Flips in Odd Matchings*

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Abstract

Let P be a set of $n = 2m + 1$ points in the plane in general position. We define the graph GM_P whose vertex set is the set of all plane matchings on P with exactly m edges. Two vertices in GM_P are connected if the two corresponding matchings have $m - 1$ edges in common. In this work we show that GM_P is connected.

1 Introduction

Reconfiguration is the process of changing a structure into another—either through continuous motion or through discrete changes. Concentrating on plane graphs and discrete reconfiguration steps of bounded complexity, like exchanging one edge of the graph for another edge such that the new graph is in the same graph class, a single reconfiguration step is often called an *edge flip*. The *flip graph* is then defined as the graph having a vertex for each configuration and an edge for each flip. Flip graphs have several applications, for example morphing [6] and enumeration [8]. Three questions are central: studying the connectivity of the flip graph, its diameter, and the complexity of finding the shortest flip sequence between two given configurations. The topic of flip graphs has been well studied for different graph classes like triangulations [3, 14, 15, 16, 17, 19, 20], plane spanning trees [11, 12], plane spanning paths [2, 5], and many more. For a nice survey see [10].

For matchings usually other types of flips were considered since a perfect matching cannot be transformed to another perfect matching with a single edge flip. A natural flip in perfect matchings is to replace two matching edges with two other edges, such that the new graph is again a perfect matching. These flips were studied mostly for convex point sets [9, 18]. While the according flip graph is connected on convex point sets it is open whether this flip graph is connected for any set of points in general position. Other types of flips in perfect matchings can be found in [1, 4, 7].

In this work we study a setting where single edge flips are possible for matchings. Let P be a set of $n = 2m + 1$ points in the plane in general position (that is, no 3 points on a line). An *almost perfect matching* on P is a set M of m line segments whose endpoints are pairwise disjoint and in P. The matching M is called *plane* if no two segments cross.

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

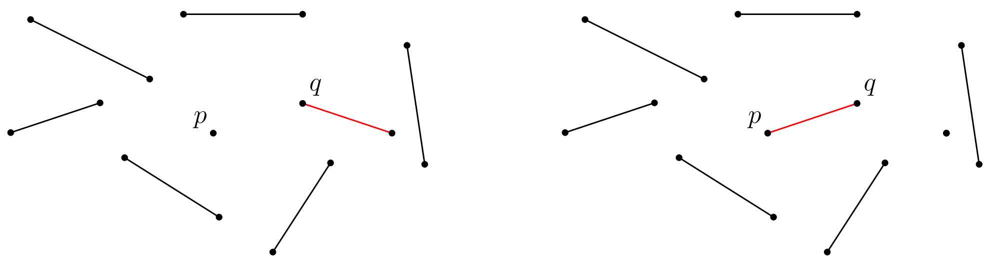

Figure 1 Flipping a matching edge: the previously unmatched point p is matched to q.

Let $\mathcal{M}_P$ denote the set of all plane almost perfect matchings on P. We define the flip graph GM_P with vertex set $\mathcal{M}_P$ through the following flip operation. Consider a matching M_1 and let p be the unmatched point. Let q be a point in P such that the segment pq does not cross any segment in M_1. The flip now consists of removing the segment incident to q from the matching and adding pq instead, see Figure 1. Note that this gives another plane almost perfect matching M_2. In the graph GM_P, the vertices corresponding to M_1 and M_2 are adjacent.

In this paper, we prove the following theorem.

▶ **Theorem 1.1.** *For any set P of $n = 2m + 1$ points in general position in the plane the flip graph GM_P is connected.*

In Section 2 we give an overview of the used techniques and the proof of Theorem 1.1. Then in Section 3 we prove the lemmata used for the proof of Theorem 1.1.

2 Overview and Proof of Theorem 1.1

In this section, we give an overview of our used techniques and the proof of Theorem 1.1.

Let $G = (V, E)$ be a graph G and let M be a matching in G. We call a path P in G an *alternating path* if the edges of P lie alternately in M and in $E \setminus M$. In the following, we consider so-called *segment endpoint visibility graphs*: graphs that encode the visibility between the endpoints of a set of segments. More precisely, given a set S of (non-intersecting) segments in the plane, its segment endpoint visibility graph is the graph that contains a vertex for every segment endpoint, and an edge between two vertices if the corresponding segment endpoints either (1) are connected by a segment in S, or (2) "see" each other, meaning that the open segment between them does not intersect any segment from S. Hoffmann and Tóth [13] proved that segment endpoint visibility graphs always admit a simple Hamiltonian polygon—this is a plane Hamiltonian cycle—, and moreover presented an algorithm to find such a polygon. This result is crucial for us, as a plane perfect matching can be considered as a set of segments in the plane. Hence, for every plane matching M there exists a plane subgraph of the segment endpoint visibility graphs of M that is the (not necessarily disjoint) union of a Hamiltonian cycle and M. Even disregarding planarity, we prove

▶ **Lemma 2.1.** *Let G be an undirected graph that is the union of a Hamiltonian cycle C and a perfect matching M. Let $e_1 = (a, b)$ and $e_2 = (c, d)$ be two matching edges. Then there exists an alternating path P that starts with the vertex a and the edge e_1 and ends with the vertex c.*

We denote the *symmetric difference* of two graphs A, B with $A \triangle B$. Given the setup of Lemma 2.1, we can compute another matching $M_2 = M \triangle P$ in which both a and d are

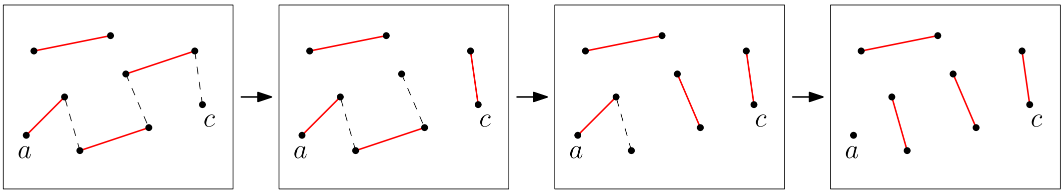

Figure 2 A plane alternating path in the visibility graph gives rise to a sequence of flips.

unmatched. Ignoring the point d, this augmentation corresponds to a sequence of flips in a point set of odd size. See Figure 2 for an illustration. This flip sequence starts with the matching $M_1 = M \setminus \{e_2\}$ and point c being unmatched, and ends with the matching M_2 and point a being unmatched.

To prove that the flip graph GM_P is connected, we show that there always exists a sequence of flips which transforms a given plane almost perfect matching into a plane almost perfect matching, where the unmatched point lies on the boundary of the convex hull.

▶ **Lemma 2.2.** *Let M_1 be a plane almost perfect matching and let t be a point on the convex hull of P. Then there exists a sequence of flips to a matching M_2 in which the unmatched point is t.*

We use Lemma 2.2 to show that we can flip every matching M to a *canonical matching* M_C, which we now define. Let $P = \{p_1, p_2, \ldots, p_{2m+1}\}$, where the points are labeled from left to right. The canonical matching M_C now consists of the edges $p_1p_2, p_3p_4, \ldots, p_{2m-1}p_{2m}$ with p_{2m+1} remaining unmatched. It follows from the ordering of the points that this matching is plane.

Proof of Theorem 1.1. Let M be any plane almost perfect matching on P. Let i be the smallest index for which the edge $p_{2i-1}p_{2i}$ is not in M. We show that there is a sequence of flips on the point set $\{p_{2i-1}, p_{2i}, \ldots, p_{2m}, p_{2m+1}\}$ after which $p_{2i-1}p_{2i}$ is in the resulting matching. In the following, for simplicity of notation, we set $i = 1$.

Using Lemma 2.2, we first flip to a matching M_2 in which the point p_1 is unmatched. As the segment p_1p_2 cannot be crossed by any other segment, we can thus do one more flip which puts p_1p_2 into the resulting matching. Now we can inductively continue the argument on the point set $P' = \{p_3, \ldots, p_{2m+1}\}$ and eventually reach the canonical matching M_C. ◀

▶ Remark. From our proof it follows directly that not more than $O(n^2)$ flips are needed to transform any plane almost perfect matching on P into any other plane almost perfect matching on P. Or in other words, the diameter of the flip graph GM_P is in $O(n^2)$.

3 Proofs of the Lemmata

In this section, we prove Lemma 2.1 and Lemma 2.2. We begin this section with presenting a procedure to find an alternating path in an abstract graph.

▶ **Lemma 2.1.** *Let G be an undirected graph that is the union of a Hamiltonian cycle C and a perfect matching M. Let $e_1 = (a, b)$ and $e_2 = (c, d)$ be two matching edges. Then there exists an alternating path P that starts with the vertex a and the edge e_1 and ends with the vertex c.*

Proof. In a first step, we reduce to the situation where no matching edge except possibly e_1 or e_2 lies on the cycle C, that is, $C \cap M \subseteq \{e_1, e_2\}$. To this end, assume that there is a matching edge $f = \{f_1, f_2\}$ lying on the path (f_0, f_1, f_2, f_3) of the cycle C. We define the graph G' with vertex set $V(G') = V(G) \setminus \{f_1, f_2\}$ by keeping all edges of G induced by $V(G')$ and adding the edge $\{f_0, f_3\}$. It follows from the construction that G' is again the union of a Hamiltonian cycle and a perfect matching and that G' contains an alternating path starting at a and ending at c if and only if G contains an alternating path starting at a and ending at c. Thus, in the following we may assume that $C \cap M \subseteq \{e_1, e_2\}$.

We now describe an algorithm that explicitly constructs a required alternating path. The algorithm constructs a sequence of graphs $G_2, G_3, \ldots, G_p$, starting with $G_2 = \{e_1\}$, with the following properties:

(1) the graph G_k has the k vertices $v_1, \ldots, v_k$;
(2) G_k has two vertices of degree 1, namely v_1 and v_k;
(3) all other vertices of G_k have degree 2 and are incident to one edge in M and one edge in $C \setminus M$;
(4) $v_1 = a$, $v_2 = b$ and $v_p = c$.

From these properties it follows that the last graph G_p is the disjoint union of cycles and the required alternating path P. It remains to describe the algorithm and prove that the constructed sequence of graphs satisfies the above properties. We start by setting $G_2 = \{e_1\}$, which trivially satisfies all the properties. In order to construct G_{k+1} from G_k we distinguish two cases, depending on whether in G_k the (unique) edge e incident to v_k is in M or not.

Case 1: $e \in C \setminus M$**.** Let $m = \{v_k, w\}$ be the matching edge incident to v_k. We define G_{k+1} by adding m to G_k. By Property (3) for G_k, all vertices in G_k except v_k are incident to an edge in M, and as M is a perfect matching, this implies that w is not a vertex of G_k. Thus, G_{k+1} has one more vertex, proving Property (1) for G_{k+1}. The only vertices whose degrees have changed are $w = v_{k+1}$, which now has degree 1, and v_k which is now also incident to an edge in M. This proves properties (2) and (3).

Case 2: $e \in M$**.** For an illustration of this case, see Figure 3. Consider the unique path Q in C from v_k to c which does not pass through a and let w be the first vertex on this path that is not a vertex of G_k. Set $v_{k+1} = w$. For any edge e in Q, add e to G_{k+1} if and only if it is not in G_k and remove it otherwise. Properties (1) and (2) follow directly by definition. For Property (3), note that the only vertices whose neighborhoods have changed are the vertices on Q. As Q is a path on C and we assumed that C contains no matching edge other than e_1 and e_2, it follows that no matching edge was removed. All vertices are thus still incident to exactly one matching edge. Further, as C is a cycle, every vertex in Q is incident to exactly two edges in $C \setminus M$. It follows from the construction that exactly one of these edges is removed while the other one is added, proving Property (3).

Finally, we stop the procedure as soon as we add the vertex c, which has to happen for some G_p, $p \leq n$, where n is the number of vertices of G. This proves the last part of Property (4) and thus finishes the proof. ◀

▶ **Lemma 2.2.** *Let M_1 be a plane almost perfect matching and let t be a point on the convex hull of P. Then there exists a sequence of flips to a matching M_2 in which the unmatched point is t.*

Proof. Let p be the unmatched point in M_1. If $p = t$ then we are trivially done, so assume for the remainder that $p \neq t$. We duplicate p such that the two points p, p' have the same neighborhood in the segment endpoint visibility graph. Moreover, we add the edge pp' to M_1. By [13] there is a plane Hamiltonian cycle C that spans all segment endpoints of M_1 and $M_1 \cup C$ is plane. Let u be the vertex that is matched to t in M_1. By Lemma 2.1, there is an

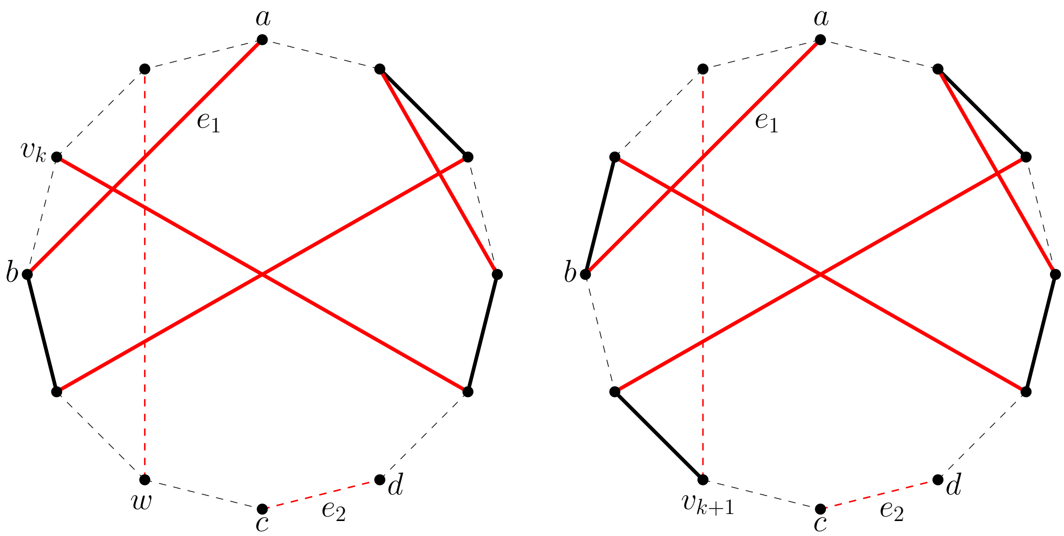

Figure 3 Constructing G_{k+1} (right) from G_k (left). The paths G_k and G_{k+1} are depicted with lines, while unused edges of G are dashed. The matching edges are red, the cycle edges are black.

alternating path P from t to p in $C \cup M_1$ which starts with the edge tu. Since the underlying graph is plane, P is also plane. If p and p' are in P, then the edge pp' is also in P because pp' is a matching edge. Hence, we can contract p and p' to a single point p such that P is still an alternating path.

Now, we construct a matching M_2 by augmenting M_1 via a sequence of flips along P to get $M_2 = M_1 \triangle P$. M_2 is an almost perfect matching in which p is matched, and t is the unmatched point. ◀

4 Conclusion

We considered the flip graph GM_P of plane matchings for point sets of odd size, and showed that GM_P is connected. In the course of the proof, we also showed that the union of a Hamiltonian cycle and a perfect matching always contains an alternating path from an arbitrary matching edge to any other arbitrary point.

While we showed that the flip graph is connected, it would be interesting to determine more precise bounds for the diameter of GM_P. Another interesting setting might be to study this problem in two colored point sets with plane almost perfect bicolored matchings and determine whether the according flip graph is still connected.

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