Oriented dilation of undirected graphs

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Abstract

Given an oriented graph $\overrightarrow{G}$ on a set of points P in the Euclidean plane, the oriented dilation of $p, p' \in P$ is the ratio of the length of the shortest cycle in $\overrightarrow{G}$ through p and p' to the perimeter of the smallest triangle in P containing p and p'. The oriented dilation of $\overrightarrow{G}$ is maximum oriented dilation over all pair of points. We show that given an undirected graph G on P, it is NP-hard to decide whether the edges can be oriented in way that the oriented dilation of the resulting graph is below a given threshold. For the case that G is complete, it is known that there is always an orientation of the edges with oriented dilation at most 2. As a first step towards improving this bound, we show that for $|P| = 4$ there is always a tournament, i.e., an oriented complete graph, with oriented dilation at most 1.5. This holds not only in the Euclidean but more generally in the metric plane. In the latter the bound is tight.

1 Introduction

Geometric spanners have may applications like wireless ad-hoc networks [4, 10], robot motion planning [5] and the analysis of road networks [1, 6]. The need to orient edges naturally arise since edges might only support one-way communication/traffic. Thus, in such applications it may be necessary to find an orientation of the edges that still provides relatively short paths between vertices. While undirected spanners are a widely researched topic during the last decades (see [2, 9] for a survey), oriented spanners have been only introduced recently [3].

Given a point set in the Euclidean plane and a parameter t, an oriented t-spanner $\overrightarrow{G}$ is an oriented subgraph of the complete bi-directed graph, such that for every pair of points, the shortest cycle in $\overrightarrow{G}$ containing those points is at most a factor t longer than their smallest triangle in the complete graph. Formally, given a point set $P \subset \mathbb{R}^d$ and a parameter $t \in \mathbb{R}^+$, an oriented graph $\overrightarrow{G} = (P, \overrightarrow{E})$ (thus a graph where $(u, v) \in \overrightarrow{E}$ implies $(v, u) \notin \overrightarrow{E}$) is called oriented t-spanner if for every two points $p, p' \in P$ the oriented dilation $\mathrm{odil}(p, p') = \frac{|C_{\overrightarrow{G}}(p,p')|}{|\Delta(p,p')|} \leq t$. Here, $C_{\overrightarrow{G}}(p, p')$ denotes the shortest oriented cycle containing p and p' in $\overrightarrow{G}$ and $\Delta(p, p')$ is the triangle $\Delta pp'p''$ with $p'' = \underset{p^* \in P}{\arg\min} |p - p^*| + |p^* - p'|$.

The problem of finding an oriented t-spanner with at most some fixed number m of edges is NP-hard [3], thus there is little hope to compute minimum oriented spanners efficiently. A natural approach for nonetheless computing an oriented spanner is to first compute a suitable undirected graph and then orienting it. For convex point sets, for instance, one can obtain an $\mathcal{O}(1)$-spanner by orienting a greedy triangulation [3]. However, no constructions are known to compute oriented spanners of small size for general point sets.

Here, we show that finding an orientation of an undirected graph such that the oriented dilation is minimal, is NP-hard even on Euclidean graphs. As our NP-hardness construction does not hold for complete graphs, we look into the oriented dilation of tournaments. As first step and potential building block for larger point sets, we show that for every point set

40th European Workshop on Computational Geometry, Ioannina, Greece, March 13–15, 2024.
This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

P with $|P| = 4$ even in a metric plane there is a tournament $\overrightarrow{K}(P)$ such that the oriented dilation of $\overrightarrow{K}(P)$ is at most 1.5. We further prove this bound to be tight.

2 Hardness

▶ **Theorem 2.1.** *Given an undirected geometric graph G and a parameter t', it is NP-hard to decide if there is an orientation $\overrightarrow{G}$ of G with oriented dilation* $\mathrm{odil}(\overrightarrow{G}) \leq t'$.

We will give a proof idea which is mainly described graphically here. A detailed proof with an explanation for the coordinates of every point can be found in the full version.

Proof sketch. We reduce from the NP-complete problem planar 3-SAT [8]. We start with a planar Boolean formula φ in conjunctive normal form with an incidence graph G_φ that can be embedded on a polynomial-size 1×1-grid [7, 8] as illustrated in Figure 1. We give a construction for a graph G based on G_φ such that there is an orientation $\overrightarrow{G}$ of G with dilation $\mathrm{odil}(\overrightarrow{G}) \leq t'$ with $t' := 1.043$ if and only if φ is satisfiable.

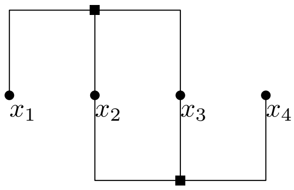

Figure 1 Example: Incidence graph of a planar 3-SAT formula embedded on a square grid

In the following, every point $p = (x, y)$ on the grid will be replaced by a so-called *oriented point*, which is a pair of points $P = \{t(p), b(p)\}$ with *top* $t(p) = (x, y + \frac{\varepsilon}{2})$ and *bottom* $b(p) = (x, y - \frac{\varepsilon}{2})$, where $\varepsilon \geq 0$ is a small constant. We will present the proof with $\varepsilon = 0$, i.e., $t(p)$ and $b(p)$ are two points with the same coordinates, while using a small positive ε in all figures for illustration purposes. This choice of ε simplifies the proof. However, the proof stays valid for a sufficiently small $\varepsilon > 0$.

We add an edge between $t(p)$ and $b(p)$, its orientation encodes whether this points represent "true" or "false". W.l.o.g. we assume that an oriented edge from $b(p)$ to $t(p)$, thus an *upwards* edge, represents "true" and a *downwards* edge represents "false". When this is not the case, we can achieve this by flipping the orientation of all edges.

Edges in the plane embedding of our formula graph G_φ will be replaced by *wire* gadgets. First, we add (a polynomial number of) grid points on the edge such that all edges have length 1. Then, we create a wire as in Figure 2. Note that wires propagate the orientation of oriented points - if two points next to each other on a wire have different orientations, their dilation would be significantly larger than $t' := 1.043$, since the shortest oriented cycle needs to go through an additional oriented point. If they have the same orientation, their dilation is 1 (since $\varepsilon = 0$; otherwise slightly larger). To switch a signal (for a negated variable in a formula), we start the wire as in Figure 3.

To ensure that all clause gadgets encode the same orientation of oriented points as "true", we add a *tree of knowledge*. This is a tree with vertices on the 1×1-grid shifted by $(0.5, 0.5)$ relative to the grid of G_φ and with wires as edges. The tree will have two leaves per clause

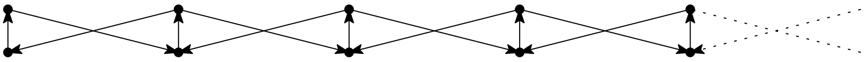

Figure 2 A wire where oriented points are oriented upwards

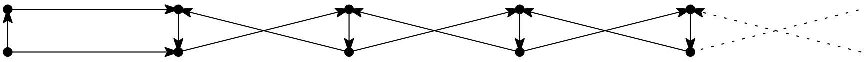

Figure 3 A wire where the orientation is switched to negate the signal

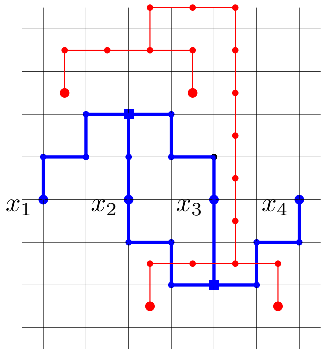

Figure 4 G_φ (blue) and its underlying grid together with a tree of knowledge (red)

(see Figure 4). W.l.o.g we assume that all oriented points of the tree of knowledge are oriented upwards (thus "true").

All oriented points, which are not direct neighbours of a wire, are linked by a $K_{2,2}$ (compare to Figures 5 and 6). This ensures dilation 1 between those points.

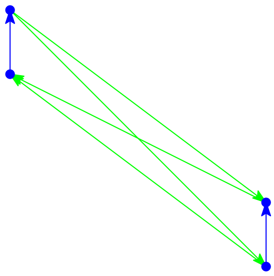

Figure 5 $K_{2,2}$ between oriented points with same orientation

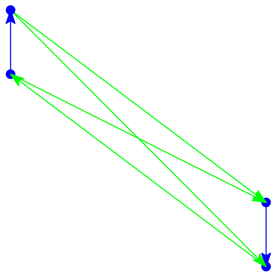

Figure 6 $K_{2,2}$ between oriented points with different orientation

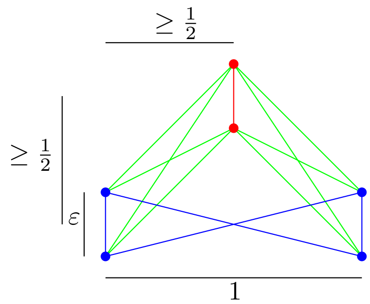

Figure 7 A wire (blue) can not be shortcut by $K_{2,2}$s (green)

Let p and p' be direct neighbours and p^* a third non-neighbouring point. Since p^* has at least distance $(0.5, 0.5)$ to p and p', the $K_{2,2}$s between p and p^* and p' and p^* do not affect that the wire between p and p' ensures equal orientation of the neighbours (compare to Figure 7).

The dilation of $t(p)$ and $b(p)$ is bounded by the dilation of p and its closest point p'. That is 1, both if p, p' are direct neighbours and not.

The two leaves of the tree of knowledge for every clause are not linked by a $K_{2,2}$. Figure 8 shows the two leaves of the tree at a clause, and G_φ at the clause. We can assume that G_φ is embedded as shown, in particular leaving the area directly above the clause empty. The two leaves are now linked by a *clause gadget*. We show how such a gadget looks like in Figure 9, more detailed in Figure 10.

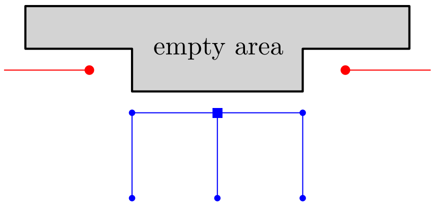

Figure 8 Embedded clause

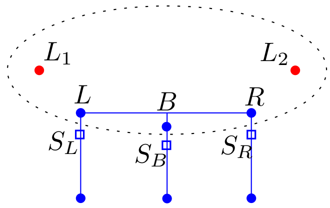

Figure 9 Clause gadget

The oriented points L (left), R (right) and B (bottom) are the ends of the variable wires of a clause. They are placed such that they lie just inside an ellipse with the locations of the leaves L_1 and L_2 as foci, and without any other points in the ellipse (compare to Figures 8 and 9). Stated differently, the triangles with endpoints L_1, L_2 and one of these three points, have nearly the same size, and any triangle with L_1, L_2, and a different oriented point has a larger perimeter. Adding edges as shown in Figure 10 guarantees that to obtain a short cycle through L_1, L_2 and one of these points, the orientation of that point has to be the same as of L_1 and L_2 (thus, the literal is "true".)

For each of $\{L, R, B\}$ there exists a *satellite* point, which is an oriented point on the variable wire, which is close but outside the ellipse. Its purpose is to make sure that the oriented dilation of L_1 (and likewise L_2) with L, B and R is stays below t' if if the corresponding literal does not satisfy the clause. We omitted all $K_{2,2}$ in the drawing. As described before, a $K_{2,2}$ exists between all unrelated oriented points, thus between all oriented points where there are no edges drawn in the figure.

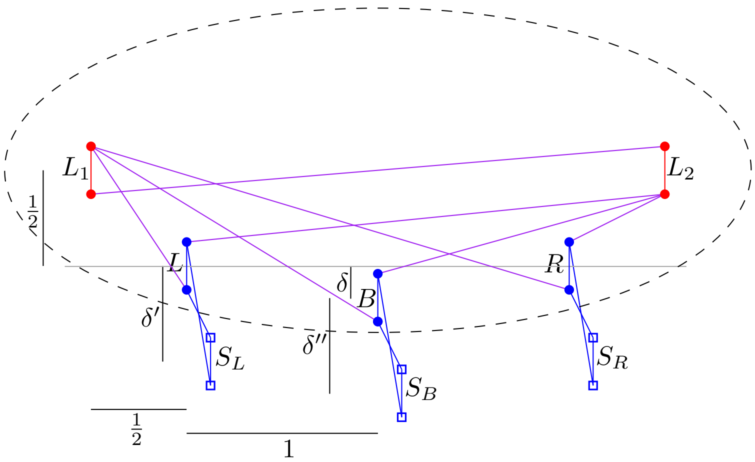

Figure 10 Detailed clause gadget

By setting $\delta = 0.1335$, $\delta' = 0.0303$ and $\delta'' = 0.35$, we obtain the following properties:

- The dilation between one of the points L_1, L_2 and one of the points L, B, R is lower or equal t', as a smallest cycle containing those points can use the related satellite point.
- If one of the oriented points L, B, R is oriented upwards, the dilation between L_1 and L_2 is smaller than $t' := 1.043$
- If none of the oriented points L, B, R is oriented upwards, the smallest cycle containing L_1 and L_2 either leaves the ellipse or takes at least two points from $\{L, B, R\}$ and thus their dilation is greater than t'.

Thus, formula φ is satisfiable if and only if there exists an orientation of our constructed graph with dilation at most 1.043. ◀

Following the construction in the proof of Theorem 2.1, Figure 11 illustrates the graph for the formula $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$ (see also Figure 4).

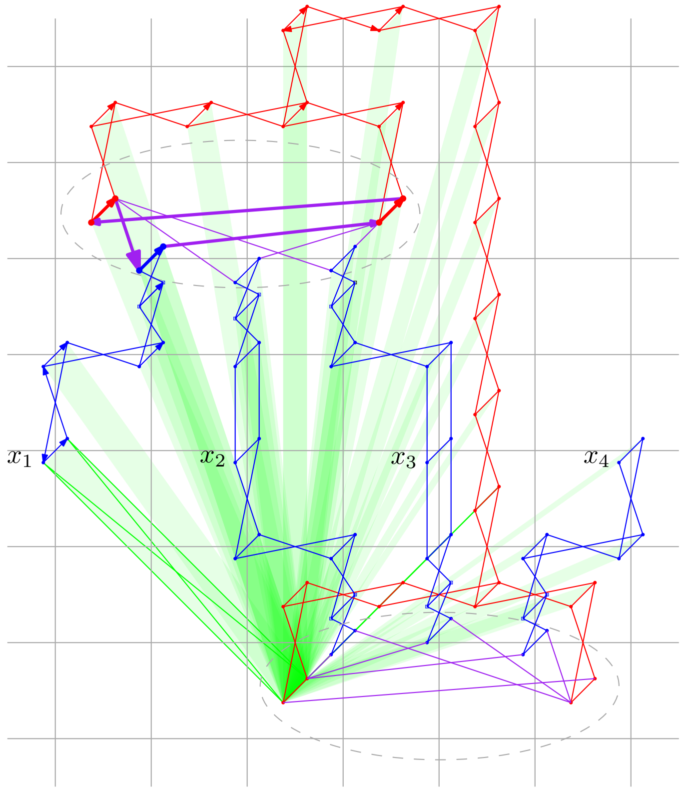

Figure 11 Graph constructed for $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$ (compare to Figures 1 and 4): For visibility, oriented points are placed diagonally instead of vertically. Only for one point, $K_{2,2}$s are indicated by green parallelograms. If the oriented point at the end of the wire from x_1 (blue) is –as indicated– oriented the same way as the tree of knowledge (red), this corresponds to setting it to true, resulting in an oriented cycle in the clause gadget (purple) that gives a dilation smaller than 1.043.

3 Bounding the dilation of tournaments

Buchin et al. [3] showed by example that there are (Euclidean) point sets for which no oriented t-spanner exists for $t < 2\sqrt{3} - 2 \approx 1.46$. For every (metric) point set P, they give an algorithm that returns a tournament $\overrightarrow{K}(P)$ on P with dilation $\mathrm{odil}(\overrightarrow{K}(P)) \leq 2$.

Our goal is to improve these bounds on the worst-case dilation $2\sqrt{3} - 2 \leq t \leq 2$ of the minimum dilation tournament. As a first step, we show a tight bound for sets of four points.

The complete graph on four points and its tournaments satisfy the following properties:

▶ Observation 3.1. For every undirected complete graph K_4 holds:

- K_4 contains $\binom{4}{3} = 4$ triangles.
- Every pair of these triangles shares exactly one edge.

- Every strongly connected tournament $\overrightarrow{K}_4$ contains exactly two *consistently* oriented triangles. This means the triangle is confined by an oriented cycle.

The following theorem gives a tight bound on the dilation of minimum dilation tournament on any metric point set of size four:

▶ **Theorem 3.2.** *For every point set P of size $|P| = 4$ embedded in a metric plane there is a tournament $\overrightarrow{K}(P)$ with dilation* $\text{odil}(\overrightarrow{K}(P)) \leq \frac{3}{2}$*. This bound is tight.*

Proof. We prove that the following algorithm computes an tournament $\overrightarrow{K}(P)$ with dilation $\text{odil}(\overrightarrow{K}(P)) \leq \frac{3}{2}$ for a point set $P = \{p_1, p_2, p_3, p_4\}$ embedded in a metric plane:

1. Let $\Delta_{p_1p_2p_3}$ be the shortest and $\Delta_{p_1p_2p_4}$ the second shortest triangle of the four triangles in K_4. Orient $\Delta_{p_1p_2p_3}$ and $\Delta_{p_1p_2p_4}$ consistently. That is always possible (compare to observation 3.1).
2. Orient the remaining edge between p_3 and p_4 such that the shortest oriented cycle $C_{\overrightarrow{K}(P)}(p_3, p_4)$ containing p_3 and p_4 is minimised.

By $d(p, p')$ we denote the weight of the edge between p and p'. Note that the weights satisfy triangle inequality.

We distinct cases by the orientation of the edge between p_3 and p_4, meaning

- $|C_{\overrightarrow{K}(P)}(p_3, p_4)| = d(p_1, p_2) + d(p_2, p_3) + d(p_3, p_4) + d(p_1, p_4)$ if
$$d(p_1, p_3) + d(p_2, p_4) \leq d(p_2, p_3) + d(p_1, p_4), \text{ or} \tag{1}$$
- $|C_{\overrightarrow{K}(P)}(p_3, p_4)| = d(p_1, p_2) + d(p_2, p_4) + d(p_3, p_4) + d(p_1, p_3)$ if
$$d(p_1, p_3) + d(p_2, p_4) > d(p_2, p_3) + d(p_1, p_4). \tag{2}$$

We show case (1), the other case can be proven analogously.

Since $\Delta_{p_1p_2p_3}$ and $\Delta_{p_1p_2p_4}$ are the shortest triangles and they are oriented consistently, the dilation of every pair of points is 1, except the pair p_3, p_4. So, we want to prove

$$t = \text{odil}(p_3, p_4) = \frac{d(p_1, p_2) + d(p_2, p_3) + d(p_3, p_4) + d(p_1, p_4)}{\min\{|\Delta_{p_3p_4p_1}|, |\Delta_{p_3p_4p_2}|\}} \leq \frac{3}{2}.$$

Assume $|\Delta_{p_3p_4p_2}| \leq |\Delta_{p_3p_4p_1}|$ otherwise the names of the points belonging to the shortest and second shortest triangle can be swapped. Since $\Delta_{p_1p_2p_3}$ and $\Delta_{p_1p_2p_4}$ are the two shortest triangles, it holds

$$d(p_1, p_2) + d(p_1, p_4) \leq d(p_3, p_4) + d(p_2, p_3), \text{ and} \tag{3}$$
$$d(p_1, p_2) + d(p_1, p_3) \leq d(p_3, p_4) + d(p_2, p_4). \tag{4}$$

Summing up the inequalities 1, 3 and 4 we achieve

$$\begin{aligned}
& 2d(p_1, p_3) + 2d(p_1, p_2) + d(p_2, p_4) && \leq 2d(p_3, p_4) + 2d(p_2, p_3) + d(p_2, p_4) \\
\Leftrightarrow \quad & 2\left(d(p_1, p_3) + d(p_1, p_2) + d(p_2, p_4) + d(p_3, p_4)\right) && \leq 4d(p_3, p_4) + 2\left(d(p_2, p_3) + d(p_2, p_4)\right) \\
& && \overset{\Delta\text{-ineq.}}{\leq} 3\left(d(p_3, p_4) + d(p_2, p_3) + d(p_2, p_4)\right) \\
\Leftrightarrow \quad & \text{odil}(p_3, p_4) = \tfrac{d(p_3,p_4)+d(p_1,p_3)+d(p_1,p_2)+d(p_2,p_4)}{d(p_3,p_4)+d(p_2,p_3)+d(p_2,p_4)} && \leq \frac{3}{2}.
\end{aligned}$$

For tightness, we show there is a point set P with $|P| = 4$, such that every strongly connected tournament on P has dilation $t = \frac{3}{2}$. The following metric give such an point set: $d(p_1, p_3) = d(p_2, p_3) = d(p_3, p_4) = 1$ and $d(p_1, p_2) = d(p_1, p_4) = d(p_2, p_4) = 2$. Taking into account mirroring and rotation, Figure 12 lists all strongly connected tournaments on P. We see that every tournament is a 1.5-spanner. ◀

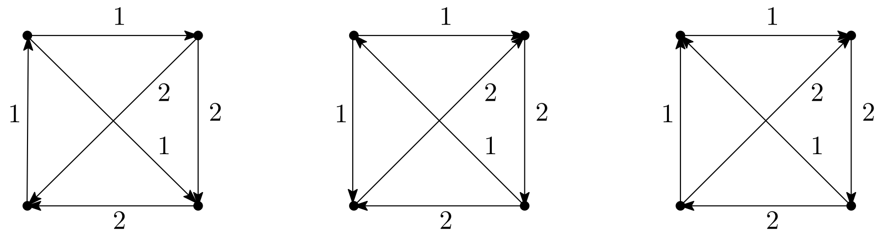

Figure 12 A metric point set where every connected tournament is a 1.5-spanner

4 Conclusion

We have shown that orienting a given geometric graph to minimise the oriented dilation is NP-hard. The complexity of this problem when restricting the graph class remains open. In particular: Is the problem NP-hard for planar graphs, or for complete graphs?

In the second part of the paper we studied the oriented dilation of metric point sets of size 4, i.e., with the K_4 as underlying graph. We proved that the oriented dilation is at most 1.5, while there are instances where it is tight. We know that in general the oriented dilation of K_n on metric instances can be upper-bounded by 2. Is it strictly less than 2 also for $n > 4$? Even for Euclidean instances this is open.

As noted in [3], in many applications some bi-directed edges might be allowed. This opens up a whole new set of questions on the trade-off between dilation and the number of bi-directed edges. Since this is a generalisation of the oriented case, our hardness proof also applies to such models.

Acknowledgements. We thank Guangping Li and Marco Ricci for helpful discussions.

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