On Orbital Labeling with Circular Contours*

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Abstract

Schematic depictions in text books and maps often need to label specific point features with a text label. We investigate one variant of such a labeling, where the image contour is a circle and the labels are placed as circular arcs along the circumference of this circle. To map the labels to the feature points, we use orbital-radial leaders, which consist of a circular arc concentric with the image contour circle and a radial line to the contour. In this paper, we provide a framework, which captures various dimensions of the problem space as well as several polynomial time algorithms and complexity results for some problem variants.

1 Introduction

Map labeling is an extensively studied topic in computational geometry [1,7,12] that typically involves annotating feature points with names or additional descriptions, ensuring non-overlapping annotations. While traditional maps often use *internal* label positions next to the feature points [11], *external* labeling models [4] place labels remotely along the contour of a bounding shape and connect them to their feature points by crossing-free leaders. This model is frequently used in applications, where feature points are dense, the details of a map or an illustration should not be obscured by labels, or labels are relatively large, e.g., in

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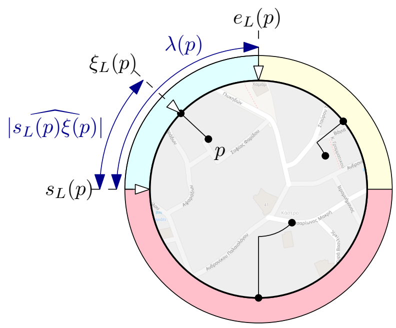

Figure 1 An orbital labeling on a map for illustrating our notation.

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This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

anatomy atlases or assembly drawings. In this paper we study a novel variant of external labeling with a circular bounding shape, e.g., for displays of smartwatches; see Figure 1. The circular map is displayed in the center of the display and each label is bent and turned into a segment of the circular boundary of the map; we call these labels *orbital* labels. This is a special case of external and boundary labeling [3]. We assume that the lengths of the orbital labels are normalized and sum up to the perimeter of the boundary of the map. Previous research on circular map display considered either multirow circular labels where the sum of label lengths does not equal the map's boundary length [8], radial labels [2, 6], or horizontal labels [6, 9, 10]. The latter two settings are relevant for circular maps on rectangular displays but not suitable for circular displays with a narrow annulus for labels.

Formally, we assume that we are given a disk D in the plane $\mathbb{R}^2$. The disk contains n points $P = \{p_1, \dots, p_n\}$. We call the set P of points *features* and we refer to the boundary of the disk as *the boundary* B. Every feature $p \in P$ has an associated *label* which represents additional information that is to be placed along a circular arc on the boundary starting at a point $b_1 \in B$ and ending at a point $b_2 \in B$. The circular arc along B is denoted as $\widehat{b_1 b_2}_B$. Usually, the start and endpoint of the label are not fixed in the input, however, the length of the arc is part of the input. We represent the associated label simply as a number $\lambda(p)$, which indicates the length of the associated label. We assume that $\sum_{i=1}^{n} \lambda(p_i)$ is equal to the circumference of D, i.e., if all labels are placed non-overlapping then there are no gaps between the arcs on B.

In a *labeling* L, every feature $p \in P$ is assigned a label with starting point $s_L(p) \in B$ and an endpoint $e_L(p) \in B$, s.t., $|\widehat{s_L(p) e_L(p)}| = \lambda(p)$. We assume that all labels are pairwise non-overlapping. Additionally, every feature p is connected to its label via a *leader*. In this paper, we consider *orbital-radial* leaders, which consist of two parts: (1) starting at the feature p with a (possibly empty) orbital circular arc that ends at a bend point q, and (2) a radial segment that connects q to the boundary B; see Figure 1. We call the leader endpoint, i.e., the point where the leader B the *port* $\xi_L(p)$ of the leader starting at p. Note that q has the same distance to the circle center as p since the first part of the leader is an orbital-radial arc. We denote the length of the leader of feature p by $l(p)$. Let the *port ratio* $\rho_L(p) = \frac{|\widehat{s_L(p)\xi(p)}|}{\lambda(p)}$ be the ratio of the arc from the starting point to the port and the arc from the start-point to the end-point. Now, we define the generic orbital labeling problem.

▸ **Problem 1.** *Given a disk D, containing n feature points P compute a labeling L, in which all leaders are pairwise non-intersecting and the sum of leader lengths is minimal.*

We discuss different variants of the problem and give an overview of the obtained running times. The paper provides detailed explanations for a subset of the results. The remaining results and proofs of statements marked with a star (*) will be presented in the full version.

2 Problem Space

In the following, we discuss the dimensions of our problem space. For the different dimensions, we use the notation based on the *COSA*-Orbital Boundary Labeling scheme and use each letter to describe the variants for the respective dimension.

- $[C]$ **Candidate port positions on the boundary.** If we are given a set C of candidate positions on B and in any valid labeling L we require that for any port $\xi \in \Xi_L$ we have $\xi \in C$, we say *the port candidates are locked* (and use the symbol $C^{🔒}$) otherwise they are *free* ($C^{🔑}$).

Table 1 A tabular overview of the problem space and our results. Empty cells remain open. Results marked with † will be presented in the full version.

<table>
<tr><th colspan="3"></th><th>$A^{\text{🔒}}_{\equiv}$</th><th>$A^{\text{📍}}_{\equiv}$</th><th>$A^{\text{🔒}}_{\Xi}$</th><th>$A^{\text{📍}}_{\Xi}$</th></tr>
<tr><td rowspan="4">$C^{\text{🔒}}$</td><td rowspan="2">$O^{\text{🔒}}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n^2|C|)$ †</td><td>$O(n^2|C|)$ †</td><td></td></tr>
<tr><td>S_{Ξ}</td><td>$O(n^2|C|)$ †</td><td></td><td>$O(n^2|C|)$ †</td><td></td></tr>
<tr><td rowspan="2">$O^{\text{📍}}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n|C|^3)$ †</td><td></td><td></td></tr>
<tr><td>S_{Ξ}</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="4">$C^{\text{📍}}$</td><td rowspan="2">$O^{\text{🔒}}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n^2)$ [Sec. 4.1]</td><td>$O(n^2)$ [Sec. 4.1]</td><td></td></tr>
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<tr><td rowspan="2">$O^{\text{📍}}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n^5)$ [Sec. 4.2]</td><td></td><td></td></tr>
<tr><td>S_{Ξ}</td><td>NP-c [Sec. 4.2.1]</td><td>NP-c [Sec. 4.2.1]</td><td>NP-c [Sec. 4.2.1]</td><td></td></tr>
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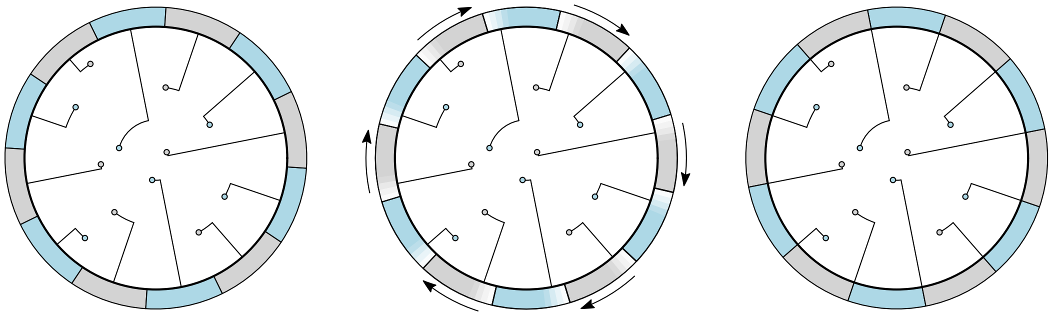
- [O] **Order.** Next, we consider the cyclic order of labels around B. If a certain label order is pre-specified we say the label order is locked ($O^{\text{🔒}}$); otherwise, for the unconstrained setting, we say the label order is free ($O^{\text{📍}}$).
- [S] **Size of labels.** Then, we distinguish the setting where $\forall p \in P : \lambda(p) = 1$, in which case we say that the label size is *uniform* ($S_{\equiv}$), otherwise the label size is *non-uniform* (S_{Ξ}).
- [A] **Port position on labels.** Lastly, we distinguish different positions of the ports on the labels. We differentiate between *uniform* port ratios, where $\forall i, j\ \rho_L(p_i) = \rho_L(p_j)$, and *non-uniform* port ratios. We also distinguish between the ratios being predefined as part of the input, in which case we call the ratios *locked*, or not, in which case we call them *free*. We obtain the following four settings:
  - Ratios are uniform and locked to a value $k \in [0, 1]$ given in the input ($A^{\text{🔒}}_{\equiv}$).
  - Ratios are uniform and free, i.e., we have to find a value $k \in [0, 1]$ for the ratios ($A^{\text{📍}}_{\equiv}$).
  - Ratios are non-uniform and locked, meaning, we are given a set $K = \{k_1, \dots, k_n\}$ of ratios, s.t., in a valid labeling L, we have $\rho_L(p_i) = k_i$ ($A^{\text{🔒}}_{\Xi}$).
  - Ratios are non-uniform and free, i.e., ports can be chosen freely and independently ($A^{\text{📍}}_{\Xi}$).

For our problem variants, we use the notation based on the $COSA$-ORBITAL BOUNDARY LABELING scheme where we substitute C, O, S and A with $C^{\text{📍}}/C^{\text{🔒}}$, $O^{\text{📍}}/O^{\text{🔒}}$, $S_{\equiv}/S_{\Xi}$ and $A^{\text{📍}}_{\Xi}/A^{\text{📍}}_{\Xi}/A^{\text{🔒}}_{\equiv}/A^{\text{🔒}}_{\Xi}$, respectively. An overview of all variants and our results can be seen in Table 1. Whenever a statement applies to all variants along a certain dimension of the problem space, we drop the sub- or superscript of C, O, S, or A. For example, $C^{\text{📍}}O^{\text{🔒}}SA^{\text{🔒}}_{\equiv}$ refers to the variants where the port candidates are free ($C^{\text{📍}}$), the order is locked ($O^{\text{🔒}}$), the label sizes could be fixed to be uniform or they could be non-uniform (S) and all port ratios are fixed to a given value ($A^{\text{🔒}}_{\equiv}$). Therefore, $C^{\text{📍}}O^{\text{🔒}}SA^{\text{🔒}}_{\equiv}$ covers a set of two problem variants.

3 Uniformly Spaced Ports

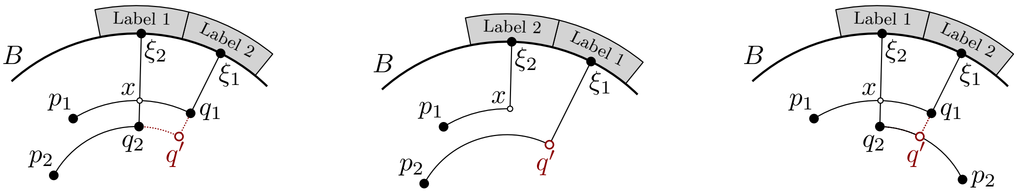
Using a simple argument about shifting labels illustrated in Figure 2, we can show the following equivalence.

▸ **Observation 1.** *All problems in $COS_{\equiv}A^{\text{🔒}}_{\equiv}$ are equivalent over all $k \in [0, 1]$. Similarly all problems in $COS_{\equiv}A^{\text{📍}}_{\equiv}$ are equivalent over all $k \in [0, 1]$.*



(a) Uniform ratio of 0.5. **(b)** Clockwise rotation. **(c)** Uniform ratio of 0.

Figure 2 Any solution with uniform label sizes and a uniform ratio (e.g., 0.5) (a) can be rotated (b) to obtain a solution of any other ratio, e.g., 0 (c).



(a) Case 1 **(b)** Rerouted leaders **(c)** Case 2

Figure 3 Given a free label order O^f we can reroute the leaders to arrive at a crossing-free solution with a shorter total leader length.

This equivalence is based on the fact that the ports in these problems are necessarily equally spaced, which is only the case if both the label size and the port ratio are uniform. Based on the same property, we make the following statement, visualized in Figure 3.

▸ **Lemma 3.1 (*).** *Given an instance of a problem variant in $COSA_{\equiv}$ any leader-length minimal labeling L is crossing-free, assuming that all feature points in P lie on circles of different radii concentric with D.*

4 Free Candidates

In this section, we consider the problem set $COSA$. Intuitively, these are problem sets, where solutions can be continuously rotated around B. Let $g : P \times [0, 2\pi] \to \mathbb{R}$ be a function which maps a feature $p \in P$ and an angle θ to the length of a leader that connects p and a port on B, s.t., the orbital segment of the leader spans the angle θ. Let r be the radius of the circle containing p concentric with D. If D has a circumference of $\mathcal{C}$ it has a radius of $\frac{\mathcal{C}}{2\pi}$. Then $g(p, \theta) = \frac{\mathcal{C}}{2\pi} - r + r\theta$; see Figure 4.

▸ **Observation 2.** *The function $g(p, \theta)$ is linear in θ.*

The total leader length of a labeling L can obtained as $h(L) = \sum_{i=1}^{n} g(p_i, \theta(p_i))$, where $\theta(p_i)$ is the angle spanned by the orbital segment of the leader connected to p_i.

Note that by fixing a port ξ on the boundary for a feature p, there are two orbital-radial leaders by which we could choose to connect them (with a clockwise or a counter-clockwise orbital segment). We call these *clockwise* and *counter-clockwise* leader, respectively.

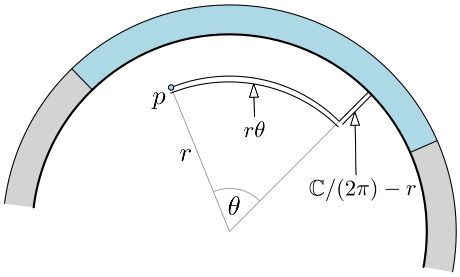


Figure 4 Determining the leader length by the length of the orbital and radial segment.

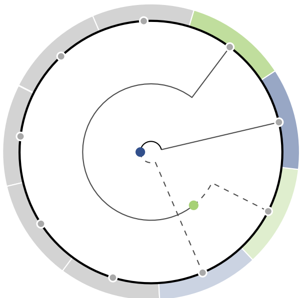


Figure 5 The port position of the inner-most feature (blue) determines labeling of other features: Depending on the labeling of the inner-most feature, the green feature point has access to different candidate ports (dark labels and solid leaders vs. light labels and dashed leaders).

▸ **Observation 3.** *The inner-most feature, i.e., the feature which lies on a circle concentric with D whose radius is smallest among all features can always be labeled with a clockwise or a counterclockwise leader.*

Observe that the leader of the inner-most feature p uses a radial line segment s starting on a circle concentric with D, which does not contain any other feature of P. Consider any other feature p' and any other point ξ' on B, then, p' and ξ' can be connected either with a clockwise or a counter-clockwise leader; see Figure 5. The orbital segments of the clockwise and the counter-clockwise leader of p' together form an entire circle concentric with D containing p and, hence, one of them has to intersect s.

▸ **Observation 4.** *A leader of the inner-most feature determines for every other leader γ connecting a feature and a point on B if γ is a clockwise or counter-clockwise leader.*

4.1 Locked Order.

Next, we consider problems in $C^{🔑}O^{🔒}SA$.

▸ **Lemma 4.1.** *For $C^{🔑}O^{🔒}S_{\equiv}A^{🔒}_{\equiv}$, $C^{🔑}O^{🔒}S_{\equiv}A^{🔒}_{\equiv}$ and by extension $C^{🔑}O^{🔒}S_{\equiv}A^{🔑}_{\equiv}$, the choice of a port point on B for the inner-most feature determines all other label placements including their port positions as well as their leaders. This also includes the length of their leaders and the angle that is spanned by the orbital segment of these leaders.*

Proof. This lemma directly follows from Observation 4 and the key point that the placement of one label not only determines the placement of others but also their port positions. ◂

With this, we state a method of solving the four problems $C^{🔑}O^{🔒}SA^{🔒}$ and by extension $C^{🔑}O^{🔒}S_{\equiv}A^{🔑}_{\equiv}$ (recall Observation 1). By Lemma 4.1 the exact position of the port of the

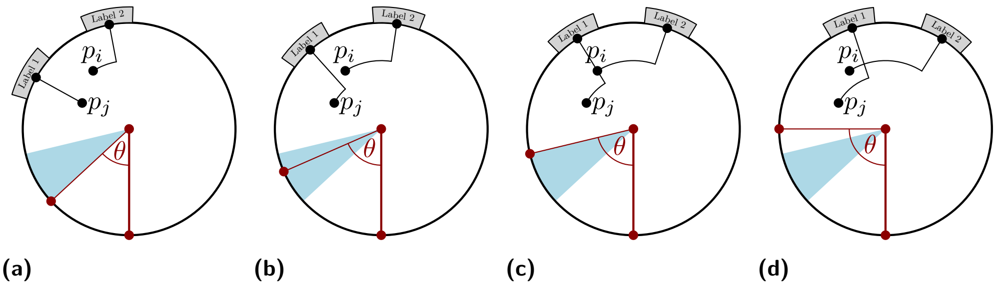


Figure 6 Two clockwise leaders whose ports are rotated, s.t., (a) the length of the orbital segment of p_j is 0, (b) the leaders are non intersecting, (c), the radial segment of p_j contains p_i and (d) the leaders intersect. The admissible range of θ is shown in blue.

inner-most feature p_1 is the only degree of freedom when choosing a labeling. By Lemma 4.1, we immediately obtain the angle θ_1 spanned by its orbital segment. By Lemma 4.1, we likewise obtain all angles $\theta_2, \ldots, \theta_n$ of all the other leaders. Therefore we can express the functions $g(p_2, \theta_2), \ldots, g(p_n, \theta_n)$ all as piecewise linear functions of θ_1, which consist of exactly two linear pieces. The sum over all of these functions is therefore a piece-wise linear uni-variate function and we can find the minimum of it in $O(n)$ time.

To guarantee that a solution (if it exists) found in this way is crossing free we compute an *admissible range* $I_{i,j}$ for θ_1, s.t., if $\theta_1 \in I_{i,j}$ the leaders of p_i and p_j are crossing free; see Figure 6. $I_{i,j}$ is one continuous interval and therefore we can in $O(n^2)$ time determine all ranges as well as their (also continuous) intersection if it exists. Then we either restrict our search for a minimum to this intersection or – if the intersection is empty – know that no solution exists.

4.2 Free Order.

These are the problems in $C^{\nearrow}O^{\nearrow}SA$. We will use a reduction of some of these problems to a non-circular variant called Boundary Labeling [5].

▸ **Lemma 4.2.** *In any (crossing-free) labeling of an instance of a problem in $C^{\nearrow}O^{\nearrow}SA$, there exists a point $b \in B$, s.t., db does not intersect any leader, where d is the center of D.*

Proof. Let x be the smallest angle between two points, two ports, or a point and a port in an optimal labeling L (measured with 0 as the center). Consider the radial segment of the leader of the inner-most feature p_1, and assume w.l.o.g. that the orbital segment is clockwise. Set b' to be $\xi(p_1)$ but rotate it clockwise by $x/2$. Since the leader of p_1 does not intersect any other leader and the next feature or port is at least at an angle x in clockwise position from $\xi(p_1)$, the segment db' must now be crossing free. ◂

The previous lemma argues the existence of this *splitting line* in any labeling. Next, we state that we only need to consider $O(n^2)$ possibilities for such a line.

▸ **Lemma 4.3 (*).** *For any problem in $C^{\nearrow}O^{\nearrow}S_{\equiv}A_{\equiv}$ there are only n^2 possibilities for the port of the inner-most feature.*

We obtain an algorithm for the problems $C^{\nearrow}O^{\nearrow}S_{\equiv}A_{\equiv}$ by creating an instance of Boundary Labeling for every possible port of the inner-most feature and using the $O(n^3)$ algorithm [5] to obtain a labeling in a total time of $O(n^5)$; see algorithm in the full version.

▸ **Theorem 4.4.** *Any problem in $C^{\nearrow}O^{\nearrow}S_{\equiv}A_{\equiv}$ can be solved exactly in $O(n^5)$.*

4.2.1 Non-uniform label sizes are NP-hard.

Finally, we investigate problems without candidate ports, a free order on the labels and non-uniform label sizes. For $C^{\nearrow}O^{\nearrow}S_{\Xi}A_{\equiv}$, we show NP-hardness and the hardness of $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\boxdot}_{\equiv}$ extends to $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\boxdot}_{\Xi}$. $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\nearrow}_{\Xi}$ remains open.

▸ **Theorem 4.5** (*). *Given an instance of* $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\nearrow}_{\equiv}$, $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\boxdot}_{\equiv}$ *or* $C^{\nearrow}O^{\nearrow}S_{\Xi}A^{\boxdot}_{\Xi}$ *together with* $k \in \mathbb{R}$ *it is (weakly)* NP*-hard to decide if there exists a labeling* L *with a total leader length of less than* k*.*

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