Deltahedral Domes over Equiangular Polygons*

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Abstract

A *polyiamond* is a polygon composed of unit equilateral triangles, and a *generalized deltahedron* is a convex polyhedron whose every face is a convex polyiamond. We study a variant where one face may be an exception. For a convex polygon P, if there is a convex polyhedron that has P as one face and all the other faces are convex polyiamonds, then we say that P *can be domed*. Our main result is a complete characterization of which equiangular n-gons can be domed: only if $n \in \{3, 4, 5, 6, 8, 10, 12\}$, and only with some conditions on the integer edge lengths.

1 Introduction

In the study of what can be built with equilateral triangles, the most well-known result is that there are exactly eight convex ***deltahedra***—polyhedra where every face is an equilateral triangle—with $n = 4, 5, 6, 7, 8, 9, 10, 12$ vertices. See references in [3] or Wikipedia.[1] What if coplanar triangles are allowed? In the plane, the polygons built of equilateral triangles are the ***polyiamonds***. Convex polyiamonds have 3, 4, 5, or 6 vertices. The convex polyhedra with polyiamond faces are the "non-strictly convex deltahedra", or ***generalized deltahedra***, following the nomenclature of Bezdek [3]. See the above cited Wikipedia article for some examples. There are an infinite number of generalized deltahedra, though the number of combinatorial types is finite since they have at most 12 vertices. There is no published characterization, though a forthcoming one is mentioned in [3].

Our goal (only partially achieved) is to characterize when a convex polygon can be "domed" with a convex surface composed of equilateral triangles. For a convex polygon P, if there is a convex polyhedron that has P as one face and all the other faces are convex polyiamonds, then we say that P ***can be deltahedrally domed***, or just ***domed*** for short. Here the ***deltahedral dome*** (***dome*** for short), denoted by $\mathcal{D}$, is the part of the polyhedron excluding face P, and P is called the ***base*** of the dome. Note that P itself may or may not be a polyiamond.

We assume that all the equilateral triangles have unit edge length, so P must be an integer polygon (with integer side lengths). Here is a simple example:

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[1] https://en.wikipedia.org/wiki/Deltahedron

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

▶ **Lemma 1.1.** *Every integer rectangle can be domed.*[2]

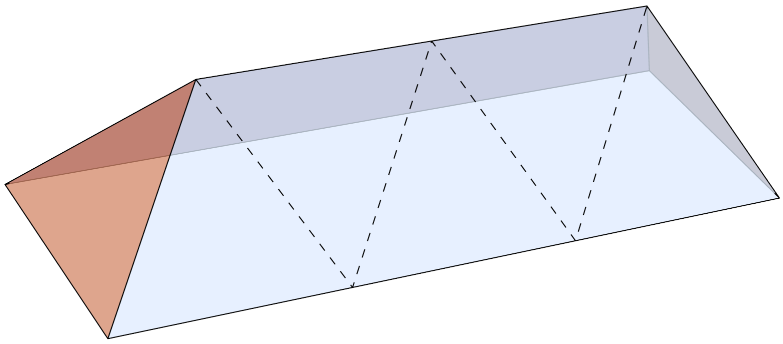

Figure 1 "Roof" dome over a 3×1 rectangle.

1.1 Main Theorem

▶ **Theorem 1.2.** *(a) The only equiangular polygons that can be domed have n vertices, where $n \in \{3, 4, 5, 6, 8, 10, 12\}$; for each such n, any regular integer n-gon can be domed. (b) Moreover, for $n = 3, 4, 5, 6$ every equiangular integer polygon can be domed, and for $n = 8, 10, 12$, an equiangular integer n-gon can be domed iff the odd edge lengths are equal and the even edge lengths are those of an equiangular $\frac{n}{2}$-gon.*

For small n, edge-length conditions for an equiangular integer polygon are known [2]: for $n = 4$ these are rectangles; for $n = 5$ there is only the regular pentagon; and for $n = 6$ the edge lengths must be integers a, b, c, a', b', c' with $a - a' = b' - b = c - c'$ (a 6-sided polyiamond).

Part (a) of Theorem 1.2 is proved in Sections 2 and 3. We have established several results beyond the main theorem (for example, that all polyiamonds, equiangular or not, are domeable, and that there is no domeable polygon with 25 or more vertices)—see Section 4.

1.2 Glazyrin and Pak

The source of our work derives from a paper by Glazyrin and Pak: "Domes over Curves" [4], which answers a question posed by Richard Kenyon in 2005.[3] In [4], a "curve" P is a closed polygonal chain in $\mathbb{R}^3$, and a dome is a PL-surface composed of unit equilateral triangles whose boundary is ∂P. Then they say that P can be *spanned*. We note the following two differences with our definitions:

(1) Our P is a 2D convex polygon; theirs is a 3D possibly self-intersecting polygonal chain.
(2) Our dome $\mathcal{D}$ is embedded (non-self-intersecting) and convex. Their PL-surface is (in general) nonconvex, immersed, and self-intersecting.

Under their conditions, they show that certain nonplanar rhombi cannot be spanned, which answers Kenyon's question in the negative.[4] More interesting for our purposes, they prove

[2] Due to space limitations, several proofs appear only in the full version of this paper.
[3] https://gauss.math.yale.edu/~rwk25/openprobs/.
[4] Recent work [1] extends the Glazyrin-Pak negative result to show that "generic" integer polygons cannot be spanned.

that every planar regular polygon can be spanned (their Theorem 1.4). In contrast, our Theorem 1.2 says that the regular 7-, 9-, and 11-gons cannot be domed, nor can any regular n-gon for $n > 12$. And here "regular" can be strengthened to "equiangular." Compared to their results, our conditions constrain the geometry and limit what can be domed.

2 Domed Regular Polygons

We prove one part of Theorem 1.2(a) by exhibiting domes over regular integer n-gons, for $n \in \{3, 4, 5, 6, 8, 10, 12\}$. We will use $\bar{P}_n$ to denote a regular integer n-gon.

- $3, 4, 5$: $\bar{P}_n$ for $n = 3, 4, 5$ can each be domed by a pyramid: Fig. 2.
- 6 : Hexagonal antiprism: Fig. 3(a).
- 8 : A slice through a gyroelongated square diprism: Fig. 3(b).
- 10 : A slice through an icosahedron: Fig. 3(c).
- 12 : A slice through a hexagonal antiprism: Fig. 3(d).

A few remarks. The pyramid pattern for $n = 3, 4, 5$ cannot be extended to $\bar{P}_6$, for that would result in a doubly-covered hexagon, not a dome by our definition. For $n = 8, 10, 12$, we show $\bar{P}_n$ as a slice of a convex polyhedron, with the dome the upper half of the surface. But we have established that not every doming of an equiangular polygon derives from a slice.

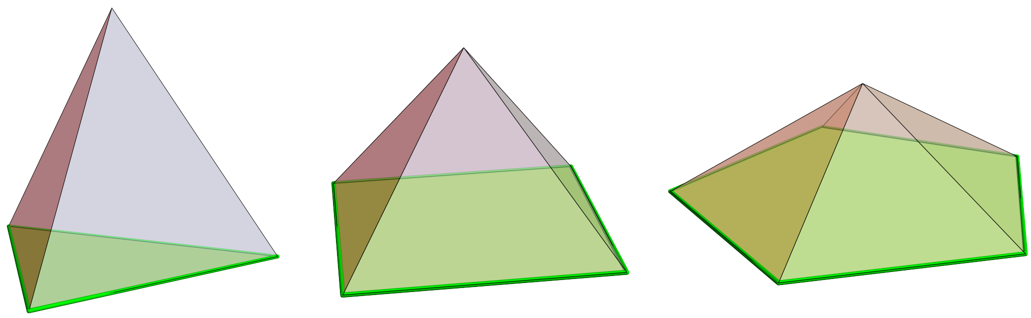

Figure 2 Pyramids over $\bar{P}_n$, $n = 3, 4, 5$.

Figures 2 and 3 show one way to dome each regular polygon $\bar{P}_n$, but there are other solutions. For example, $\bar{P}_5$ can be domed by a low slice through the icosahedron as shown in Fig. 4(a). And again, these figures illustrate regular polygons, special cases of equiangular polygons. To give a hint of the further possibilities, Fig. 4(b) shows an equiangular decagon P_{10} whose edge lengths alternate 1 and 3.

3 Proof of Theorem 1.2(a): Restrictions on n

In this section we complete the proof of the first half of Theorem 1.2: The only equiangular n-gons that can be domed have $n \in \{3, 4, 5, 6, 8, 10, 12\}$. For a dome over an equiangular n-gon, $n \geq 6$, we use the following steps:

(1) Each base vertex has three incident dome triangles.
(2) Curvature constraints imply that the number of (non-base) dome vertices is at most 6.

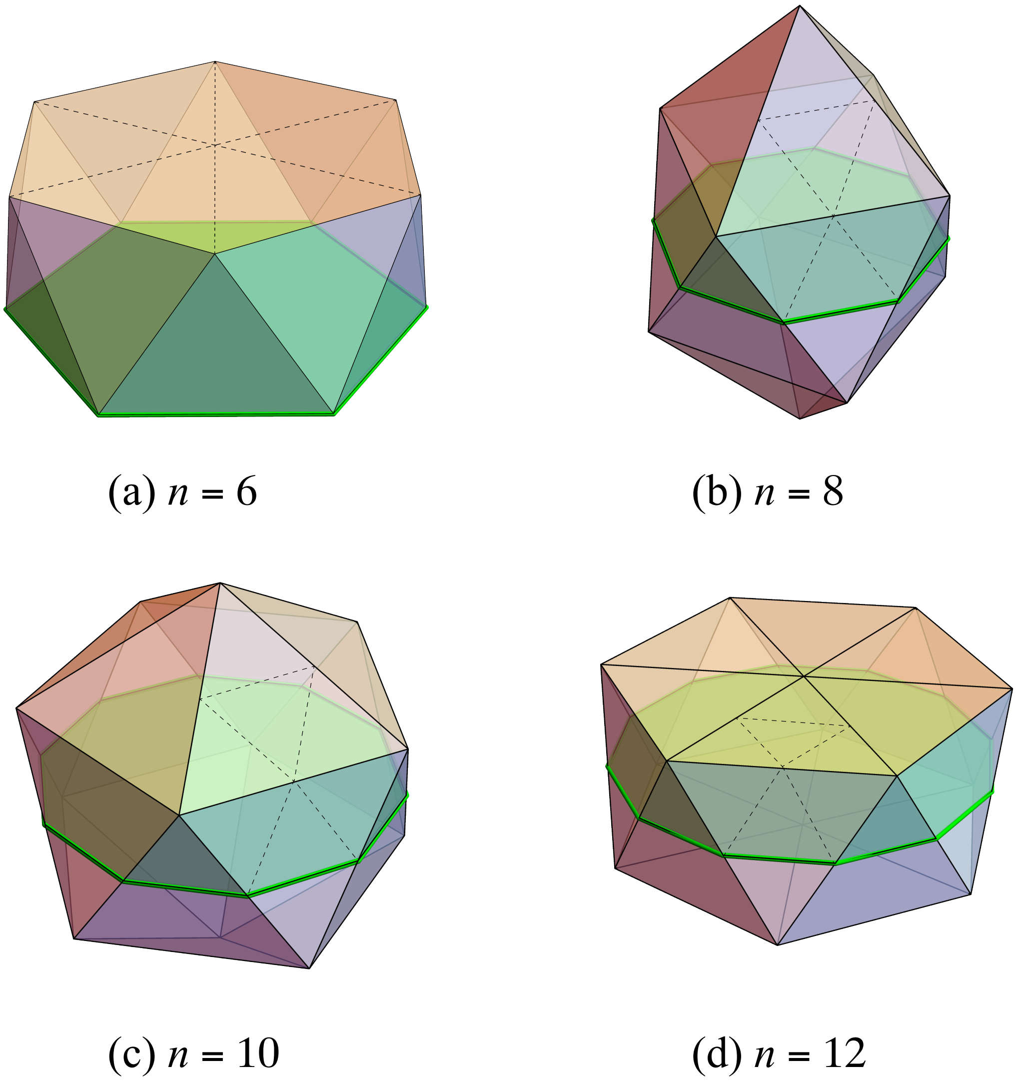

Figure 3 Examples of $\bar{P}_n$ domes for $n = 6, 8, 10, 12$.

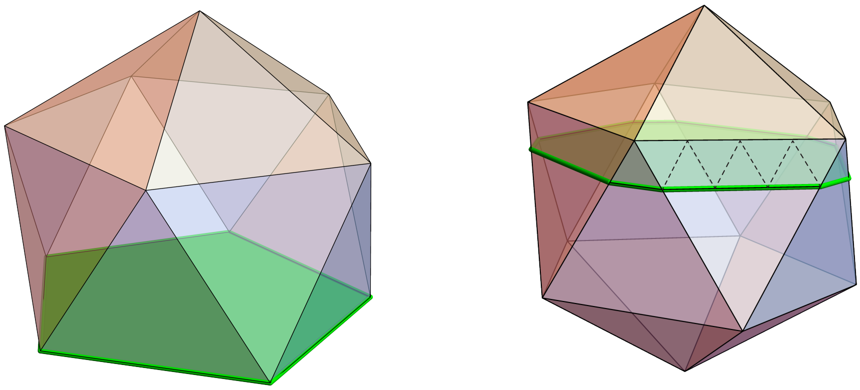

Figure 4 (a) A different dome over $\bar{P}_5$. (b) Equiangular decagon with edge lengths alternating $1, 3$.

(3) Of the n dome faces incident to base edges, at least half tilt toward the outside of the base and have a "private" dome vertex. Furthermore, for n odd we strengthen this to *all* dome faces incident to base edges.

(4) Thus, since there are at most 6 dome vertices, $n \leq 12$, and for n odd, there are no solutions for $n \geq 6$.

Note that the base angle β of an equiangular n-gon is $\frac{n-2}{n}180°$ so if $n \geq 6$, then every base angle is $\geq 120°$. This weaker assumption on a domeable convex n-gon is enough for most of our argument.

▶ **Lemma 3.1.** *If a base vertex b_i has base angle $\beta_i \geq 120°$, then it is incident to three dome vertices.*

Proof. Base vertex b_i cannot be incident to just one or two triangles, otherwise the total face angle is $\leq 120°$, which is not enough to span β_i. Vertex b_i cannot be incident to four triangles, because $\beta_i + 240° \geq 360°$, and similarly for five (or more) triangles. ◀

From this we can analyze the base curvature:

▶ **Lemma 3.2.** *If every base vertex b_i is incident to three dome triangles, then the sum of the curvatures at the base vertices is 2π.*

Proof. Let β_i be the angle of P at vertex b_i. Then the curvature at b_i is $\omega_i = 2\pi - (\beta_i + \pi)$, where the final π term follows from the assumption that b_i is incident to three triangles. Recalling that $\sum_i \beta_i = \pi(n-2)$ for any simple polygon, we have:

$$\sum_i \omega_i \;=\; \sum_i (\pi - \beta_i) \;=\; n\pi - \sum_i \beta_i \;=\; n\pi - \pi(n-2) \;=\; 2\pi.$$

◀

▶ **Lemma 3.3.** *If $\mathcal{D}$ is a dome over a convex polygon P that has all angles $\geq 120°$, then $\mathcal{D}$ has at most 6 dome vertices.*

Proof. Let V_3, V_4, V_5 be the number of (non-base) dome vertices with $3, 4, 5$ incident triangles, respectively. By the Gauss-Bonnet theorem, the total curvature of a convex polyhedron is 4π. The curvature of a V_k vertex is $2\pi - k\frac{\pi}{3}$. By Lemmas 3.1 and 3.2 (this is where we use the assumption that all base angles are $\geq 120°$) the curvature at the base vertices of any dome over P is 2π. Thus, in units of π:

$$V_3 + \frac{2}{3}V_4 + \frac{1}{3}V_5 = 2 \, .$$

Therefore the number of dome vertices of $\mathcal{D}$ is $V_3 + V_4 + V_5 \leq 3V_3 + 2V_4 + V_5 = 6$. ◀

3.1 Face Normals and Private Dome Vertices

It remains to show step (3): that, of the n dome faces incident to base edges, at least half of them [and for odd n, all of them] tilt toward the outside of the base and each have a "private" dome vertex. We say that a dome vertex v is ***private*** if there is a unique dome face incident to v and to a base edge.

Orient the dome with the base in the horizontal xy-plane. A dome triangle/face has an ***upward normal*** if its normal has a positive z-component, and a ***downward normal*** if its normal has a negative z-component. (This formalizes "tilting towards the outside").

▶ **Lemma 3.4** (±Normals)**.** *Consider a base vertex b_i with base angle $\geq 120°$. Suppose the three dome triangles incident to b_i are t_1, t_2, t_3 where t_1 and t_3 are incident to the base edges at b_i (possibly t_2 is coplanar with t_1 or with t_3, but not both). Then t_2 has an upward normal and at least one of t_1, t_3 has a downward normal.*

Lemma 3.4 is proved in the full version of this paper using the Gauss map and spherical trigonometry.

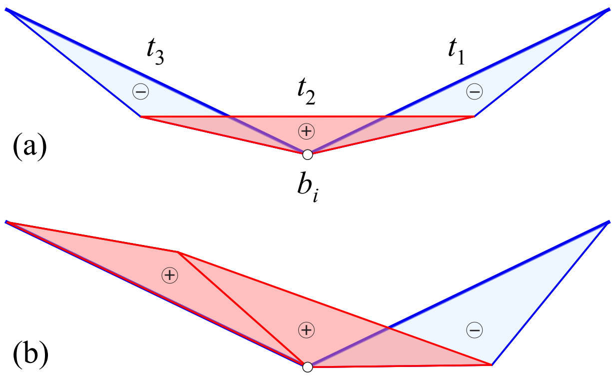

Figure 5 Overhead view of three triangles incident to base vertex b_i. Triangles with upward normals pink, downward normals blue. (a) Both t_1 and t_3 downward. (b) Only t_1 downward.

▶ **Observation 3.5.** *In Lemma 3.4, if t_1 has a downward normal, then it cannot be coplanar with t_2 (which has an upward normal), and thus the dome face containing t_1 has a face angle of 60° at the base vertex b_i.*

▶ **Lemma 3.6.** *If P is a domeable convex n-gon with all angles $\geq 120°$, then $n \leq 12$. Furthermore, there is no domeable equiangular n-gon for odd $n \geq 6$.*

Proof. For the second statement in the lemma, note that an equiangular n-gon, $n \geq 6$ has all angles $\geq 120°$. So consider an n-gon P with all angles $\geq 120°$, and suppose P has a dome $\mathcal{D}$. By Lemma 3.3, $\mathcal{D}$ has at most 6 dome vertices. We will prove that $n \leq 12$, and derive a contradiction for odd $n \geq 6$.

Consider the n faces of $\mathcal{D}$ incident to a base edge. Let d be the number of those faces with downward normals. By Observation 3.5, a downward pointing face f incident to base edge e has $60°$ face angles at the the endpoints of e, so it must include a dome vertex whose projection to the xy-plane lies in the equilateral xy-triangle on edge e. See Fig. 5 and Fig. 6. Such a dome vertex is unique to base edge e so we call it a "private" dome vertex. Thus there are at least d private vertices. Since $\mathcal{D}$ has at most 6 dome vertices, we have $d \leq 6$.

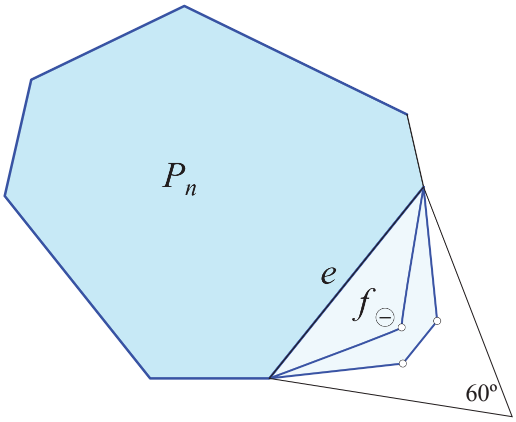

Figure 6 Projection of downward face f lies within an equilateral triangle outside base edge e. Here $P_n = P_7$.

Now Lemma 3.4 implies that $d \geq \frac{n}{2}$. Thus $\frac{n}{2} \leq d \leq 6$ so $n \leq 12$. Furthermore:

▶ **Lemma 3.7.** *For any dome over an equiangular n-gon with odd $n \geq 6$, all the dome faces incident to base edges have downward normals, i.e., $d \geq n$.*

Therefore there is no domeable equiangular n-gon with odd $n \geq 6$, since we would need $n \leq d \leq 6$. ◀

Proof outline for Lemma 3.7. Each base vertex b_i has four incident faces and thus, considered in isolation, has only one degree of freedom for the dihedral angles of incident edges. Let δ_i be the dihedral angle of $\mathcal{D}$ at base edge $e_i = b_i b_{i+1}$. Because b_i and b_{i+1} share edge e_i and have the same face angles, we can show that $\delta_{i-1} = \delta_{i+1}$. Since n is odd, this implies that all the δ_i's are equal. Lemma 3.4 implies that at least one δ_i is $> 90°$, so they all are. ◀

4 Further Results and Open Questions

We have characterized equiangular domeable polygons. Many open problems remain. Here are a few.

(1) Is there any convex n-gon with $n > 12$ that can be domed? A rough bound is $n \leq 55$: every dome vertex has curvature at least $\frac{\pi}{3}$ so there are at most 11 dome vertices; every

base vertex must be adjacent to a dome vertex and dome vertices have degree at most 5. We can prove the stronger bound $n \leq 24$.

(2) Is there any convex 7-gon that can be domed? We have constructed 9- and 11-gons (non-equiangular) that can be domed. See Fig. 7.

(3) Is there any non-equilateral triangle that can be domed? Glazarin and Pak conjectured [4] that, even under their looser conditions, an isosceles triangle with edge lengths $2, 2, 1$ cannot be spanned.

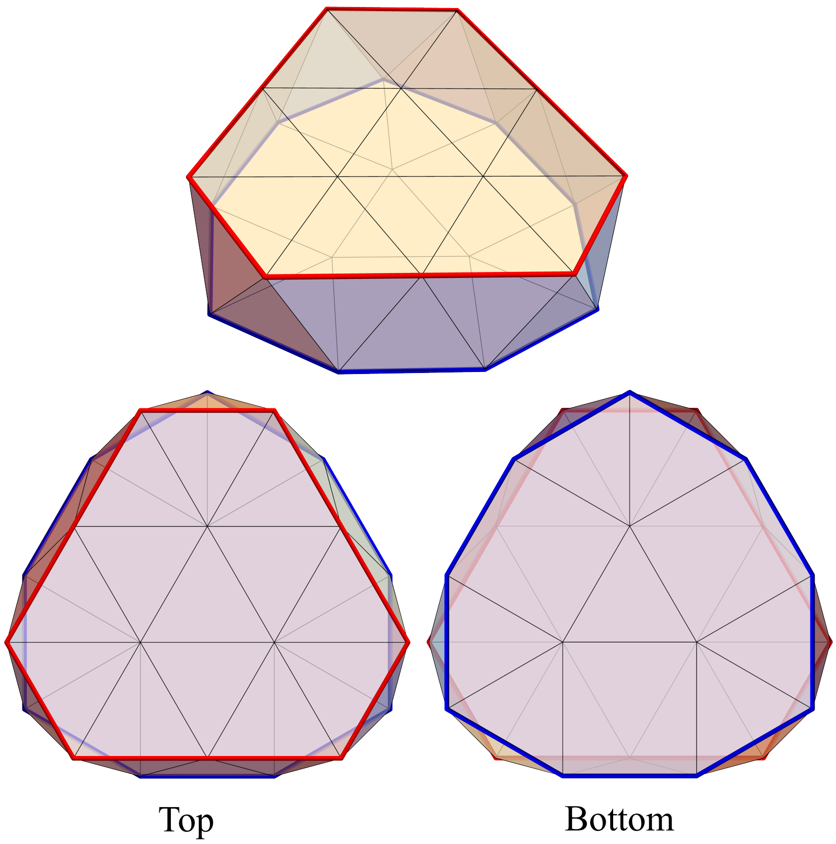

Figure 7 The top (red) is a polyiamond of 13 equilateral triangles. The base (blue) is a 9-gon with base angles 120° and 150°.

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