R \in \mathcal{R}, u \in R\}$.

Given an integer k, we construct a histogram P such that k-visibility between points encodes the edges of the graph G: it contains a set of points $S = \mathcal{U}$, where point $u \in \mathcal{U}$ is k-visible from a region $R \in \mathcal{R}$ if and only if $(u, R) \in E(G)$.

The points $u_1, \ldots, u_{|\mathcal{U}|} \in \mathcal{U}$ are placed in the top of "towers", so-called *defensive towers*, that lie in the right part of P. Similarly, the regions $R_1, \ldots, R_{|\mathcal{R}|} \in \mathcal{R}$, in the following also called *observation regions*, are placed from left to right in this order, in *watch towers* that lie close to the starting point s and to the left of the defensive towers. To ensure that the points in $\mathcal{U}$ are far to reach from s we place a long corridor between the two sets of towers. We call the part to the left of this corridor the *observational part of* P, and the part to the right of the corridor the *defensive part of* P. See Figure 1 for an example.

The horizontal boundary edges between the towers are all collinear. The longest line segment in P that contains all of these edges defines the so-called *supporting line.* We place the starting point s on the left end of the supporting line.

The watch towers are high enough such that walking up each of them is expensive. Moreover, we need to ensure that every point $u \in \mathcal{U}$ is k-visible from the observation regions whose corresponding subsets contain u. Therefore, we place the points and regions on different heights, such that the watch towers have decreasing height from left to right, and the defensive towers have increasing height. This means that R_1 will be in the highest tower in the observational part of P, and $u_{|\mathcal{U}|}$ will be in the highest tower in the defensive part of P. Let h be the height of the watch tower containing R_1, and let the height of the watch towers differ by $\varepsilon_1 \ll h$ only. Furthermore, let the width of the watch towers be $w < \varepsilon_2$, and denote the difference of the x-coordinates of R_1 an $R_{|\mathcal{R}|}$ by ℓ. We choose the horizontal distance of the watch towers such that ℓ is significantly smaller than h, $\ell \ll h$. The height of the defensive towers is adapted to the height of the watch towers to ensure k-visibility between observation regions and points in $\mathcal{U}$.

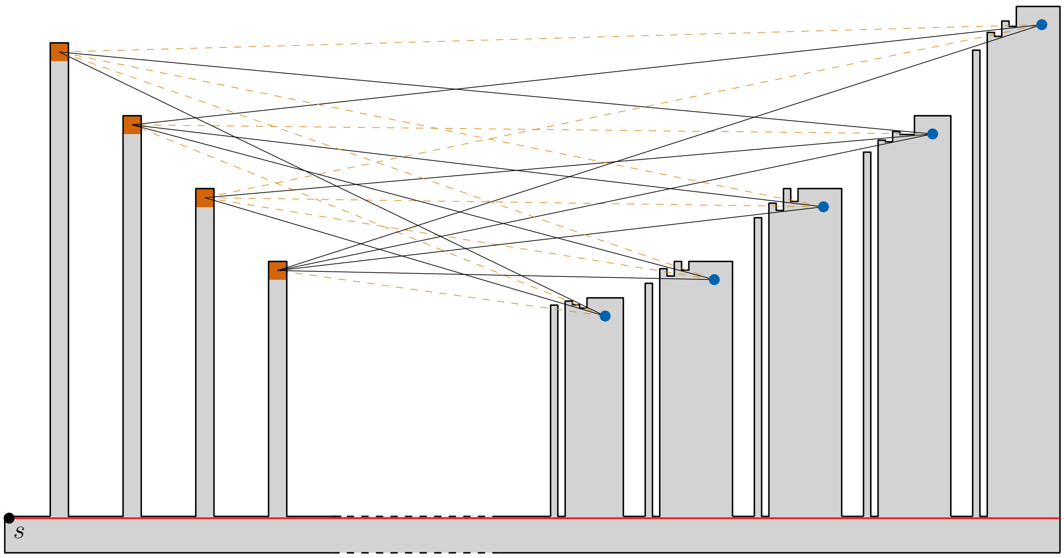

Figure 1 Construction of a histogram for $k = 2$. The regions $R_1, \ldots, R_{|\mathcal{R}|}$ are marked in orange. The points in $\mathcal{U}$, which lie in the defensive part of P, are colored blue. The continuous black line segments represent the edges of the graph G: the lines of k-visibility between the observation points and the points $\mathcal{U}$. If $(u, R) \notin E(G)$, then u is not k-visible from R. This is indicated by a dashed orange line segment. The red horizontal line segment in the bottom of P is the supporting line.

Let the length of corridor between the watch towers and the towers containing the points in $\mathcal{U}$ be $L \gg (h \cdot |\mathcal{R}| \cdot |\mathcal{U}| \cdot k + \ell) \log n$. Thus, walking through the corridor to reach the right part of P (and hence see the defensive points from their proximity) is much more expensive than climbing up every watch tower.

Since a point $u \in \mathcal{U}$ shall only be k-seen from an observation region R if $(u, R) \in E(G)$, and from its proximity, we need to block the k-visibility from every other point that lies within a feasible walking distance from s. To do this, we add battlements to the defensive towers (see Figure 2). Consider the visibility cone from a point u to an observation region R, meaning the cone that arises when connecting u with every point in R by a straight line. For the sake of simplicity, let each observation region be a square of height w, and assume the visibility cone to be just a straight line. Then, between every two such lines emanating from u, we add $k/2$ battlements. We also add $k/2$ battlements in front of the tower, which stretch from the supporting line up to the first line of visibility, and—in case $(u, R_1) \notin E(G)$—we add $k/2$ battlements after the uppermost line of visibility, which intersect the line of visibility between u and R_1. Every battlement is sufficiently close to the next line of visibility, but will not touch it. For an example of this construction, see Figure 2.

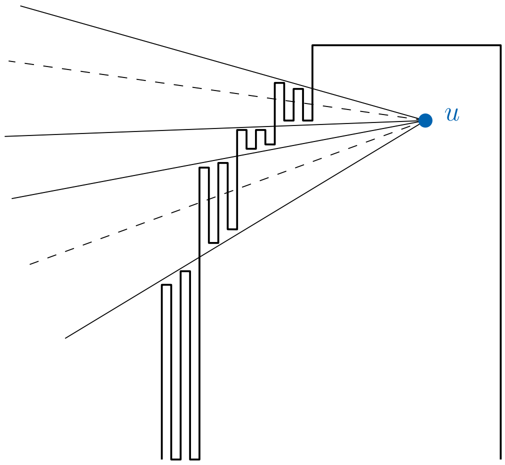

Figure 2 Adding battlements in front of a tower to block k-visibility. The lines suggest the line segment between u and the points corresponding to subsets in $\mathcal{R}$, where the continuous lines indicate $(u, R) \in E(G)$, and dashed lines indicate $(u, R) \notin E(G)$, for a region R representing a subset.

▶ Observation 3.1.1. The only points in the left part of the histogram P that k-see a point $u \in \mathcal{U}$ are exactly those that lie in the regions whose corresponding subsets contain u (that is, those $R \in \mathcal{R} : (u, R) \in E(G)$).

Since the watch towers are rather high, we obtain the shortest watchman route by climbing only few of the watch towers. More precisely, the watchman will visit exactly those regions in $\mathcal{R}$ that correspond to a minimum set cover of $\mathcal{U}$.

It remains to show that we can compute the histogram in polynomial time. To be more precise, this means that the histogram needs to have integer coordinates. The ratio between the distance of an observational point to its closest battlement and the total width of the polygon is at most $1/O\big(|\mathcal{R}| + k \cdot |\mathcal{U}| \cdot |\mathcal{R}|\big) = 1/O\big(k \cdot |\mathcal{U}| \cdot |\mathcal{R}|\big)$. We can construct the horizontal battlement edges using integer coordinates if we have $O(k \cdot |\mathcal{R}|)$ integral y-levels for each battlement. To achieve this, it is sufficient for the polygon to fit in a square of size $O\big(k^2 \cdot |\mathcal{U}| \cdot |\mathcal{R}|^2\big)$, hence we can restrict the construction to use coordinate values from 0 to $O\big(k^2 \cdot |\mathcal{U}| \cdot |\mathcal{R}|^2\big)$.

From the construction shown above, we conclude that our reduction is gap preserving, and thus, assuming $\mathsf{P} \neq \mathsf{NP}$, the problem cannot be polynomially approximated within a factor of $c \log n$, for any constant $c > 0$, where n is the total number of vertices. To see this, note that we can assume that $|\mathcal{R}| = |\mathcal{U}|^\alpha$, for some sufficiently large positive constant α, see also [6]. The number of vertices is bounded by $4 \cdot |\mathcal{R}| + 4k \cdot |\mathcal{U}| + 4 < n \leq 4 \cdot |\mathcal{R}| + 4k \cdot |\mathcal{R}| \cdot |\mathcal{U}| + 4$. Hence, $\Omega(n^{1/(2\alpha+1)}) \ni |\mathcal{U}| \in O(n^{1/\alpha})$ and since $|S| = |\mathcal{U}|$, the bound follows. ◀

We can substitute the starting point by the k-visibility region of a point u' and set $S = \mathcal{U} \cup \{u'\}$: We add $k/2$ towers of height $\gg h$ to the left of all watch towers, and one more tower of almost the same height to the left of these, in which we locate u', see Figure 3. All points in the k-visibility region of u' have a smaller x-coordinate than the leftmost watch tower. We then need to visit this region to k-see u'. Because of the length of the corridor, we again cannot afford to visit the defensive part of P. Thus, we have:

▶ **Corollary 3.2.** *For any $k \geq 2$, k-TrWRP(S, P) is* NP*-hard for histograms and cannot be approximated within a factor $c \log n$, for any $c > 0$.*

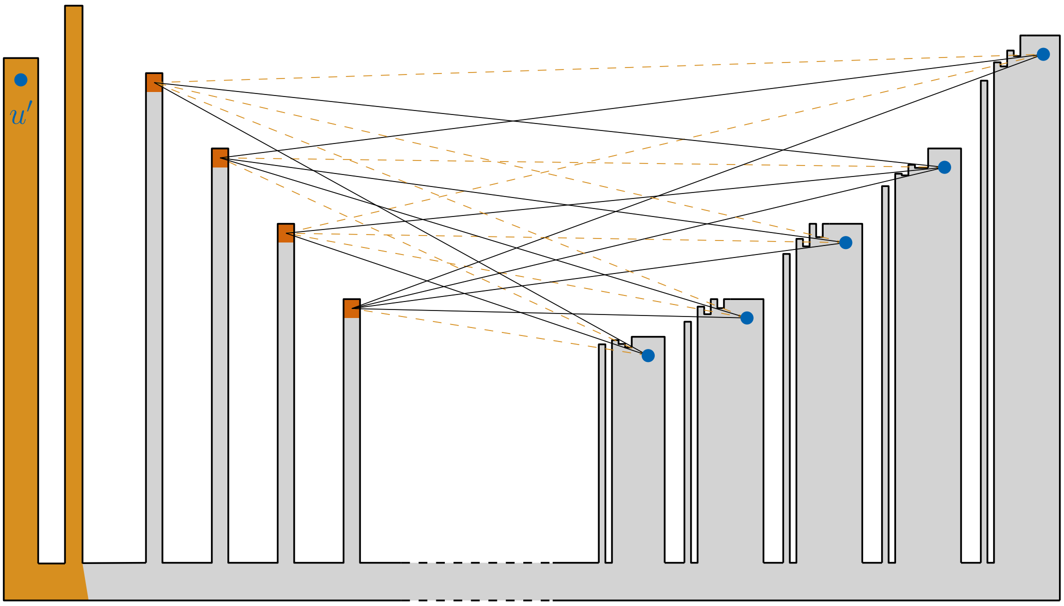

Figure 3 A modification of the histogram for $k = 2$: We add two more towers to the left of the watch towers and place u' in the leftmost tower. The 2-visibility region of u' is colored light orange.

A polygon P is called *x-monotone* if any vertical line intersects the boundary of P in at most two connected components. The boundary of an x-monotone polygon can be decomposed into two chains, splitting it at the (lowest) leftmost point and the (lowest) rightmost point of the boundary. An x-monotone polygon is called *uni-monotone* if either the upper or the lower chain is a horizontal segment. Clearly, a histogram is uni-monotone, hence, Theorem 3.1 and Corollary 3.2 yield:

▶ **Corollary 3.3.** *For any $k \geq 2$, k-TrWRP(S, P, s) and k-TrWRP(S, P) are* NP*-hard for uni-monotone polygons and cannot be approximated within a factor $c \log n$, for any $c > 0$.*

4 NP-Hardness for Star-Shaped Polygons

A polygon P is star-shaped if it contains a region (possibly a single point), called the *kernel*, from which every point in P is 0-seen. Given the histogram from the proof of Theorem 3.1, we can modify it into a star-shaped polygon for which the k-visibility properties again encode the bipartite graph G.

First, we stretch the towers such that they all “point” towards a common viewpoint—the kernel of P, which lies very far below the supporting line—preserving the k-visibility

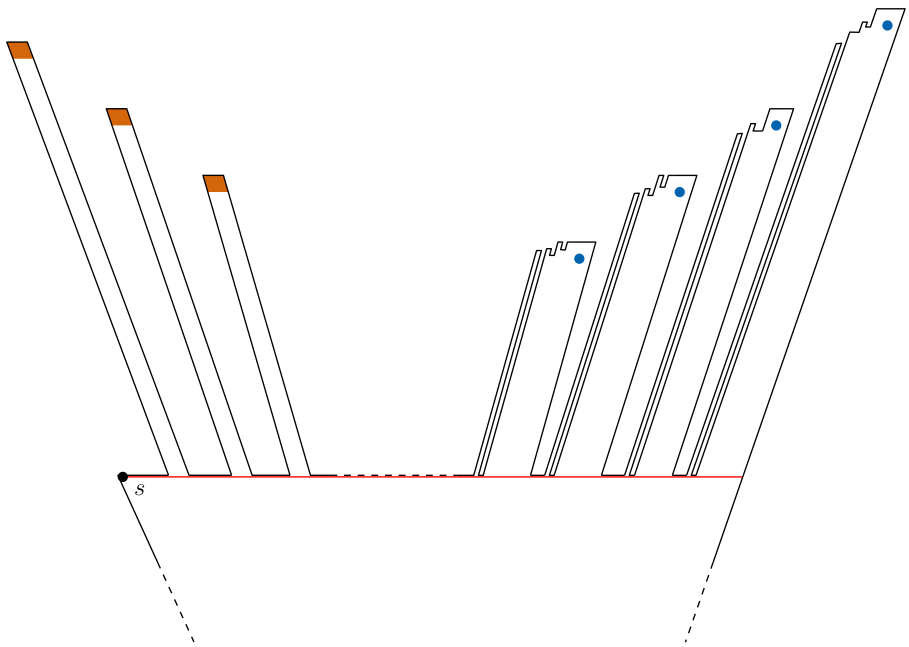

Figure 4 Sketch of a star-shaped polygon that evolves from slanting the histogram constructed previously. Note that the kernel is far below the supporting line.

properties. Moreover, we replace the base edge by two almost vertical edges that are slightly tilted towards the kernel and place the starting point s onto the left end of the supporting line. See Figure 4 for a sketch of this transformation.

To guarantee that the shortest watchman route will not be the direct path from s to the kernel of P, we ensure that the kernel is way too far away from s by choosing the angles along which the base edges and the towers are tilted accordingly. The construction can be made using integer coordinates inside a bounding box of polynomial range, similarly as in the proof of Theorem 3.1, yielding:

▶ **Theorem 4.1.** *For any $k \geq 2$, k-TrWRP(S, P, s) is* NP*-hard for star-shaped polygons and cannot be approximated within a factor $c \log n$, for any $c > 0$.*

Without a fixed starting point, k-TrWRP(S, P) clearly is easy to solve as then the shortest watchman route will be a route of length 0 somewhere in the kernel of P.

5 Conclusion

We establish NP-hardness of k-TrWRP(S, P, s) for histograms, uni-monotone polygons, and star-shaped polygons, as well as NP-hardness of k-TrWRP(S, P) for histogram and uni-monotone polygons. This is rather surprisingly, since these polygon classes seem to be fairly simple. The hardness reduction from Set Cover moreover yields inapproximability within a logarithmic factor in polynomial time. It would be of interest whether this result can be adapted to more polygon classes, like x-y-monotone polygons for example.

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