Intersection patterns of geometric set systems

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Inria

The pitch...

- ▷ Can we generalize some of the benefits of **convexity**?

E.g. algorithmic.

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E.g. algorithmic.

- ▷ This talk: three **methodologies** to do so.

  1. *Mapping simplicial complexes into* $\mathbb{R}^d$*.*
  2. *Patterns in hypergraphs.*
  3. *Homological properties of nerve complexes.*

One benefit of convexity:

Helly's theorem...

... and why we keep generalizing it.

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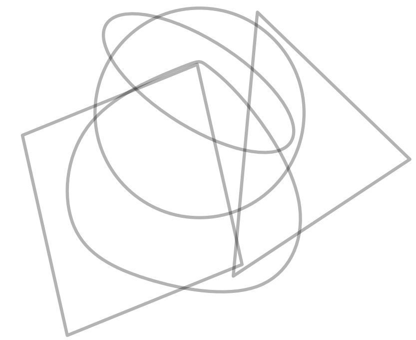

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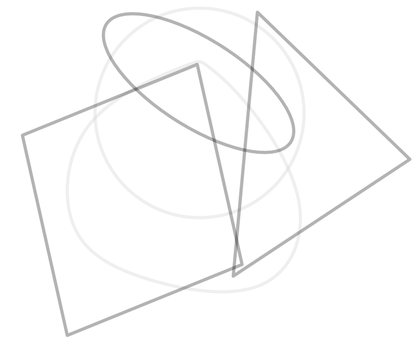

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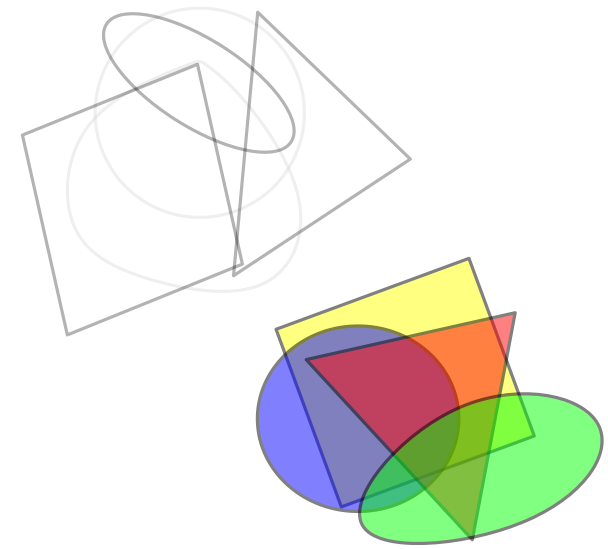

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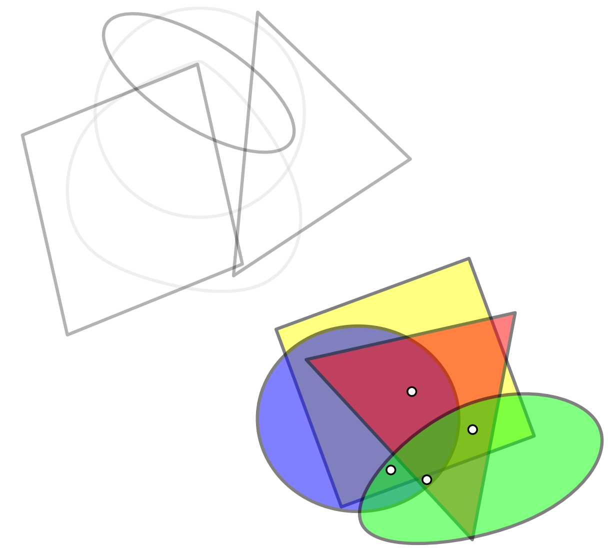

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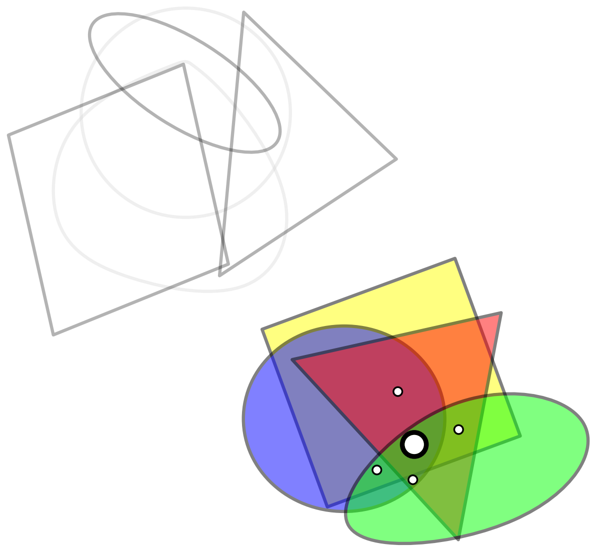

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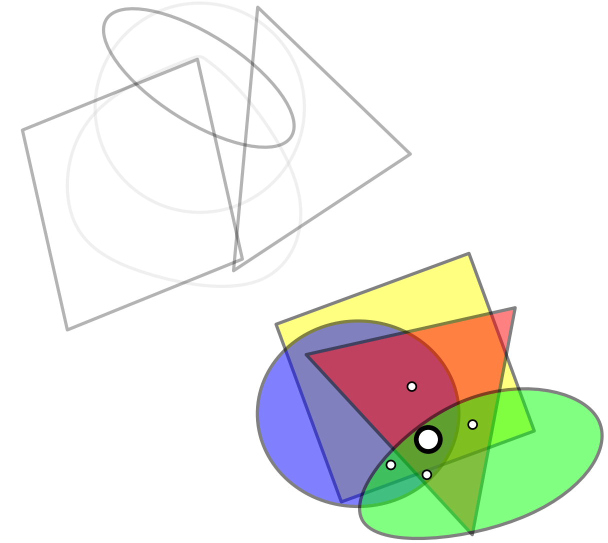

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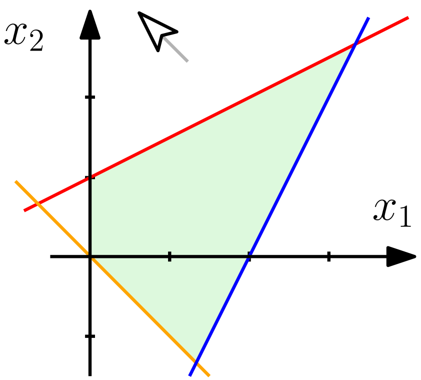

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d constraints **suffice**.

$d = \#$ *variables.*

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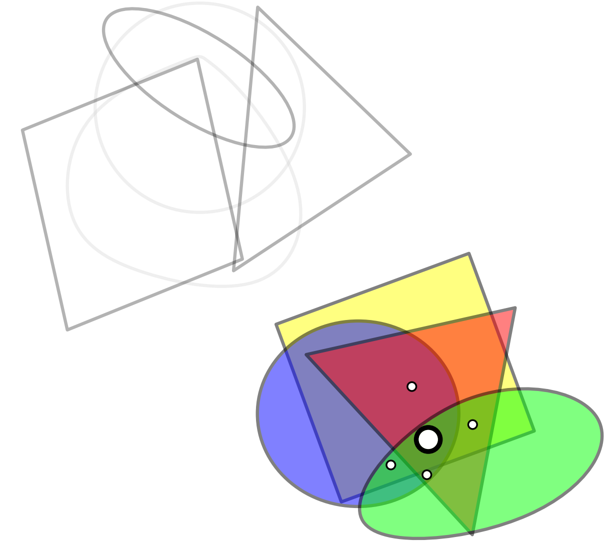

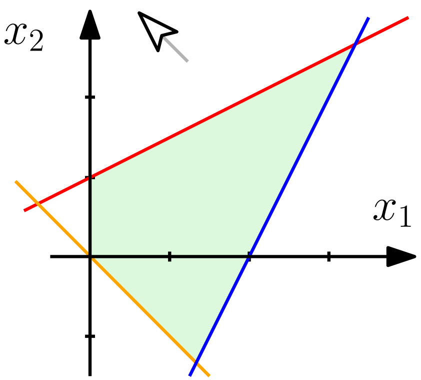

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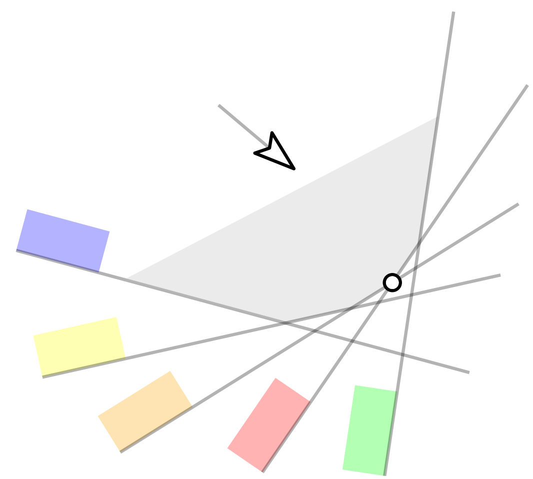

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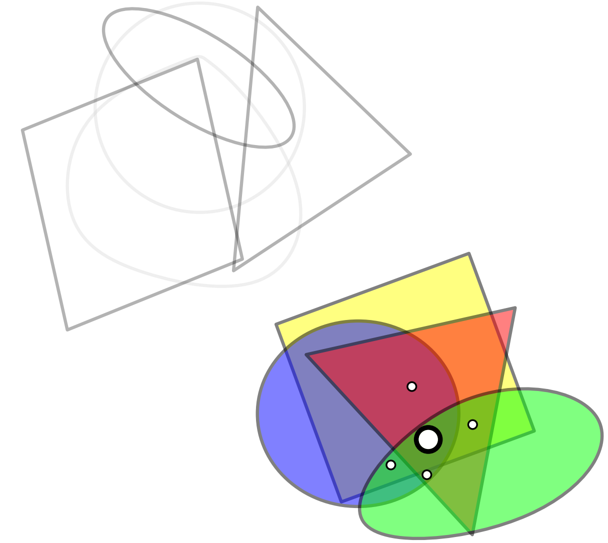

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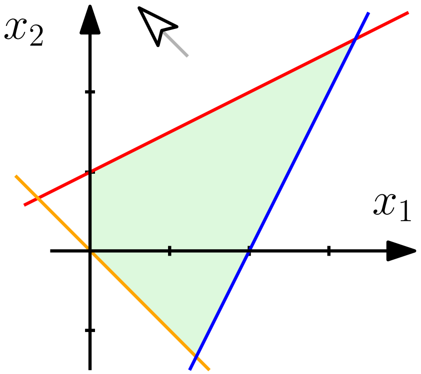

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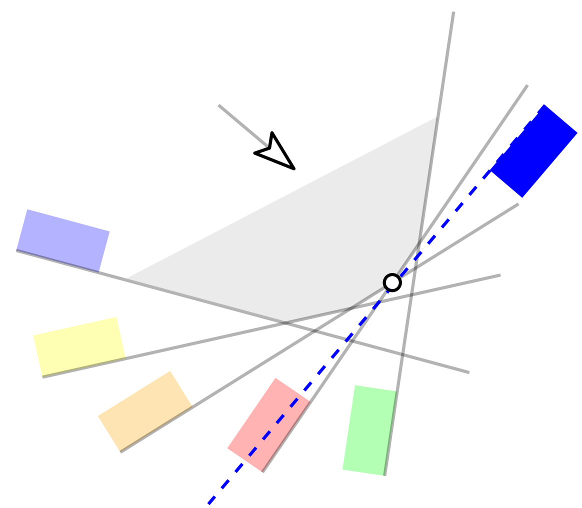

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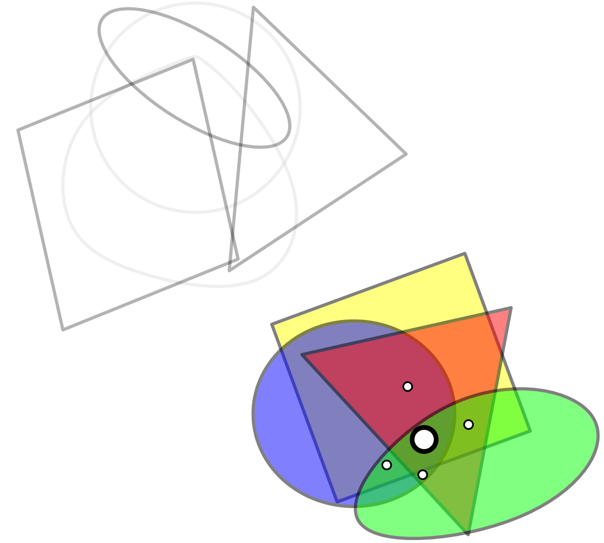

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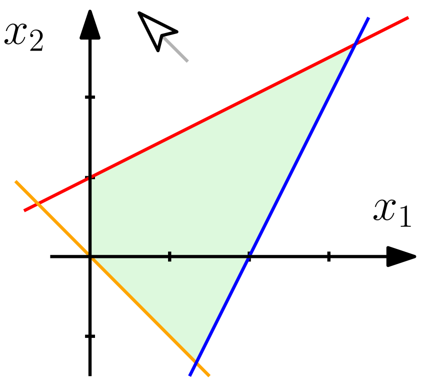

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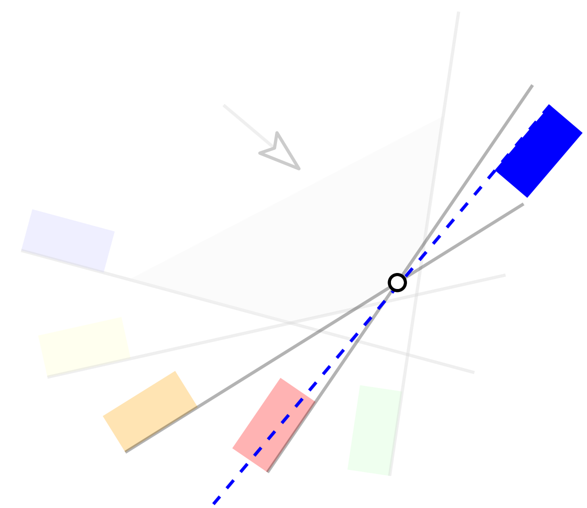

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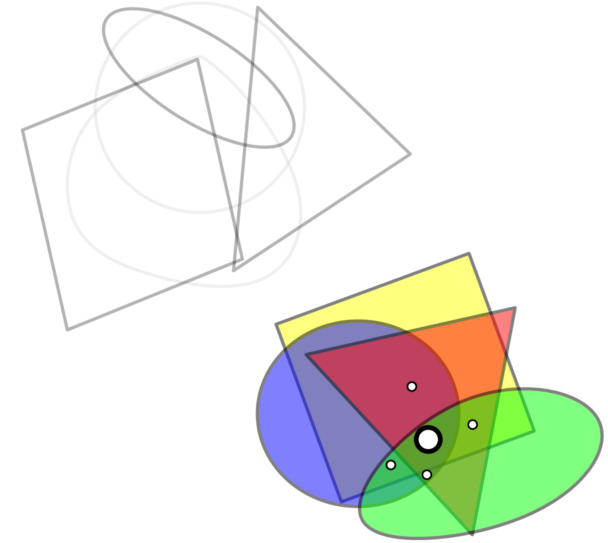

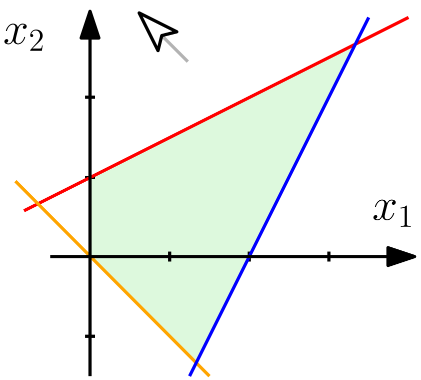

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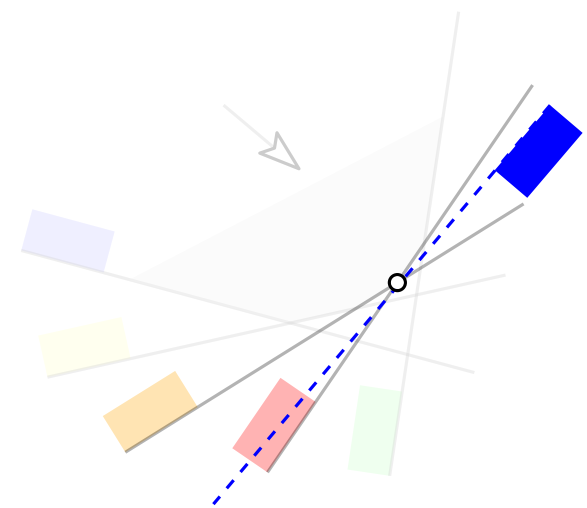

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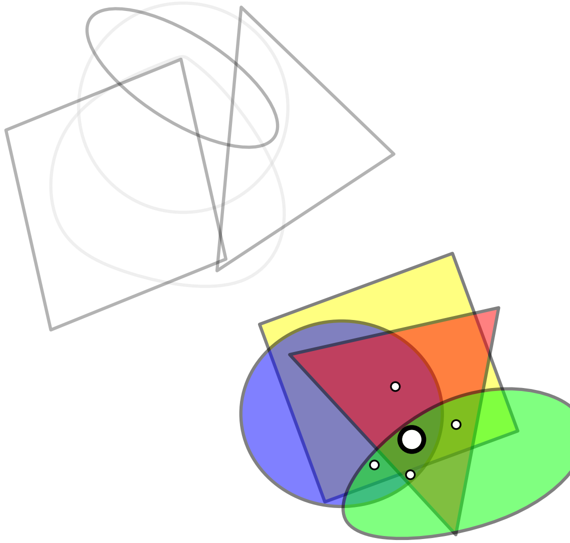

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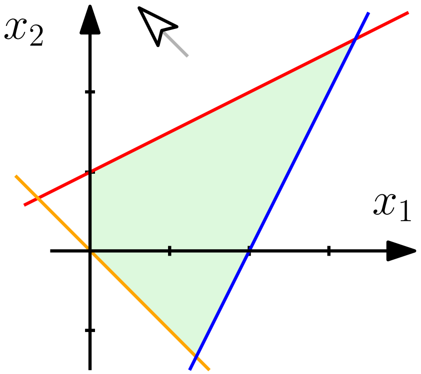

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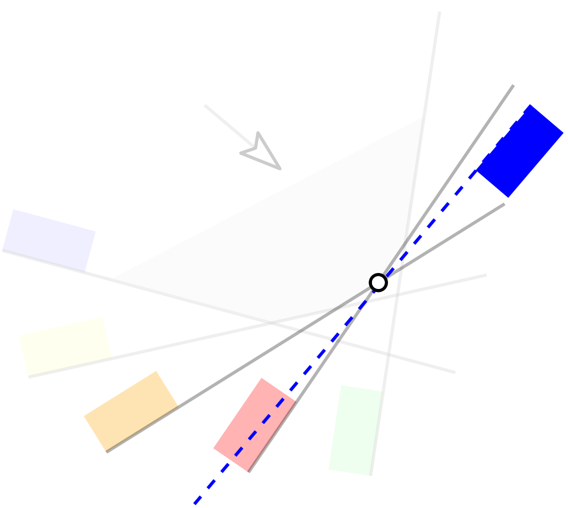

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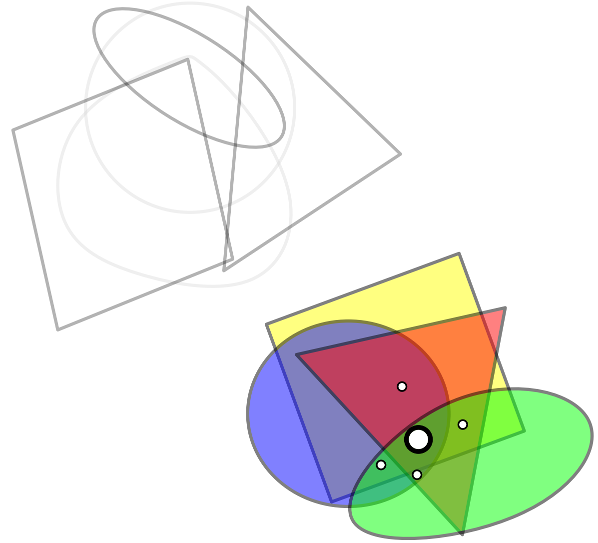

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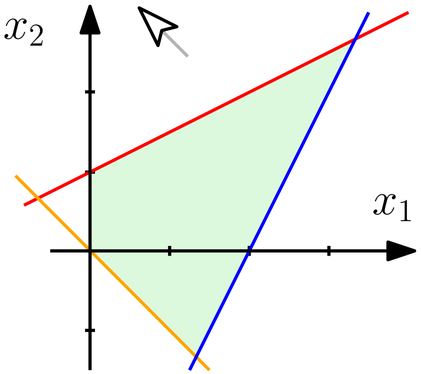

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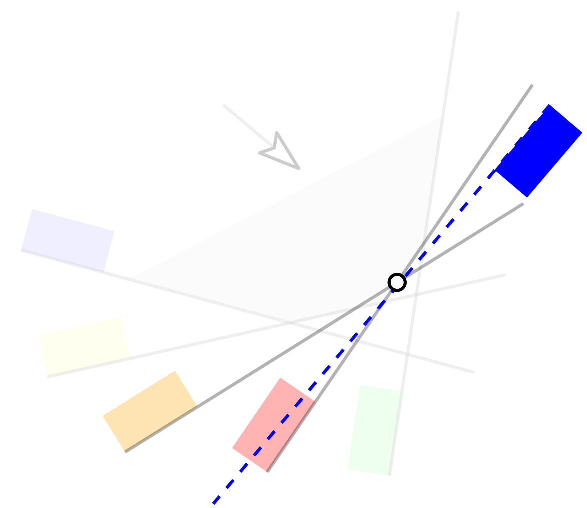

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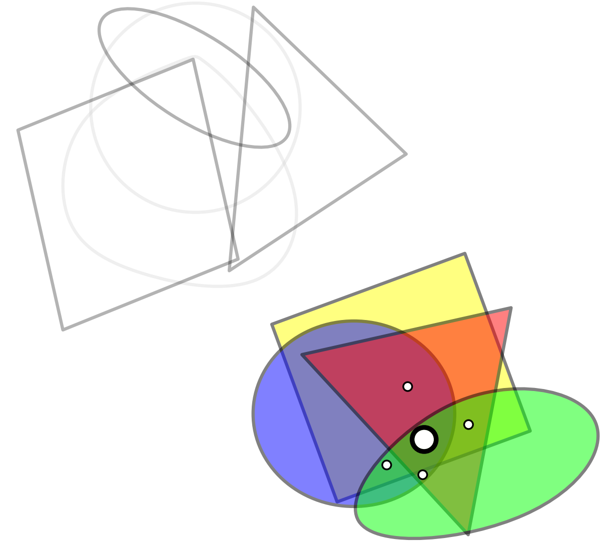

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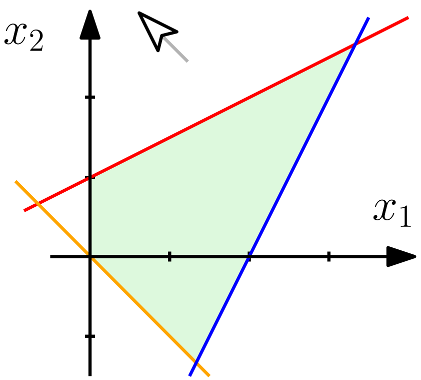

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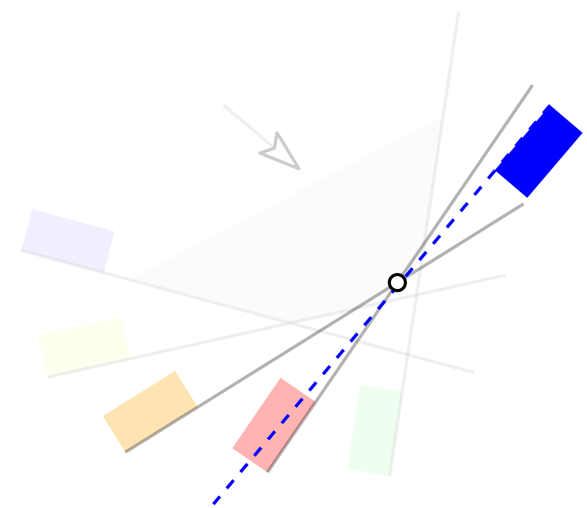

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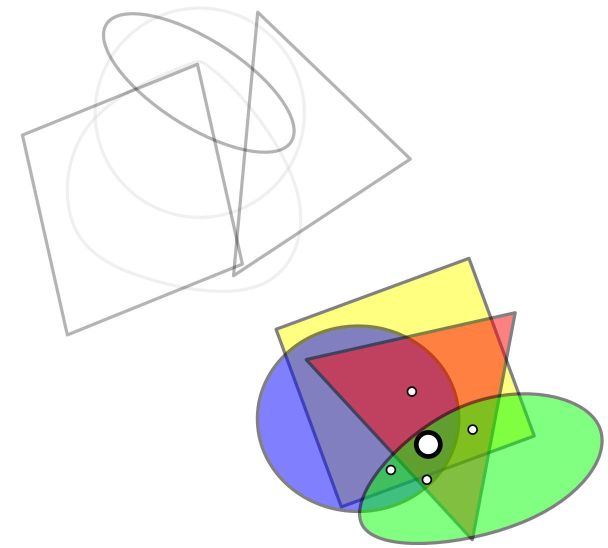

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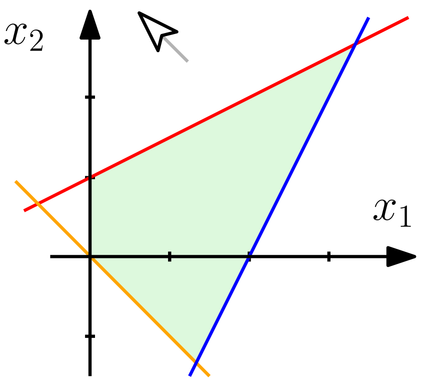

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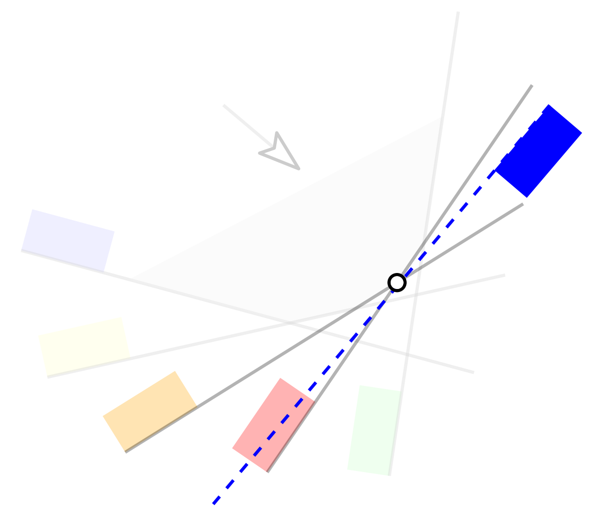

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Some subset B of $\leq d$ constraints is involved in **every** doubling.

Double only if weight(unsatisfied)$< \frac{1}{2d}$weight(all).

$d2^{\frac{k}{d}} \leq$ weight$(B) \leq$ weight(all) $\leq \left(1 + \frac{1}{2d}\right)^k n$

(Non-doubling rare for samples of size $4d^2$.)

min

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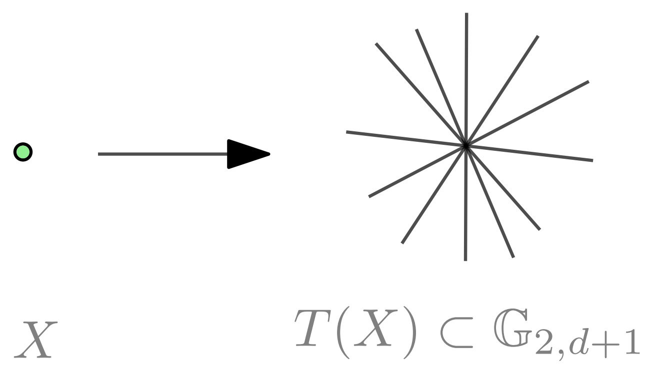

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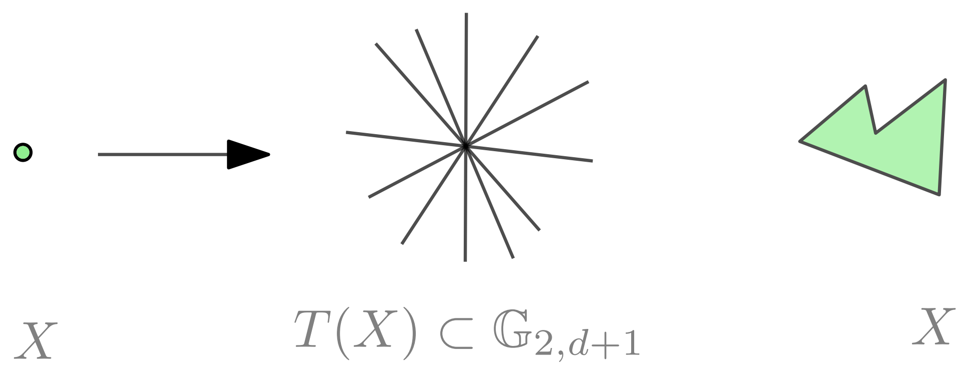

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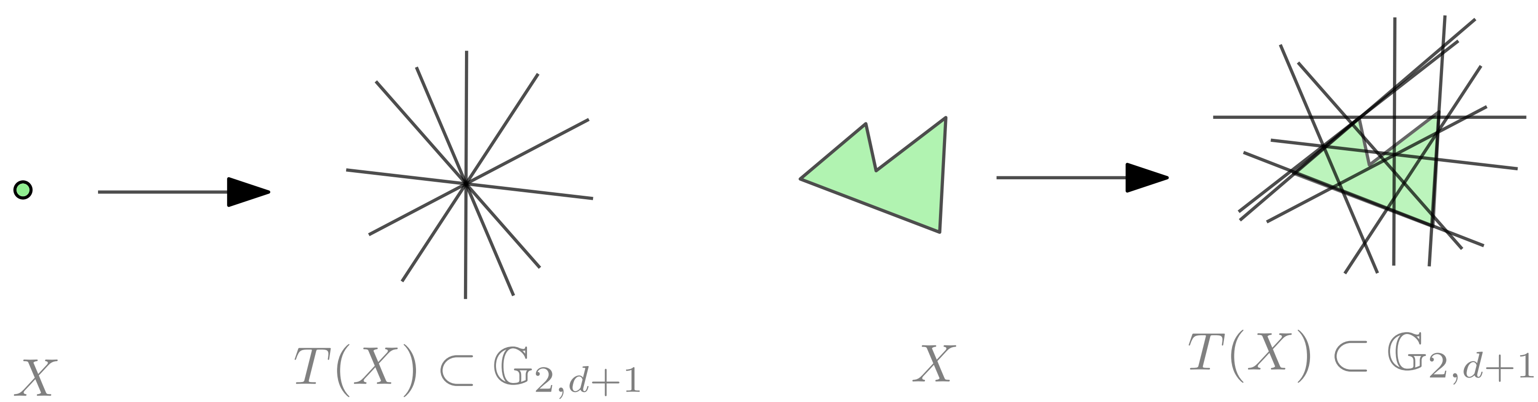

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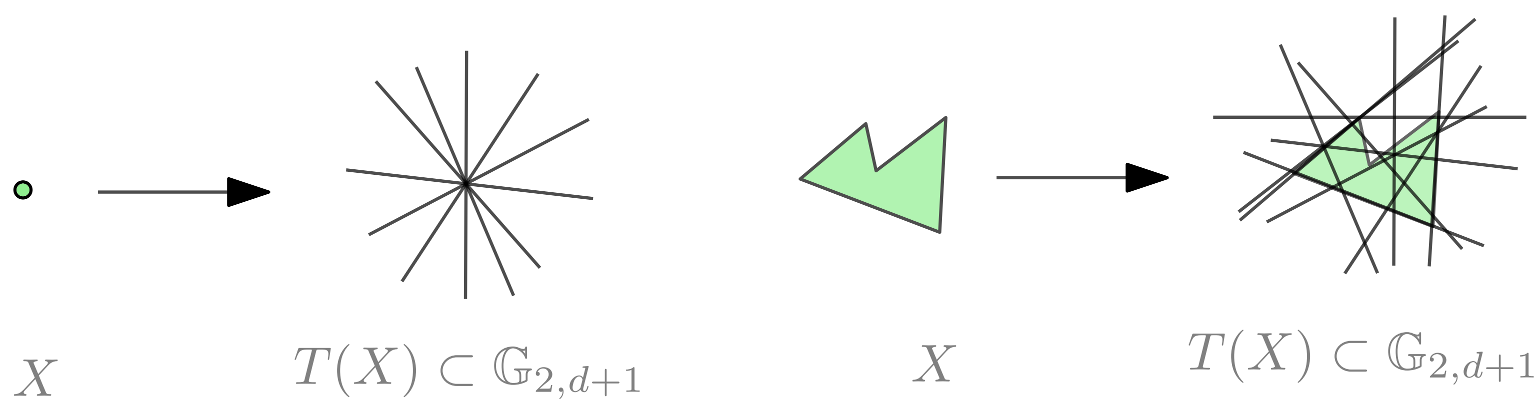

For which X do the sets $T(X)$ satisfy a **Helly-type** theorem?

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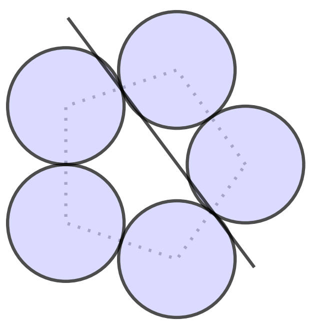

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[Danzer 1957]

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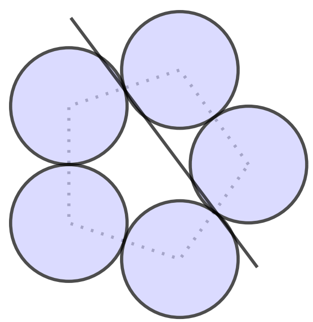

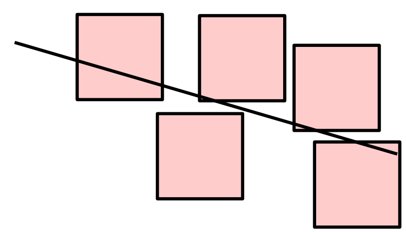

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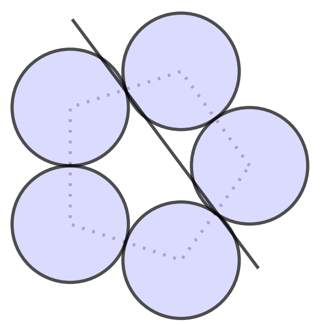

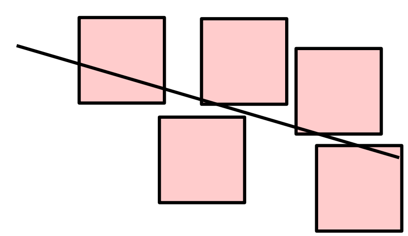

→ translates ?

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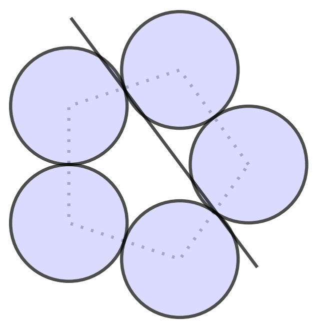

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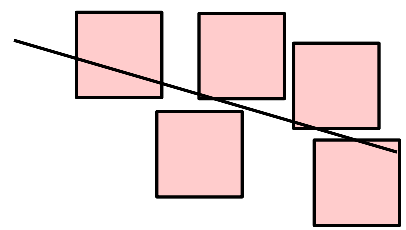

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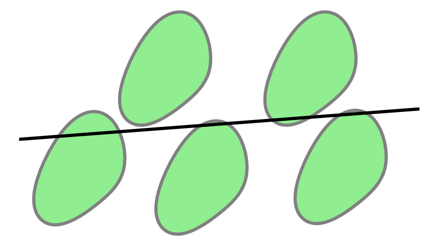

[Tverberg 1989]

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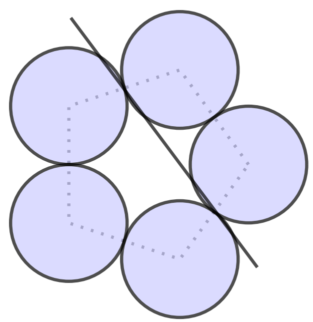

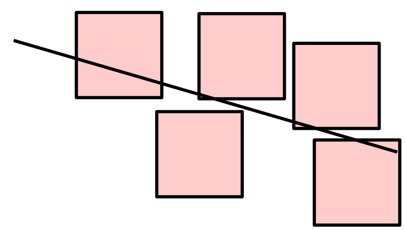

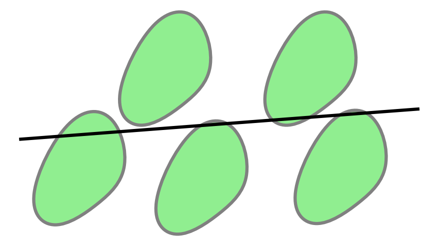

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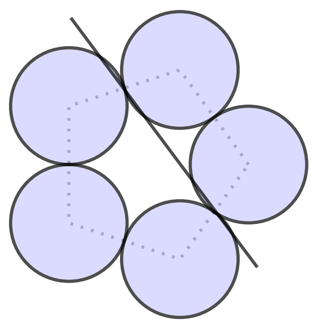

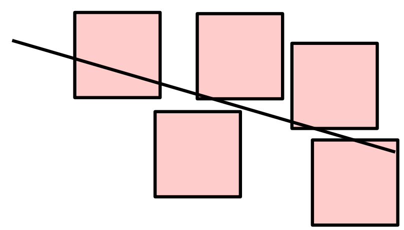

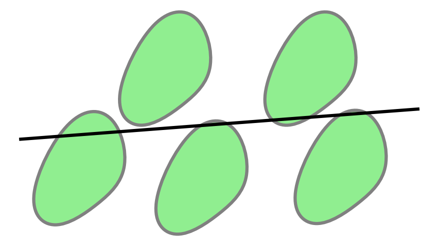

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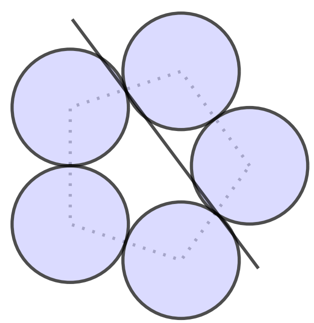

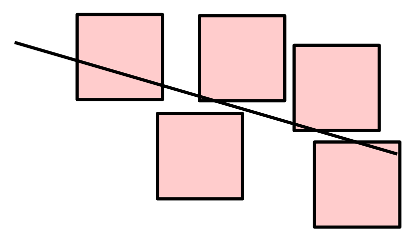

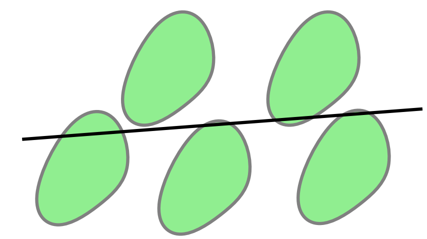

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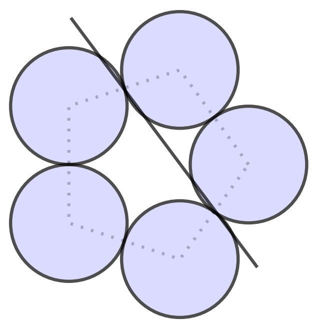

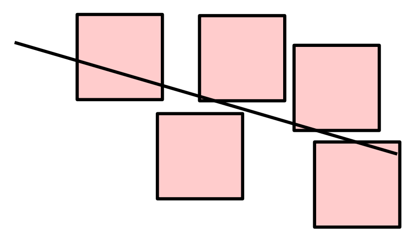

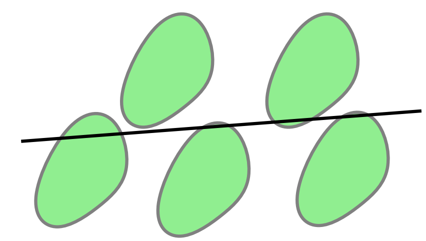

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All very ad hoc... What about **structural** results?

More benefits of convexity...

Combinatorial convexity

A wealth of **combinatorial** properties of convexity in $\mathbb{R}^{\mathbf{d}}$.

- ▷ If $p \in \text{conv}(X)$ then p is in a simplex with vertices in X. [Carathéodory 1905]
- ▷ Any $d + 2$ points contain two disjoint parts with overlapping convex hulls. [Radon 1921]
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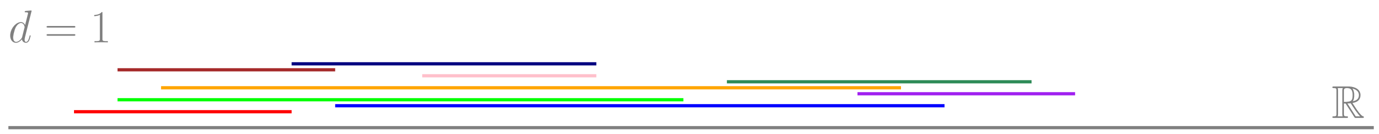

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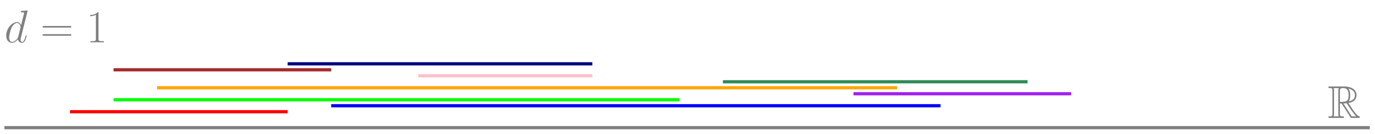

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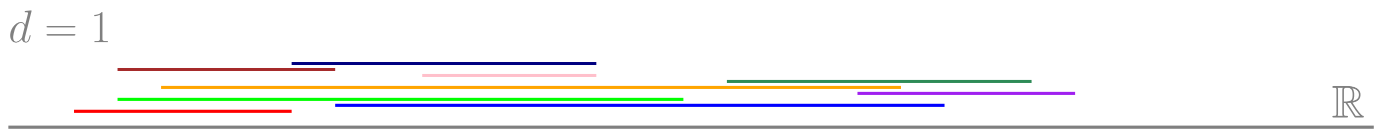

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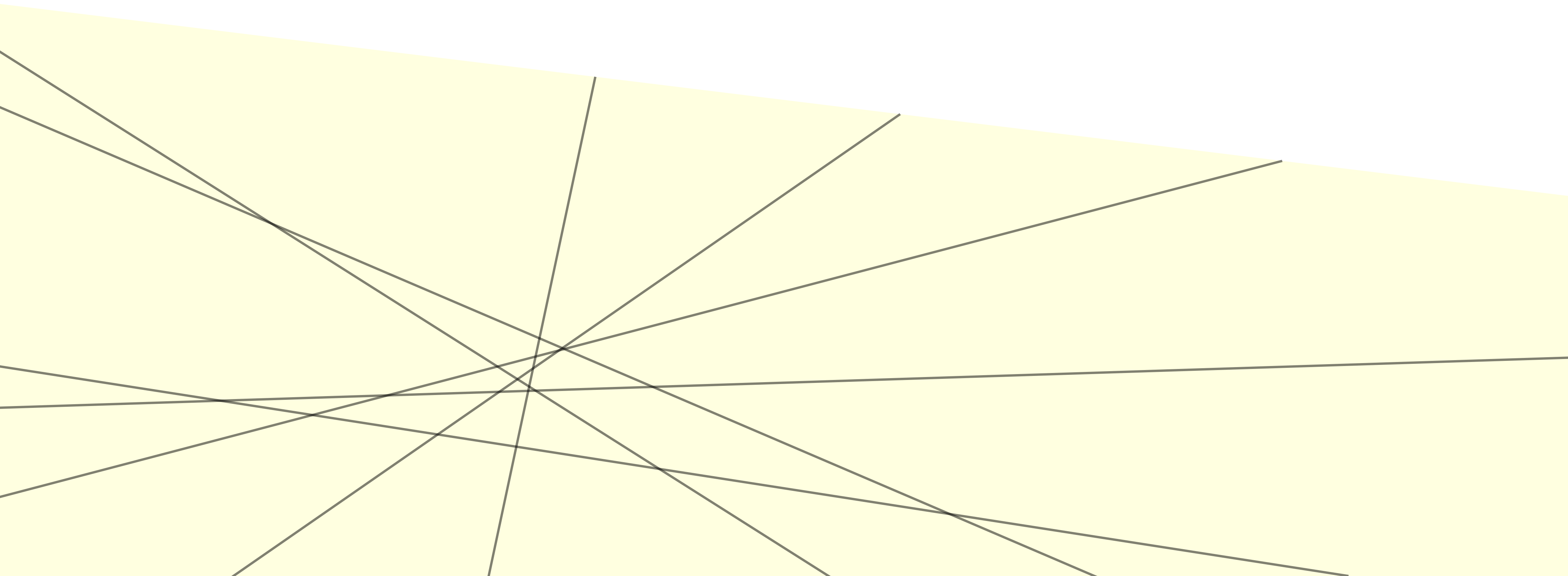

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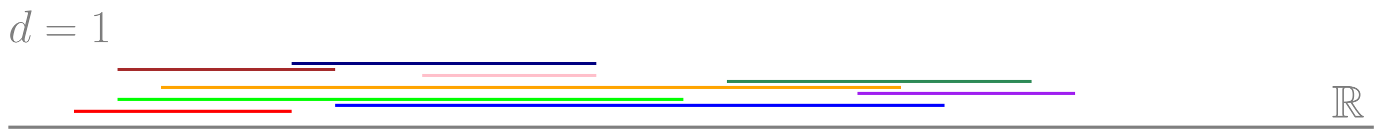

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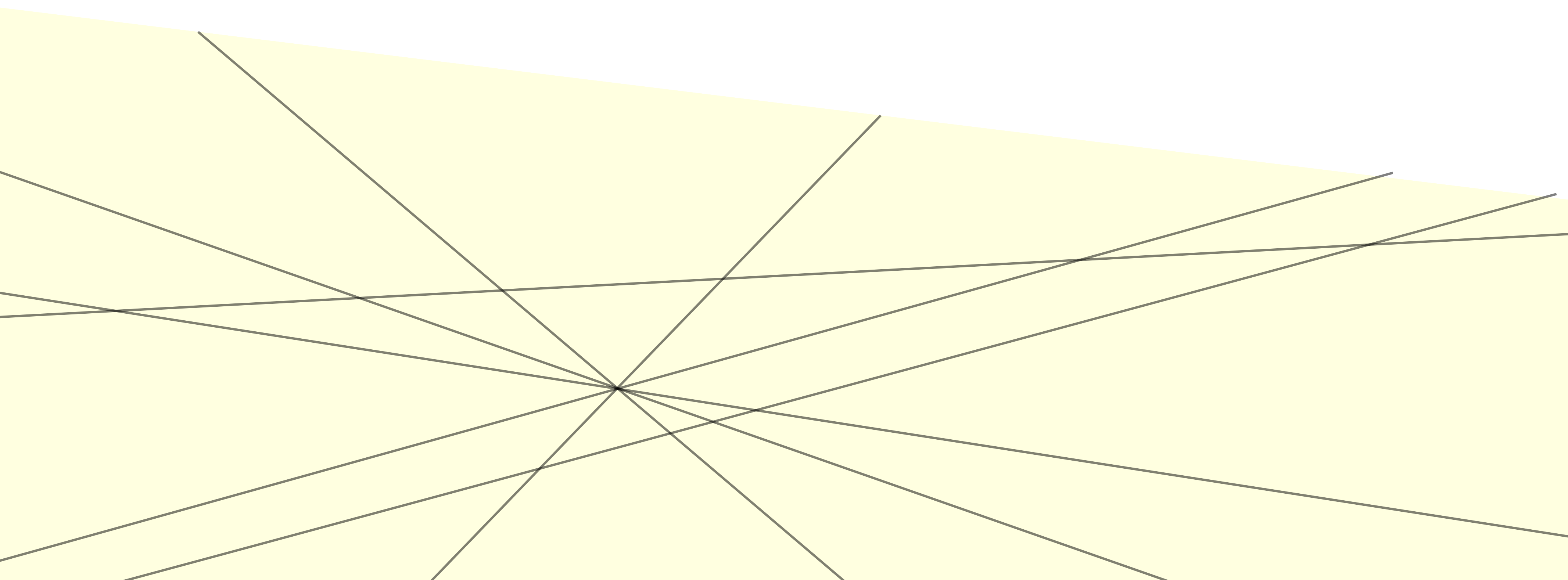

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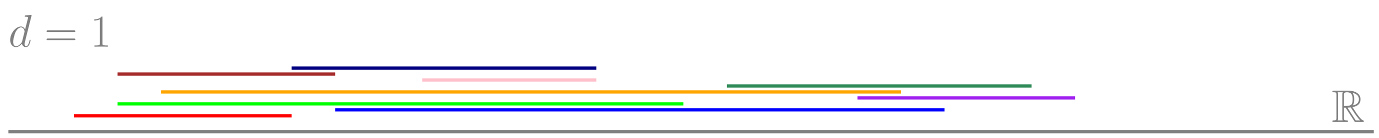

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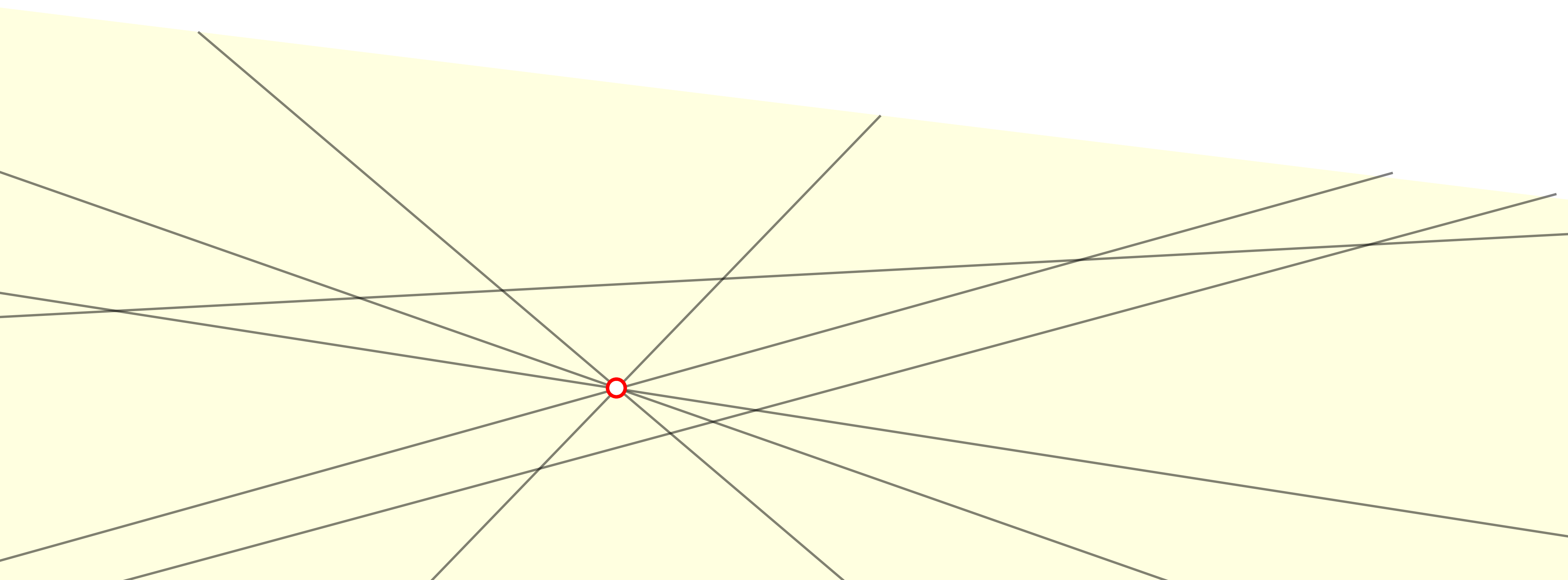
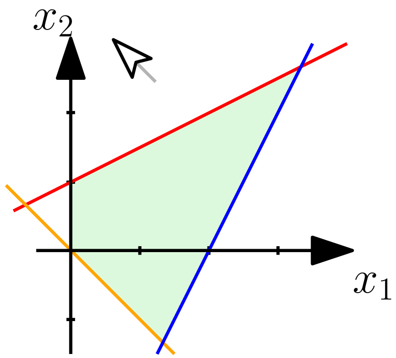


Also providing an algorithm...

Question: is the value of a given LP better than a given threshold τ?



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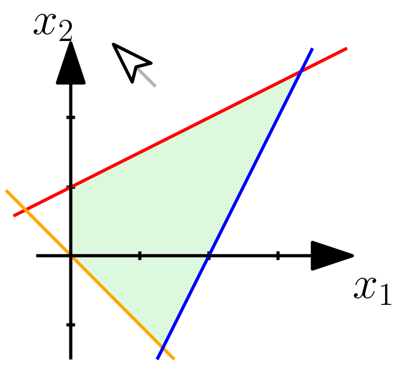

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Repeat k times...

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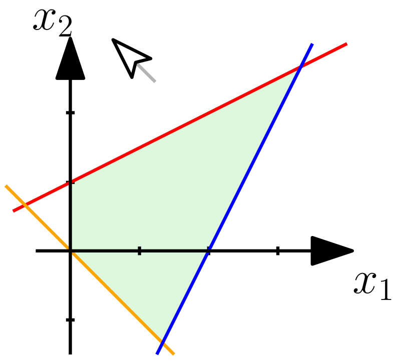

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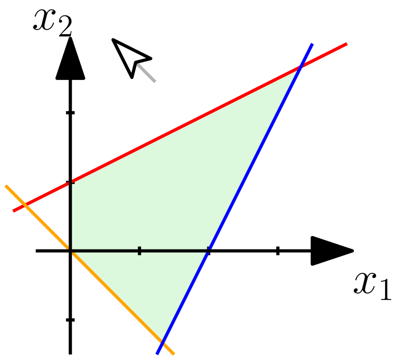

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ϵ-**far from** τ = *every point as good as* τ *violates at least an* ϵ *fraction of the constraints.*

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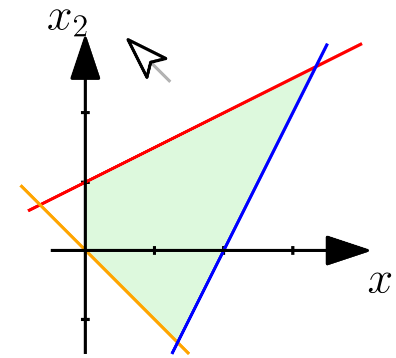

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Fractional Helly

$\rightsquigarrow$

Property tester

[Chakraborty et al. 2018]

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So how to generalize any of these beyond convexity?

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Methodology #1

Convexity and maps of simplicial complexes into $\mathbb{R}^d$

Convexity $\simeq$ **linear maps** from simplicial complexes into $\mathbb{R}^d$.

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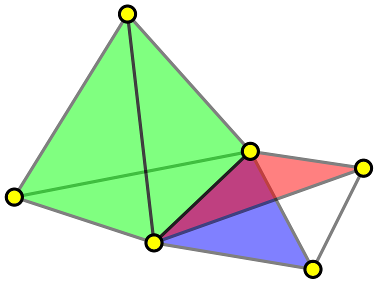

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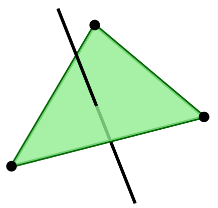

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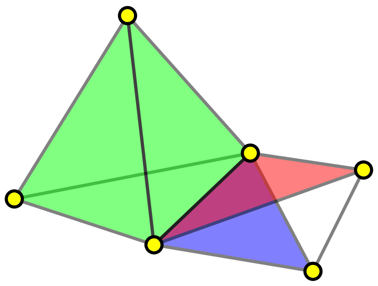

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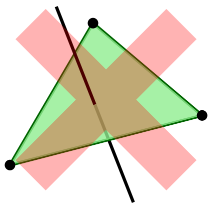

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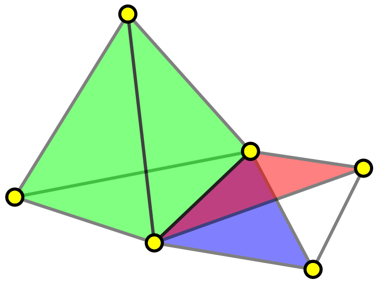

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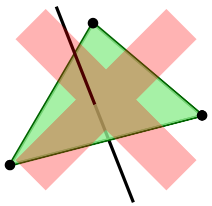

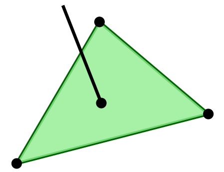

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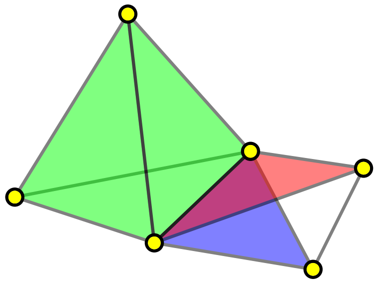

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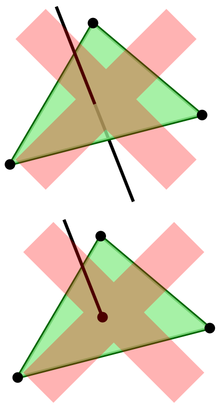

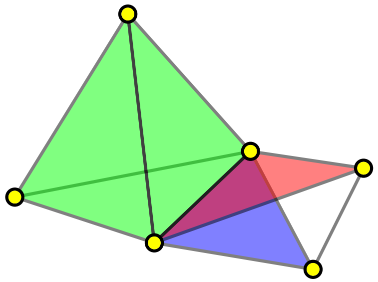

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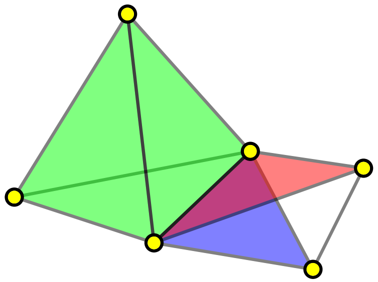

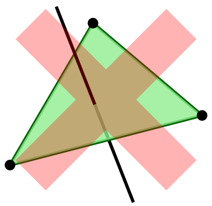

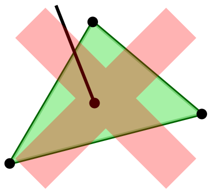

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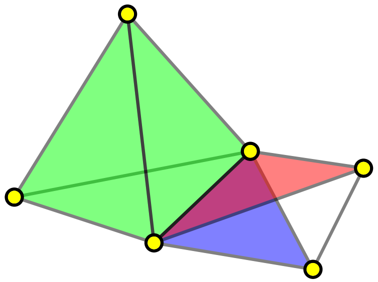
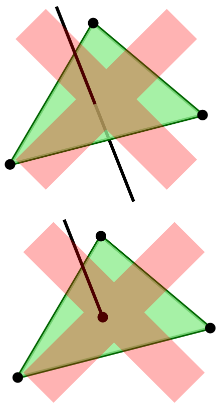
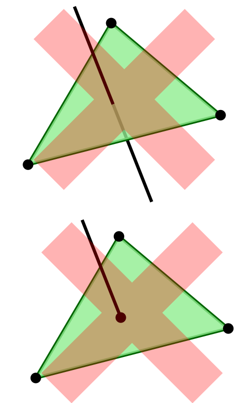




Analogue of graph planarity:

- For which d does $|\mathcal{K}|$ **embed** into $\mathbb{R}^d$?

Convexity $\simeq$ **linear maps** from simplicial complexes into $\mathbb{R}^d$.



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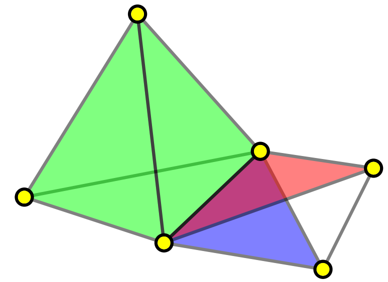

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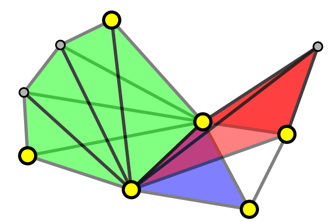

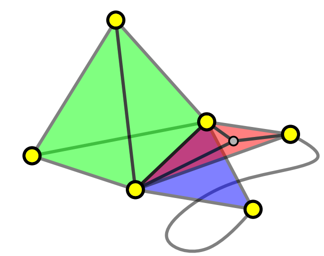

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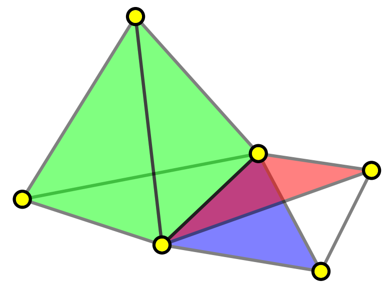

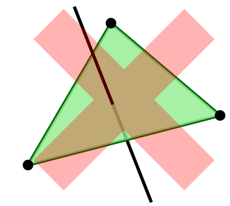

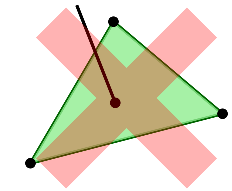

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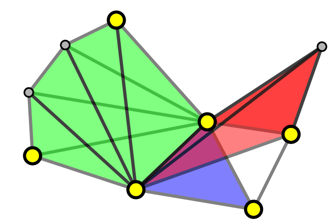

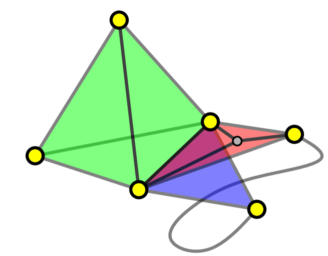

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$\Delta_n^{(\delta)}$ = the $(\leq \delta)$-dimensional faces of the n-dimensional simplex.

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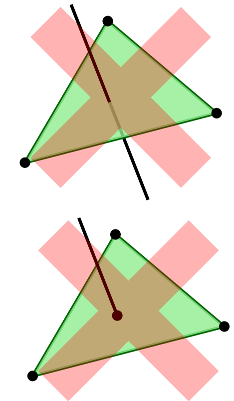

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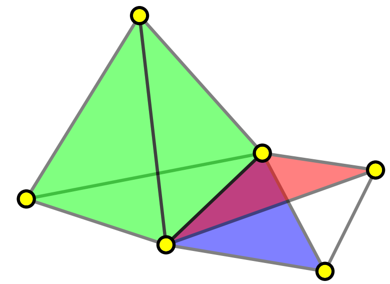

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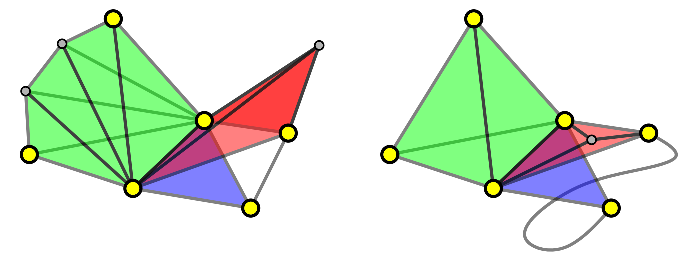

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Back to combinatorial convexity...

- $\triangleright$ Any $d+2$ points can be divided into 2 parts with overlapping convex hulls.

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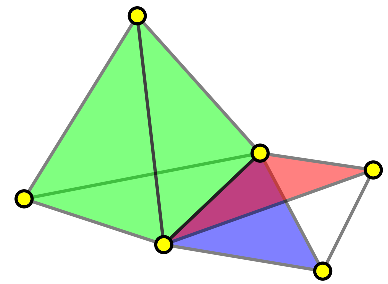

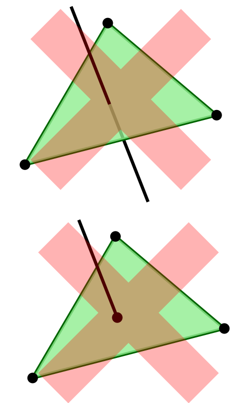

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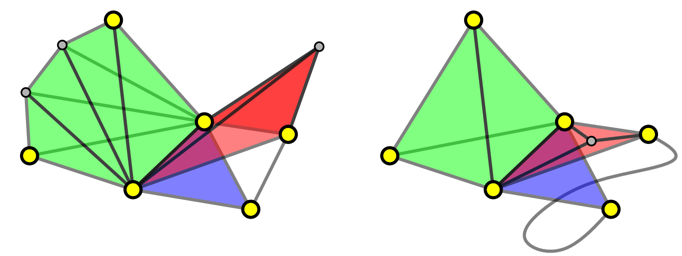

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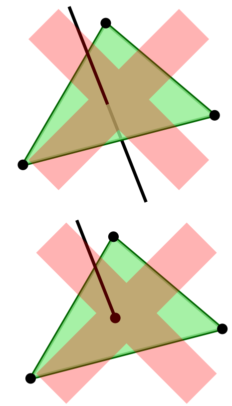

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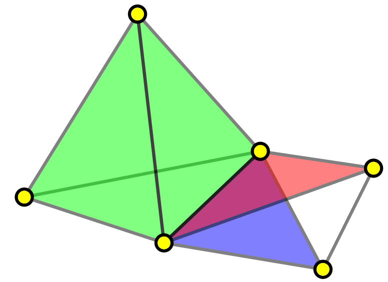

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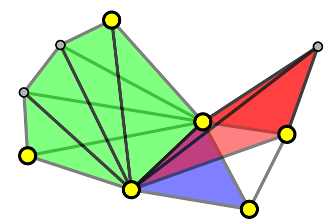

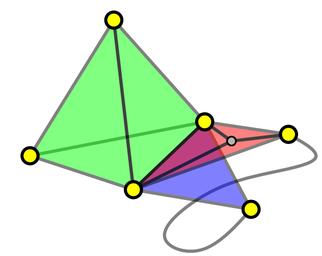

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continuous

[Bajmóczy-Bárány 1979]

Zoom: "For any map $f : |\mathcal{K}| \to \mathbb{R}^d$, two **disjoint** faces have overlapping images."

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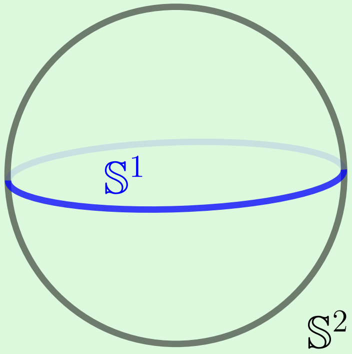
There does not exist a continuous antipodal map $\mathbb{S}^k \to \mathbb{S}^{k-1}$.



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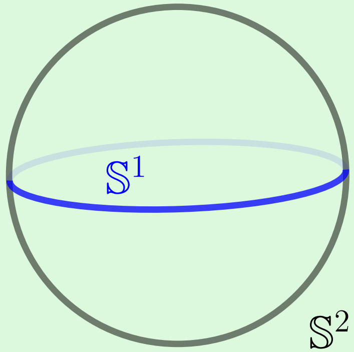

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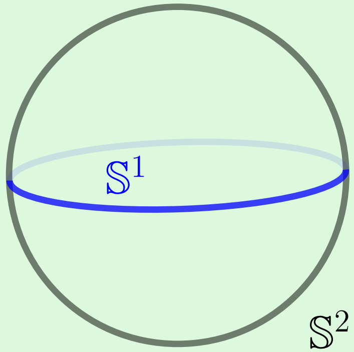

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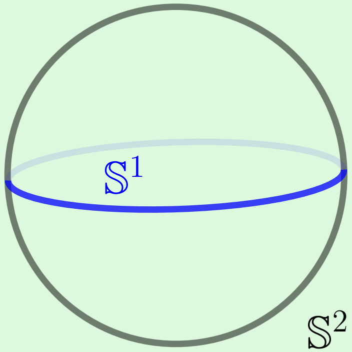

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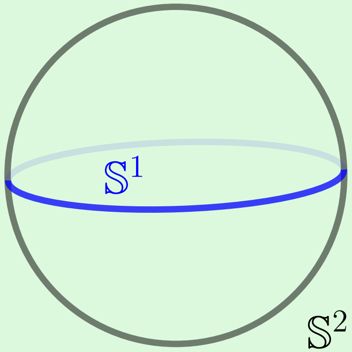

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but not in general. [Mabillard-Wagner 2015, Frick 2015]

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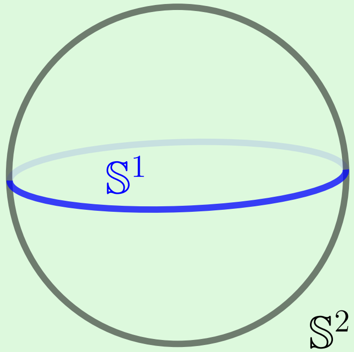

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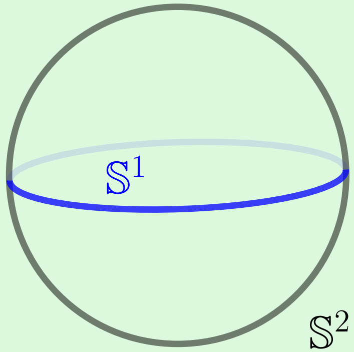

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"Linear" can be dropped. [Gromov 2010].

- ▷ If $p \in \text{conv}(X)$ then p is in a simplex with vertices in X. [Carathéodory 1905]
- ▷ Any $d+2$ points contain two disjoint parts with overlapping convex hulls. [Radon 1921]
- ▷ Any $(r-1)d+r$ points contain r disj. parts with overlap. convex hulls. [Tverberg 1966]

- ▷ Any point that is in the convex hull of $d+1$ color classes is in a colorful simplex. **Colorful Carathéodory** [Bárány 1976]
- ▷ For convex sets of $d+1$ colors, if each colorful subset intersects, then one color class has a point in common. **Colorful Helly** [Lovász 1976]
- ▷ Any $2d+2$ points, 2 of each color, can be partitioned into colorful subsets with overlapping convex hulls. **Colorful Radon** [Lovász 1992]

- ▷ If a positive fraction of the $(d+1)$-tuples of intersect, then a positive fraction has a point in common. [Katchalski-Liu 1979, Kalai 1985]
- ▷ For any point set, a fraction c_d of the simplices overlap. [Boros-Füredi, Bárány 1984]
- ▷ For any $p \geq q \geq d+1$ there exists $N(p,q,d)$ s.t. any family satisfying "among any p some q overlap" has a hitting set of size N. [Hadwiger-Debrunner 1957] [Alon-Kleitman 1992]

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Do some generalizations imply others?

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Methodology #2

Convexity and patterns in hypergraphs

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A family $\mathcal{F}$ of convex sets $\rightsquigarrow$ a sequence of hypergraphs $\mathcal{H}_{\mathcal{F}}(m)$

vertex set $=\mathcal{F}$*, edges* $=$ **intersecting** m*-tuples.*

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▷ Colorful Helly $=$ a **forbidden pattern** for $\mathcal{H}_{\mathcal{F}}(m)$.

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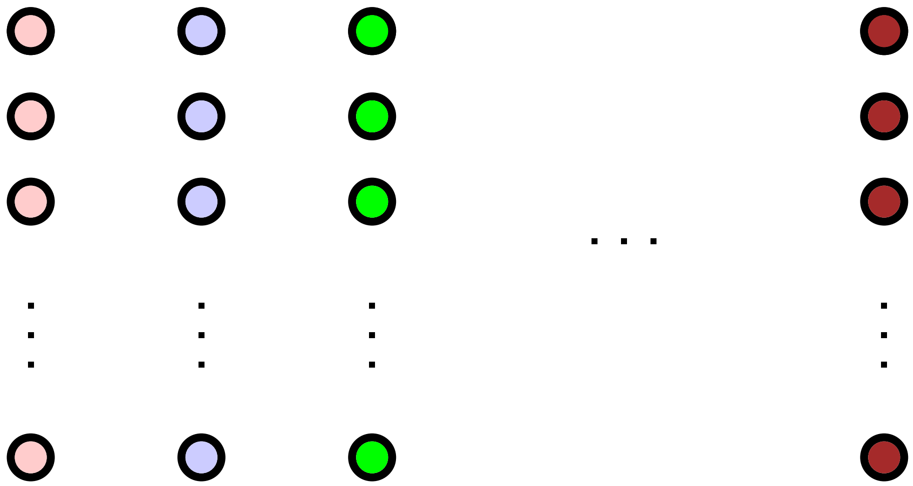
- m sets of m vertices.



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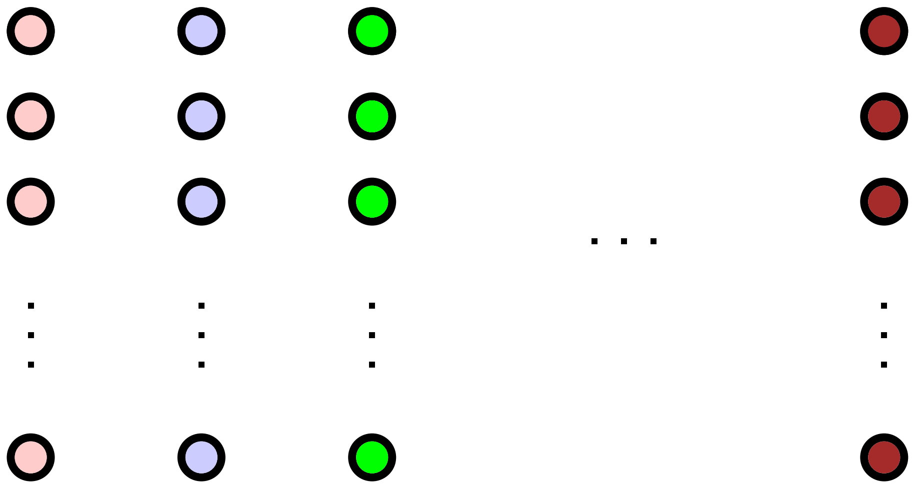

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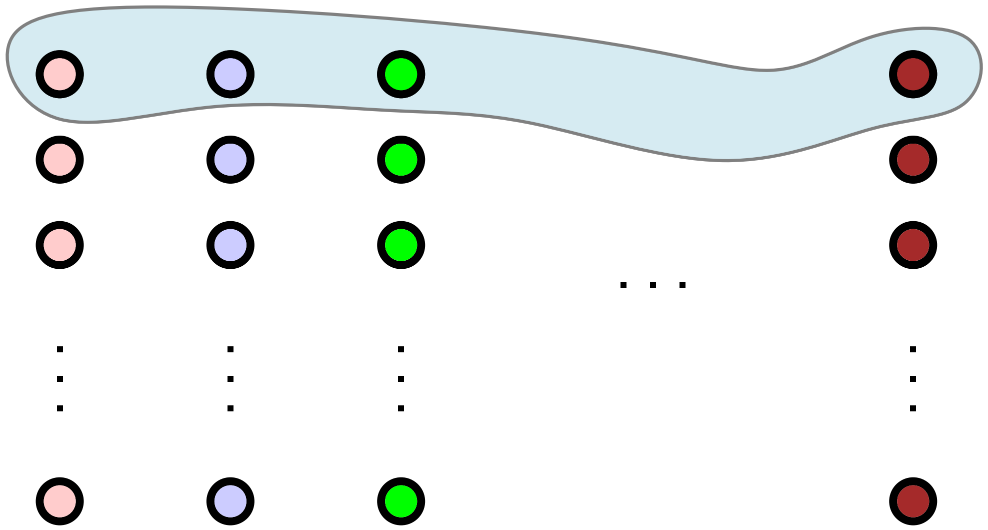

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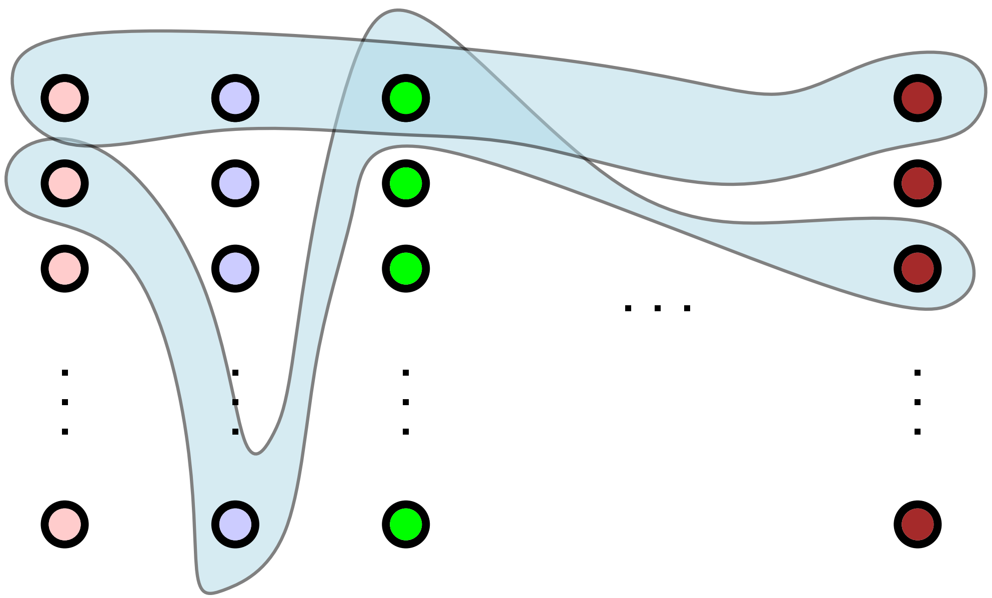

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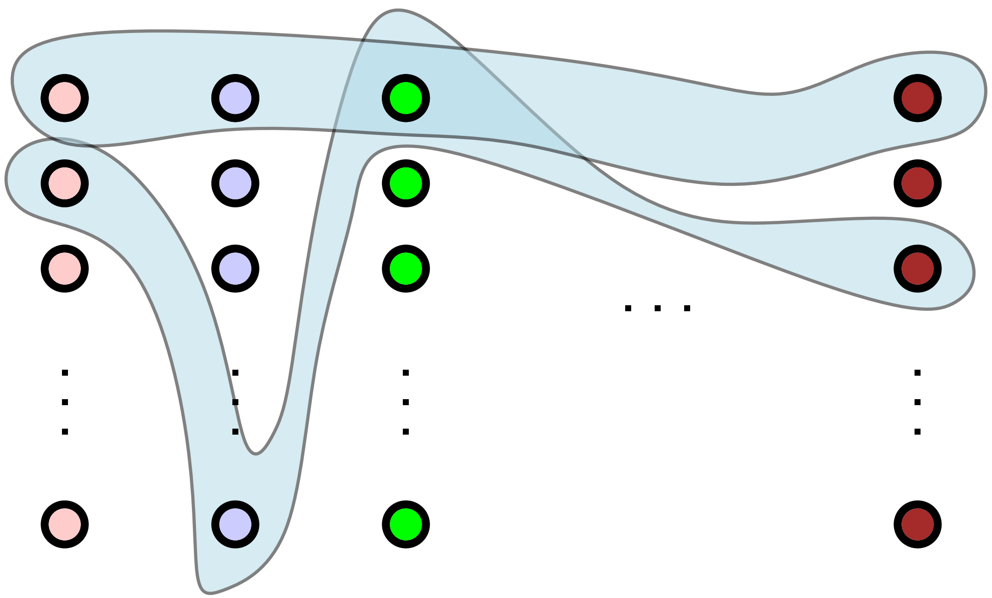

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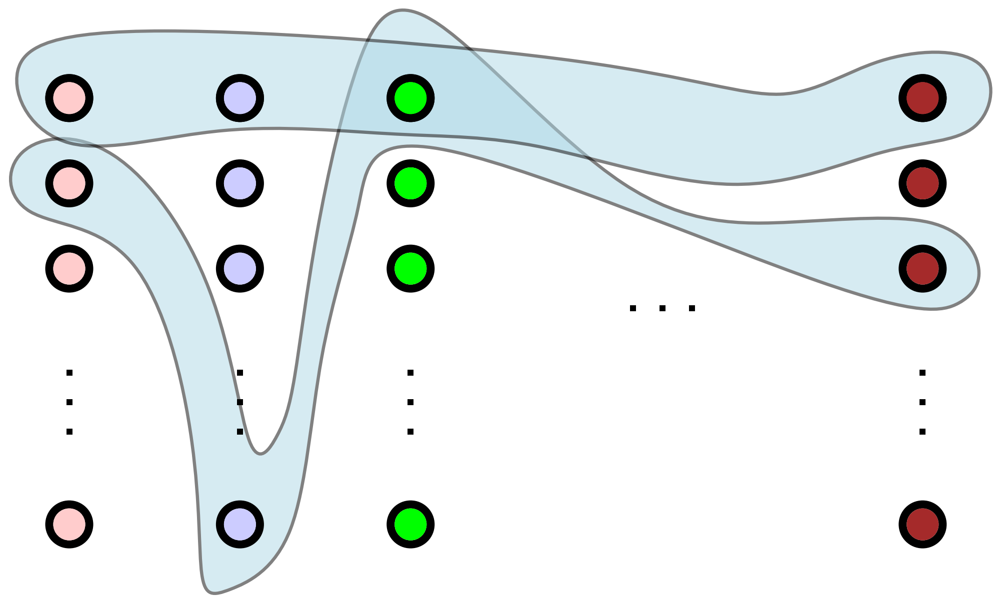

Fractional Helly holds whenever this pattern is forbidden.

Positive edge density $\Rightarrow$ *linear-size clique.* [Holmsen 2019]

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colorful m-tuples intersect

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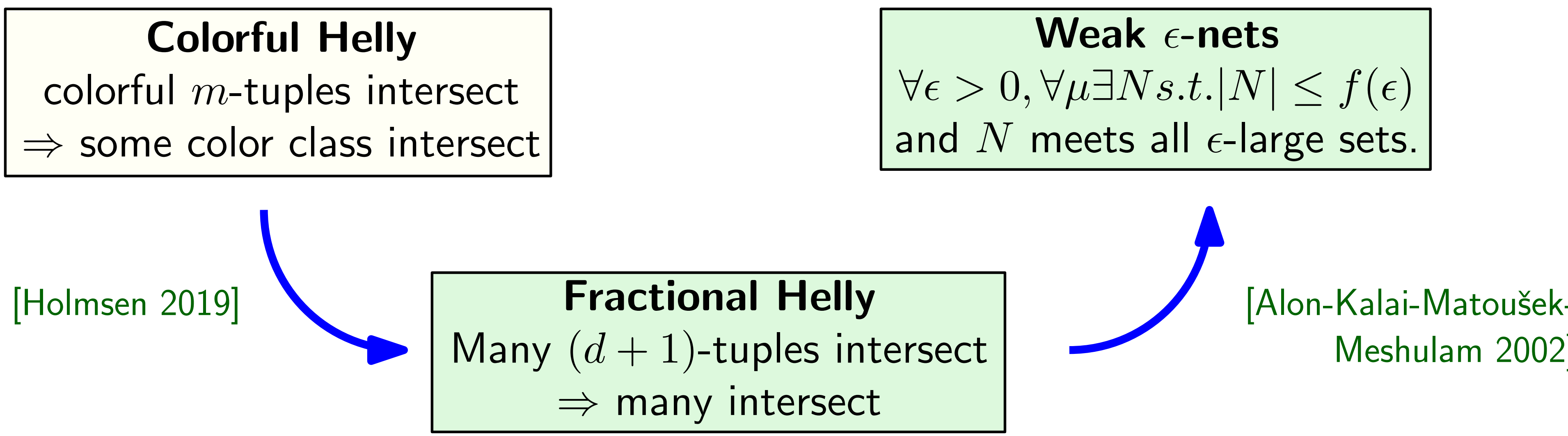

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Weak ϵ-nets
$\forall \epsilon > 0, \forall \mu \exists N s.t. |N| \leq f(\epsilon)$
and N meets all ϵ-large sets.
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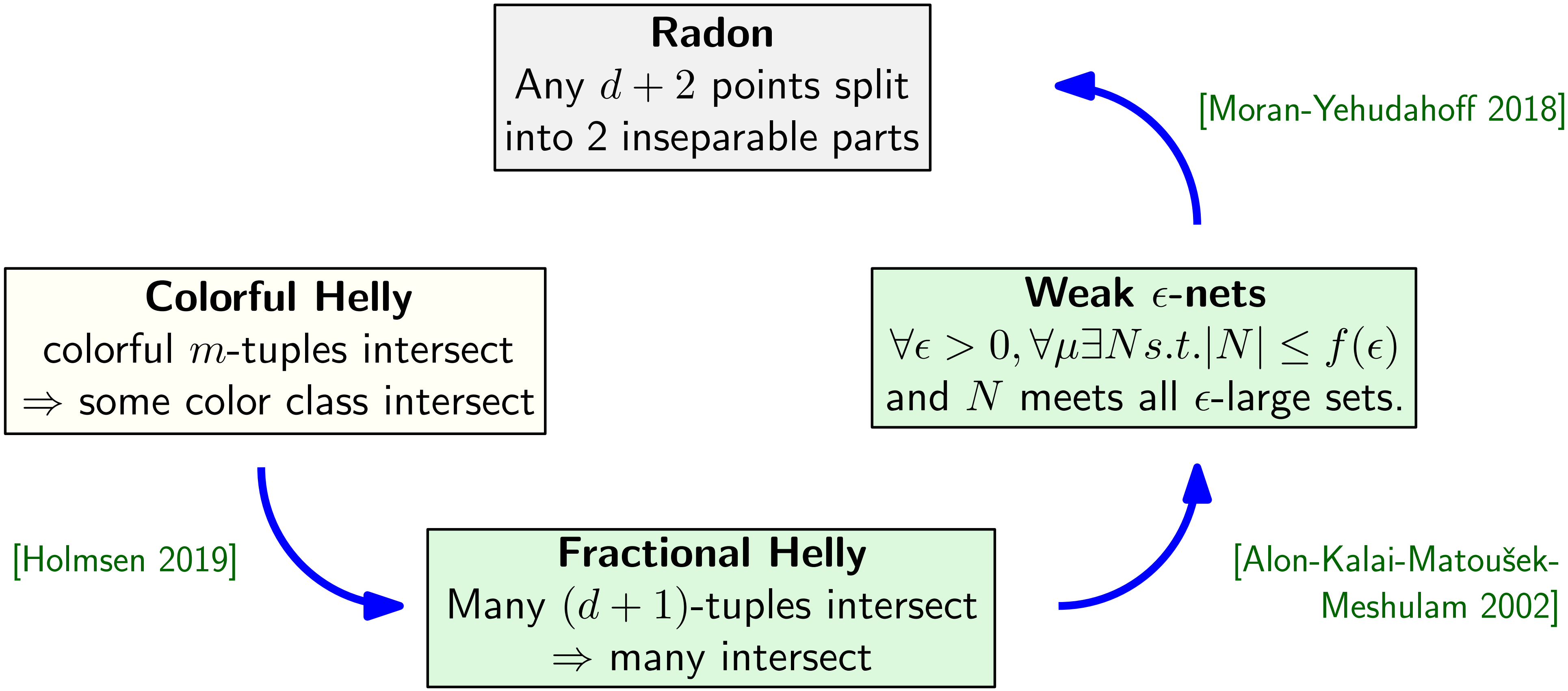
Radon
Any $d+2$ points split
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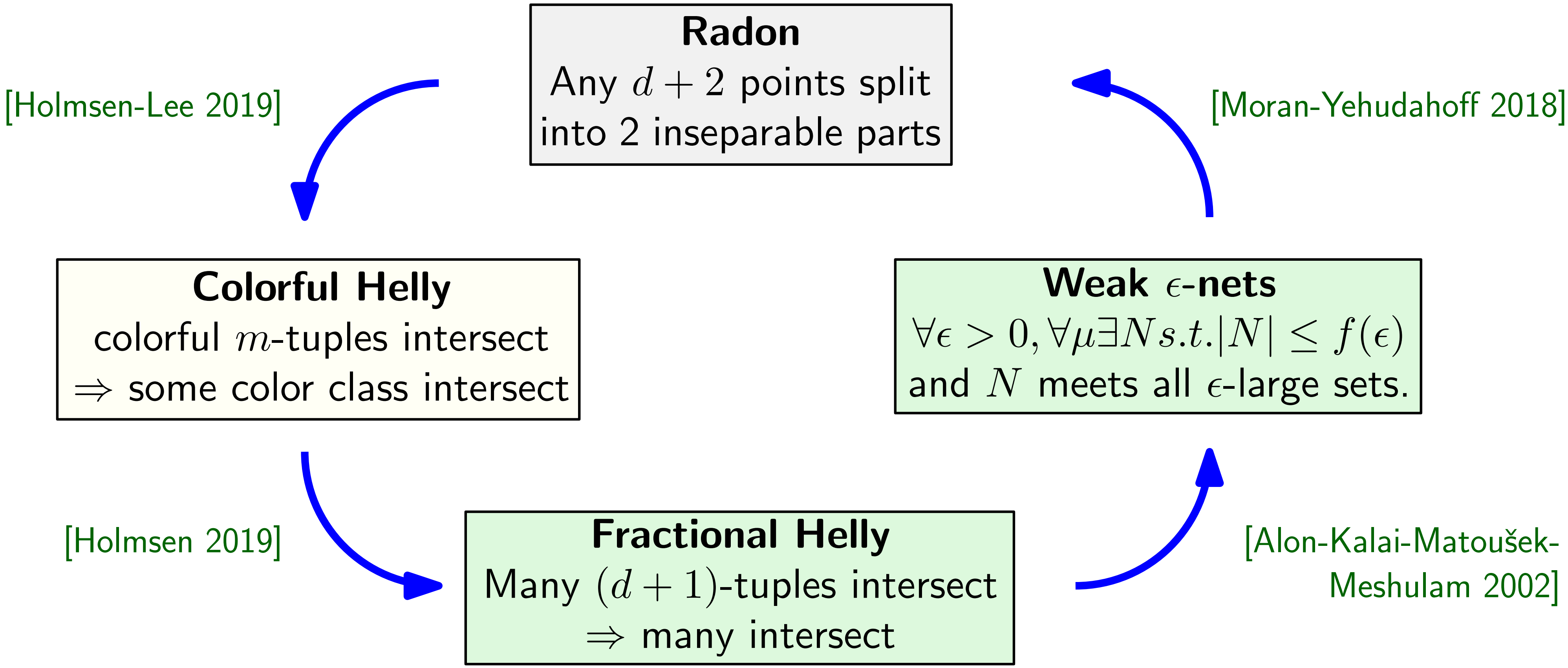
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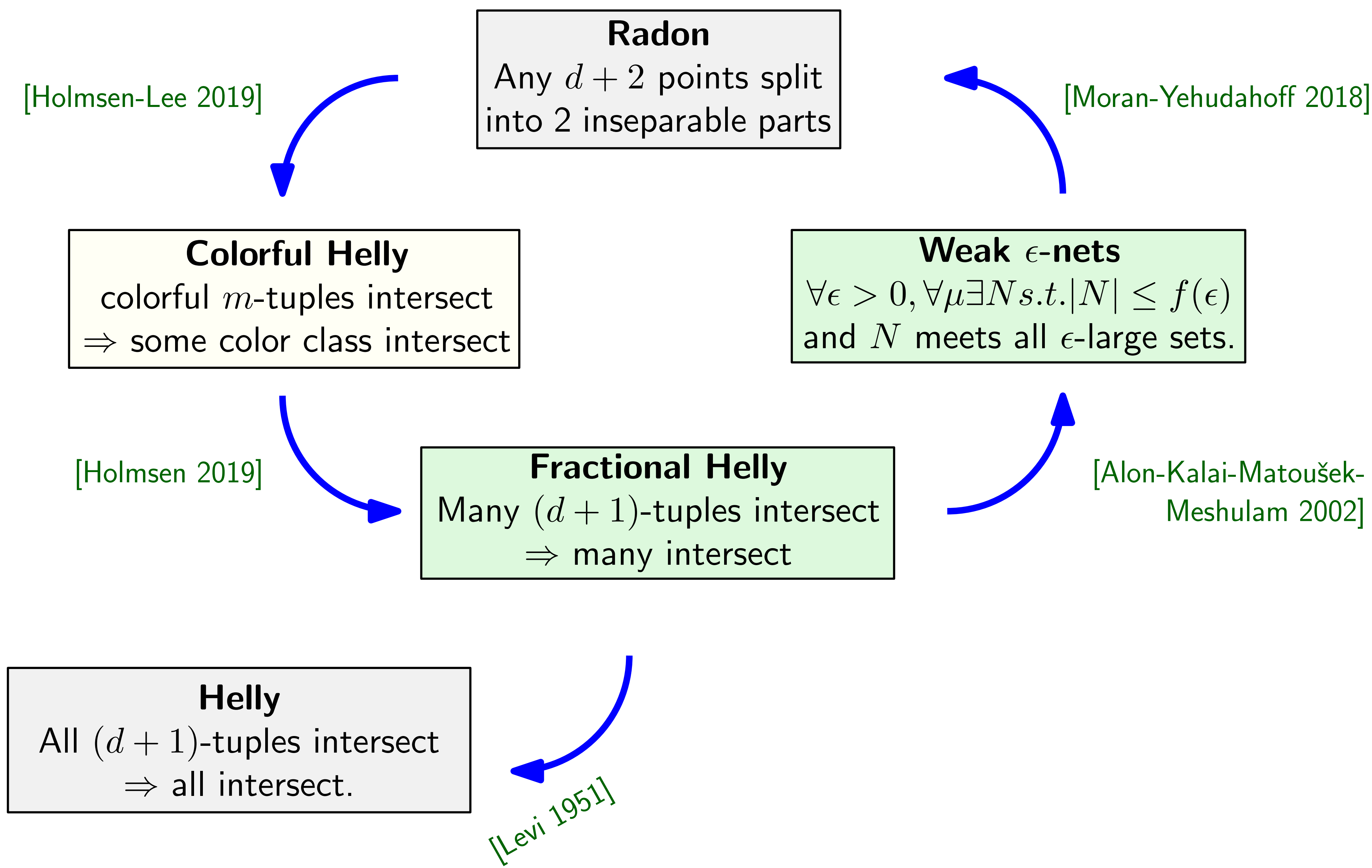

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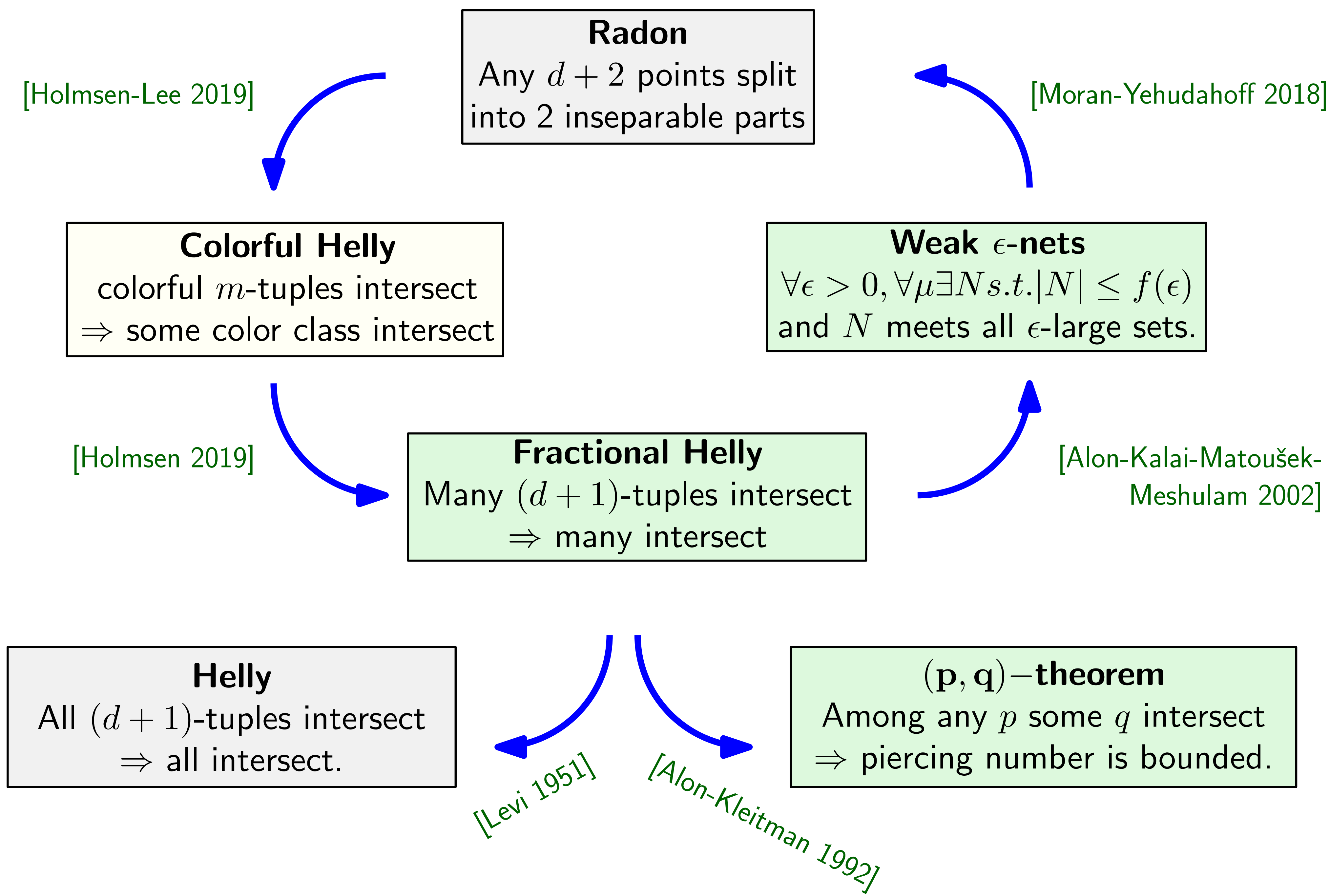
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$(\mathbf{p},\mathbf{q})-$theorem
Among any p some q intersect $\Rightarrow$ piercing number is bounded.
[Levi 1951]
[Alon-Kleitman 1992]

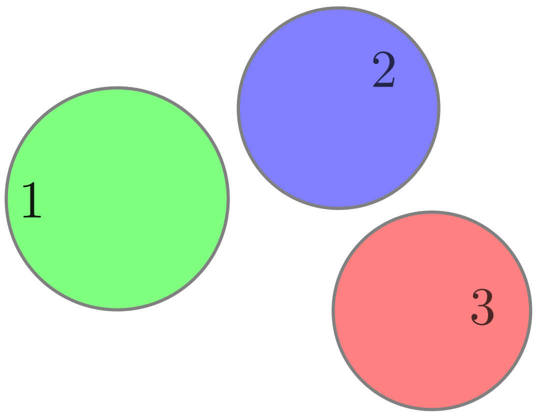
Methodology #3

Convexity and homological properties of nerves

Nerve $\mathcal{N}(\mathcal{F}) \simeq$ intersection **hypergraph** of $\mathcal{F}$

$$\mathcal{N}(\mathcal{F}) = \{G : G \subseteq \mathcal{F} \text{ and } \cap_{A \in G} A \neq \emptyset\}.$$

$\mathcal{F}$



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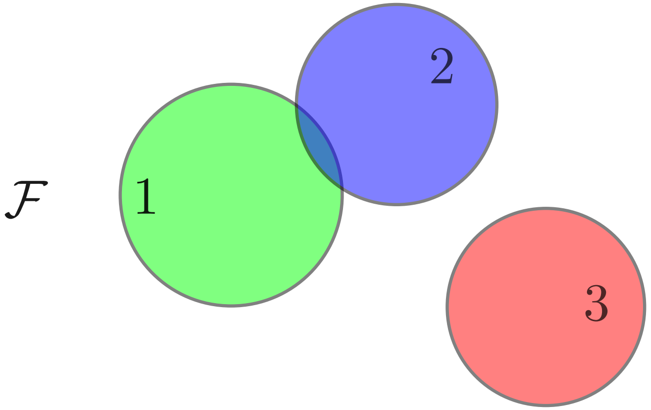

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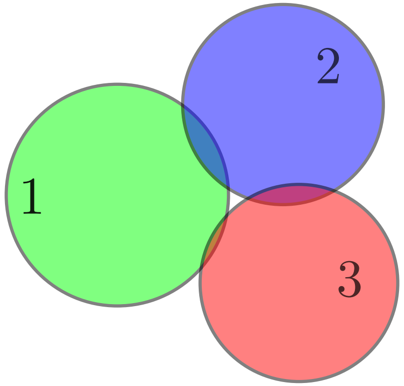

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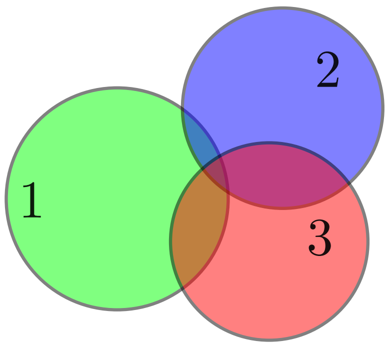

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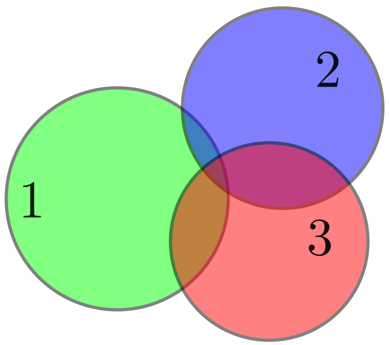

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▷ Nerves are **abstract simplicial complexes**.

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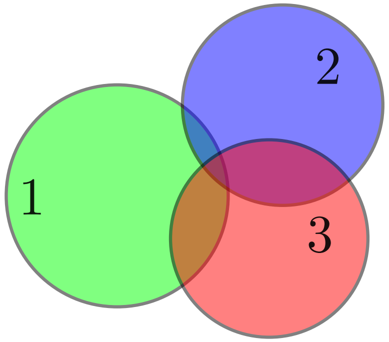

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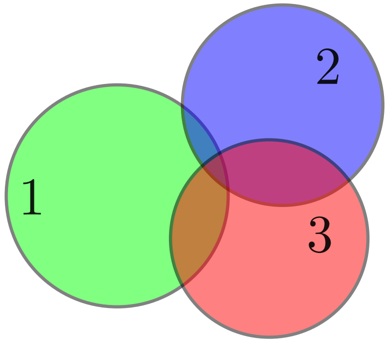

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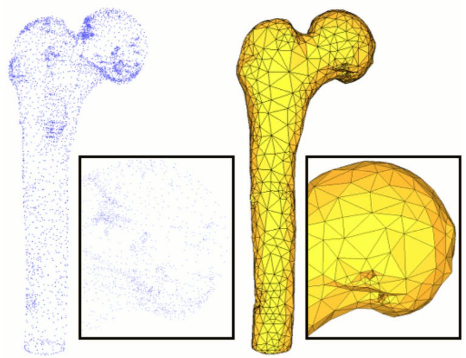

▷ Reconstruction methods.

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https://doc.cgal.org/latest/Manual/tuto_reconstruction.html

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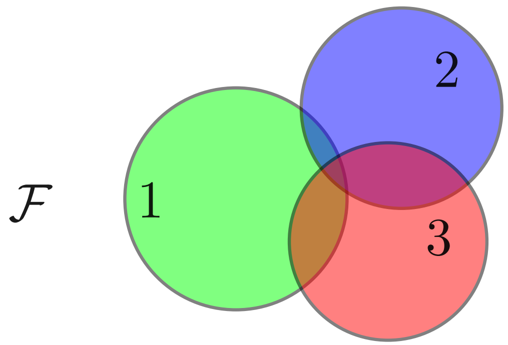

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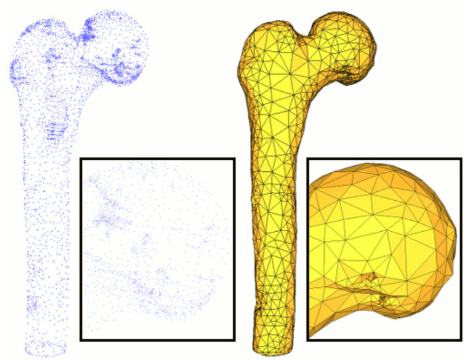

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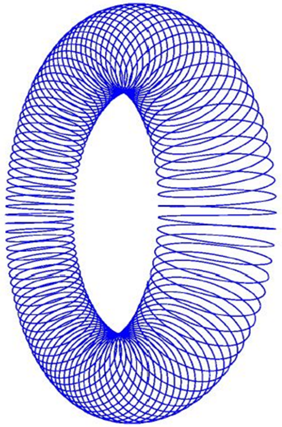

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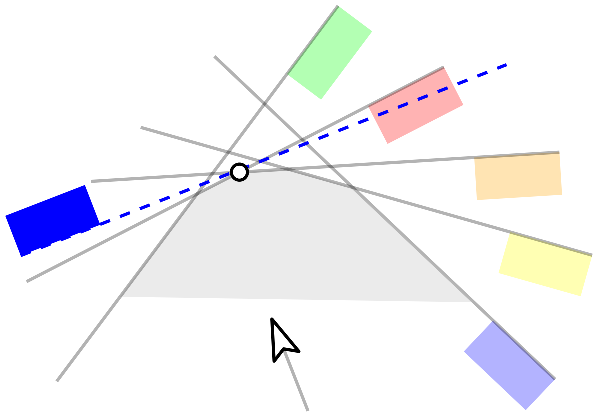

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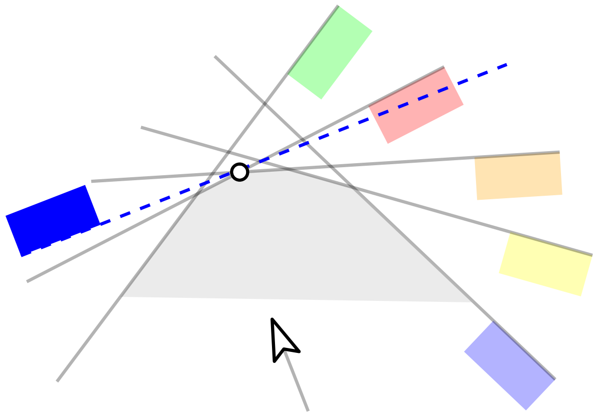

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Many $(d+1)$-tuples intersect $\Rightarrow$ many intersect. [Kalai 1985]

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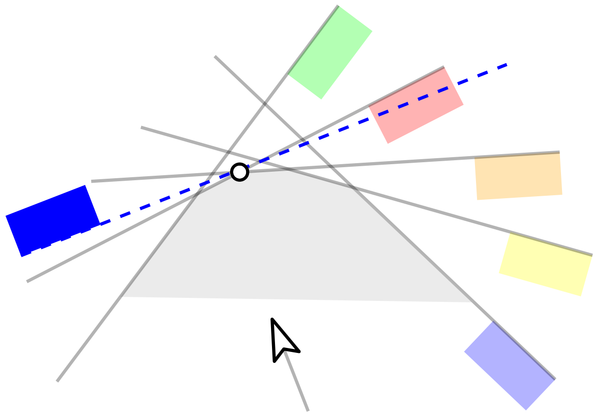

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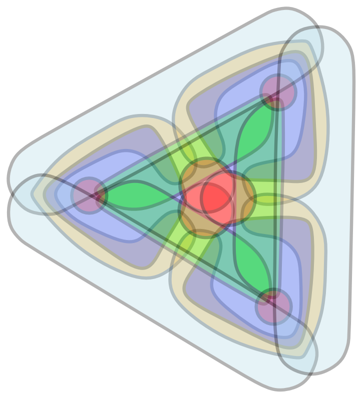

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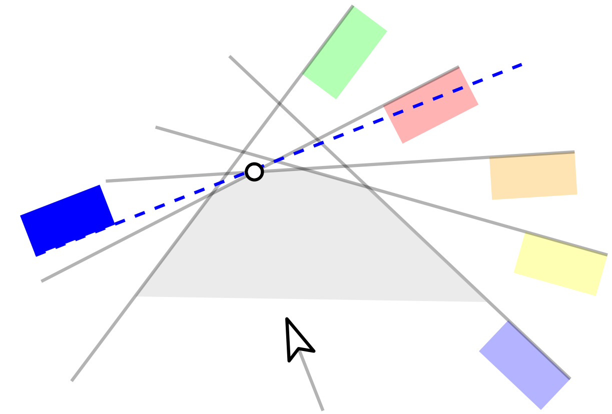

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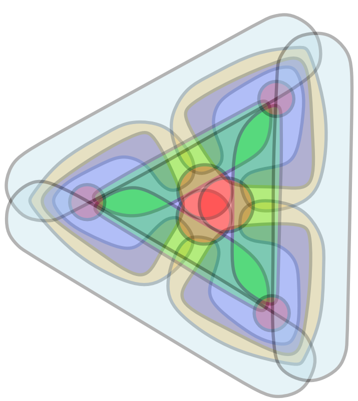

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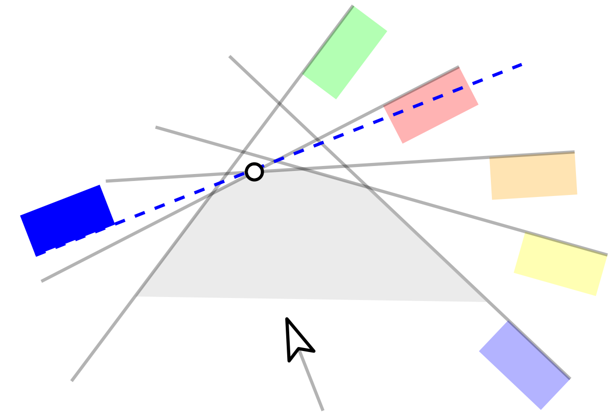

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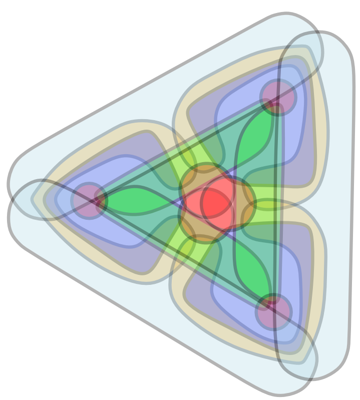

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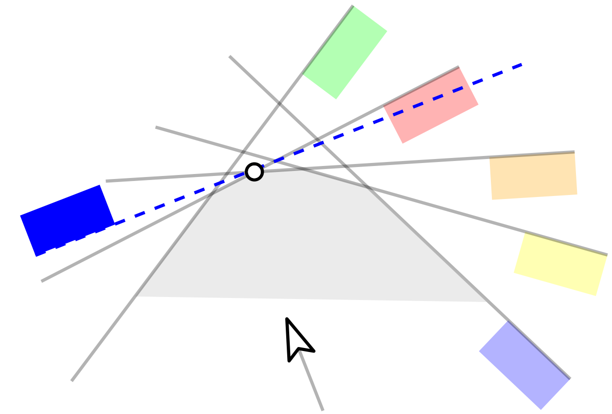

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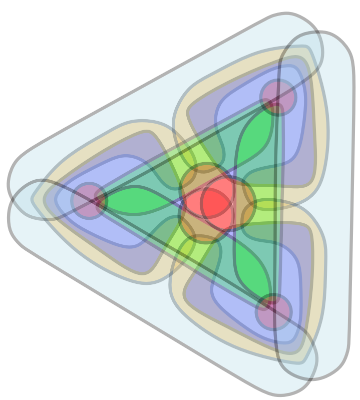

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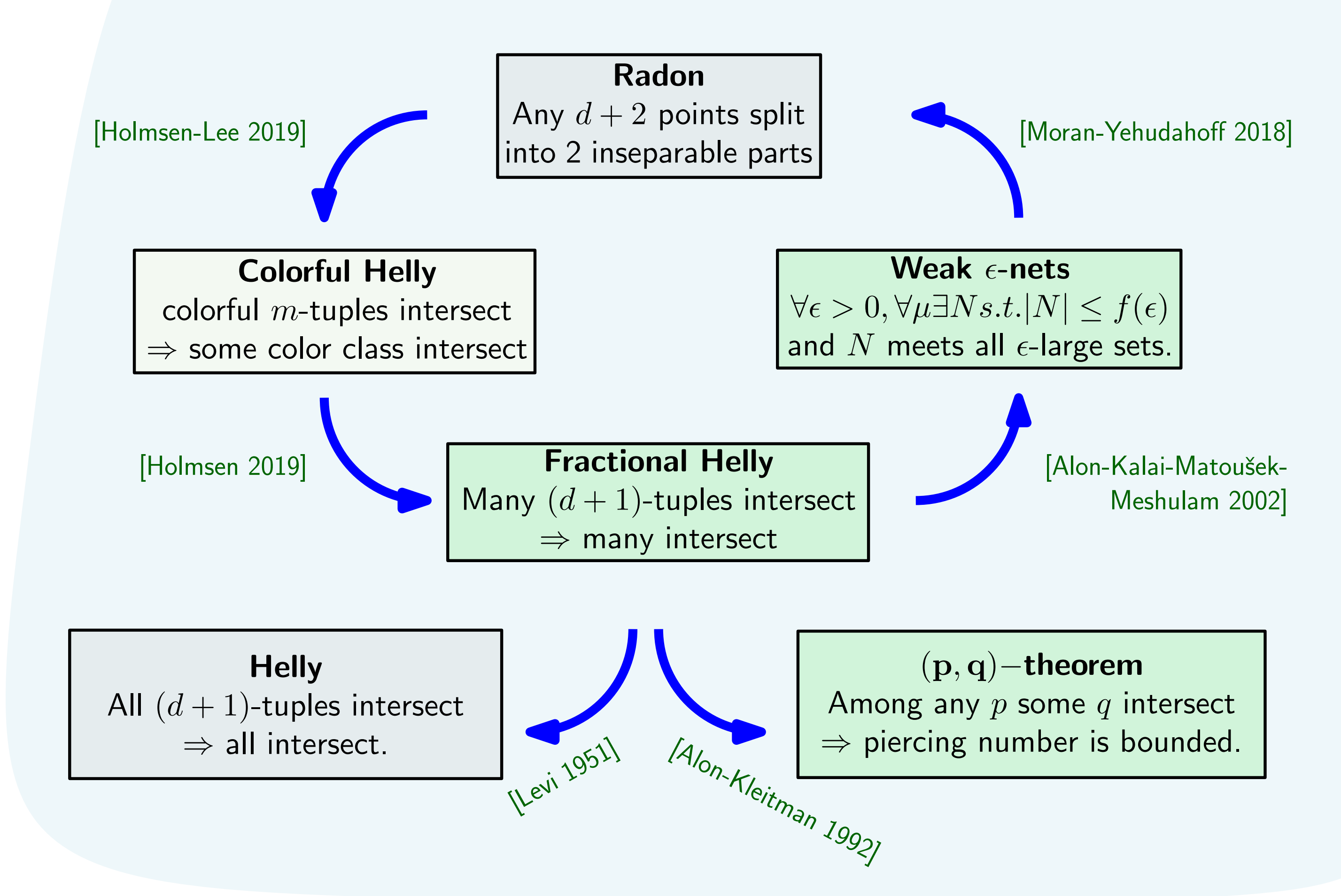

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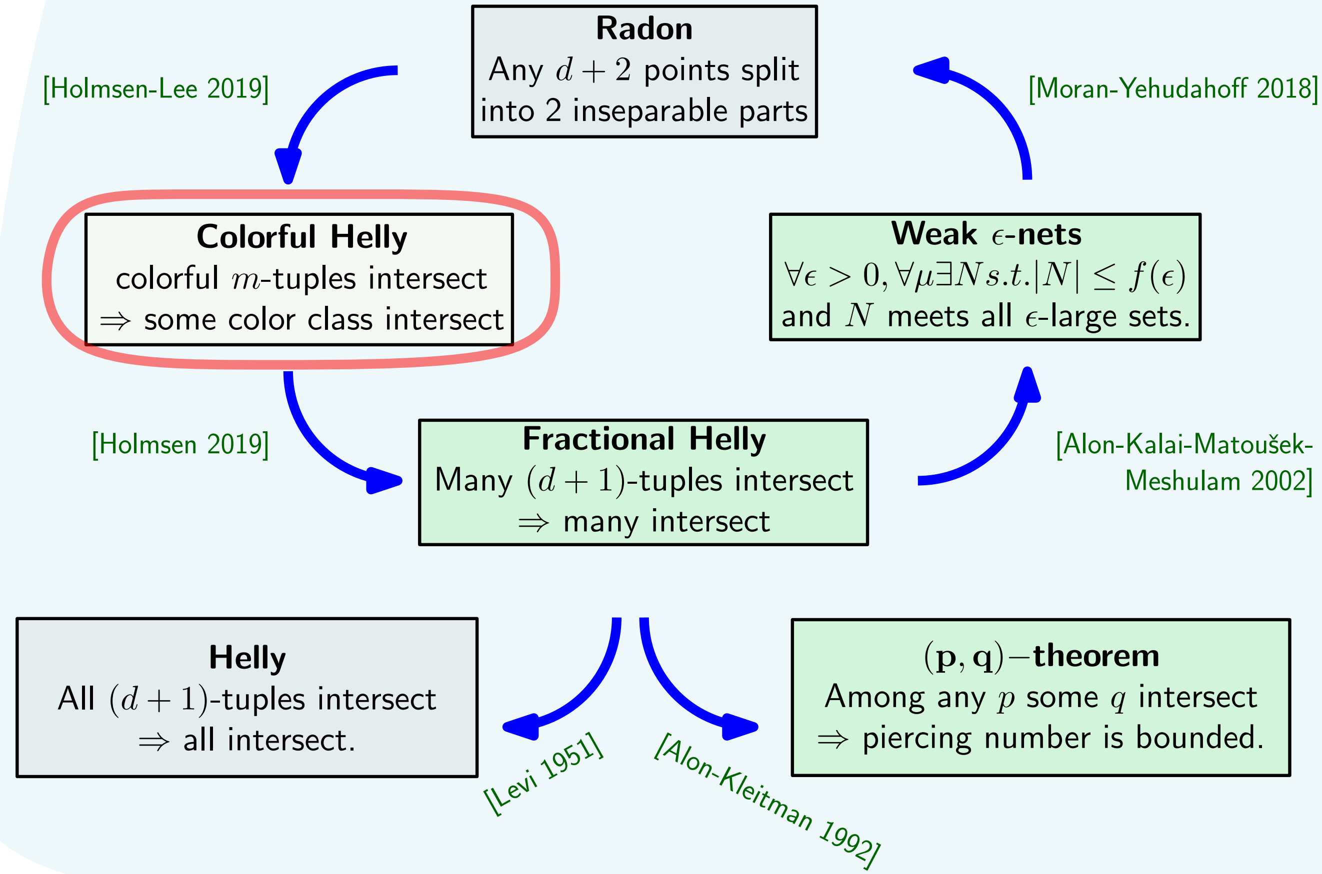

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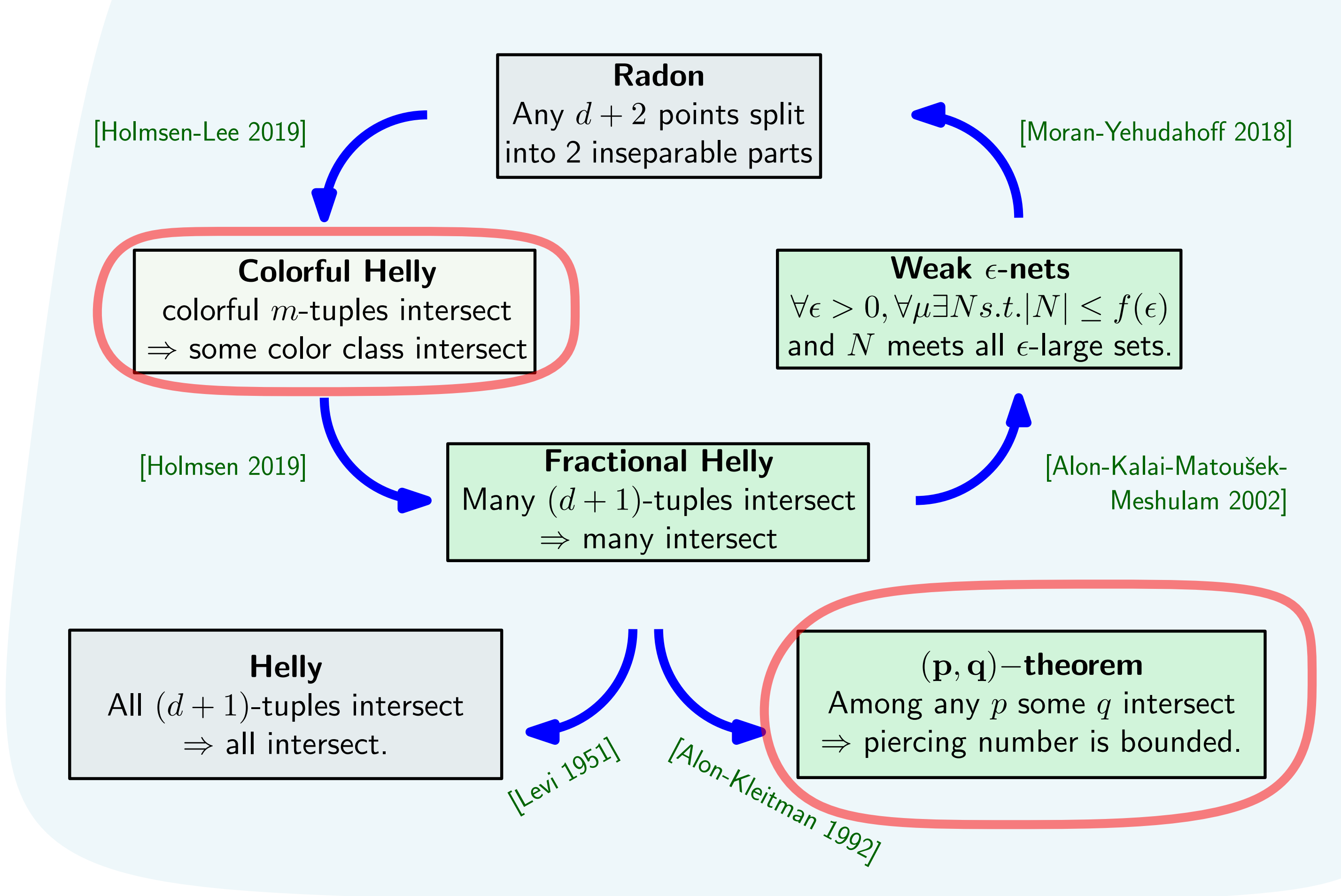

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What set systems have nerves of bounded Leray numbers?

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- Every family $\mathcal{F}$ s.t. for every $G \subseteq \mathcal{F}$ $\cap_{A \in G} A$ has $\leq b$ connected components, each one **acyclic**.

[Kalai-Meshulam 2007][Colin de Verdière-Ginot-G 2014]

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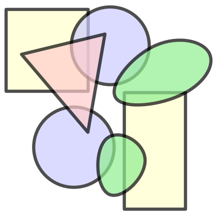

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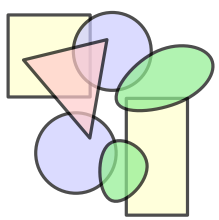

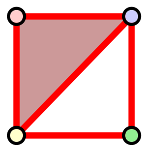

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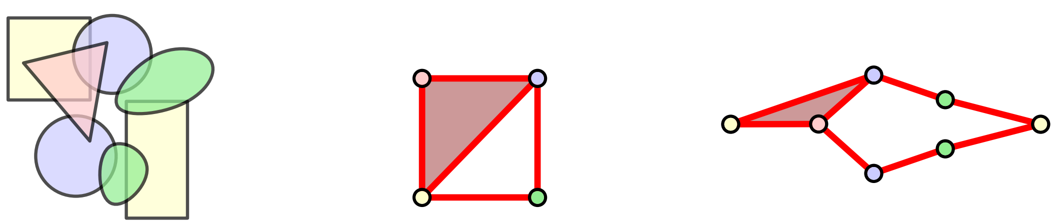

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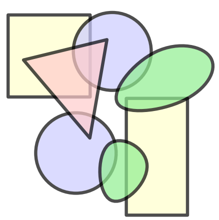

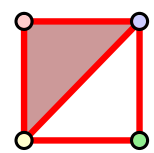

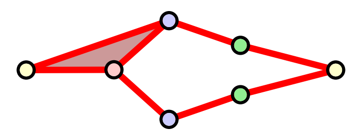

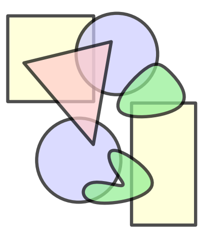

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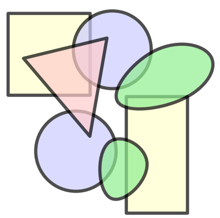
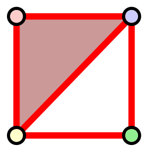
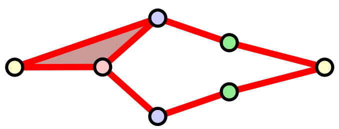
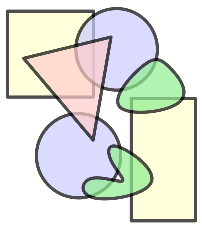
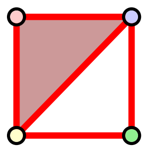

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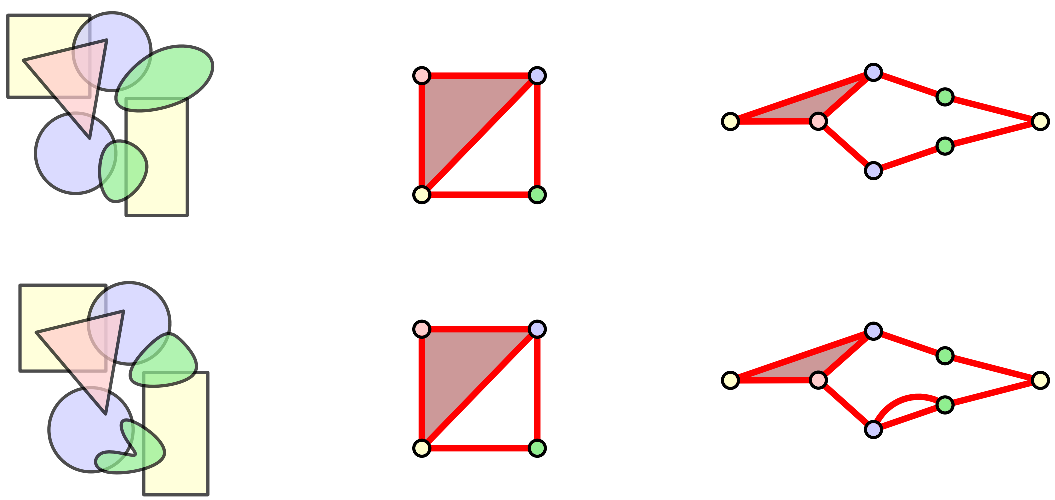

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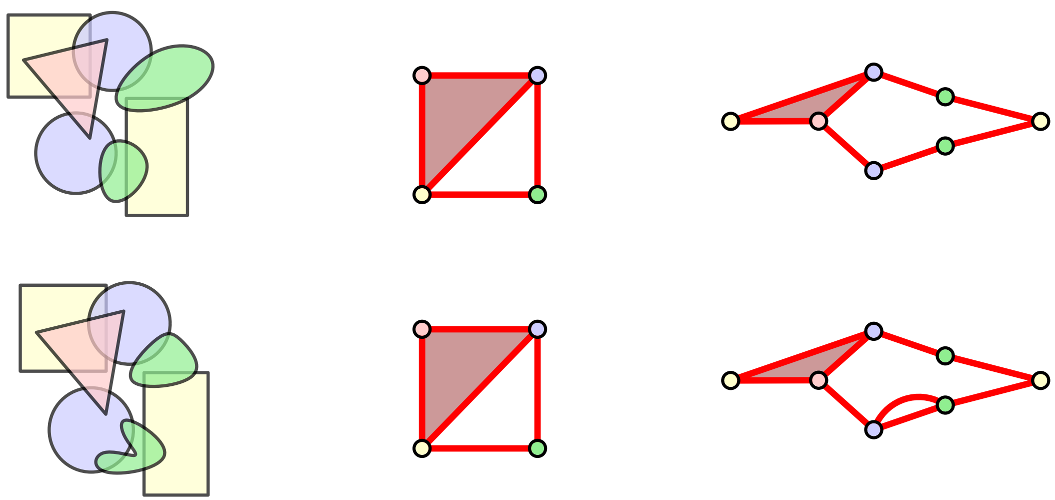

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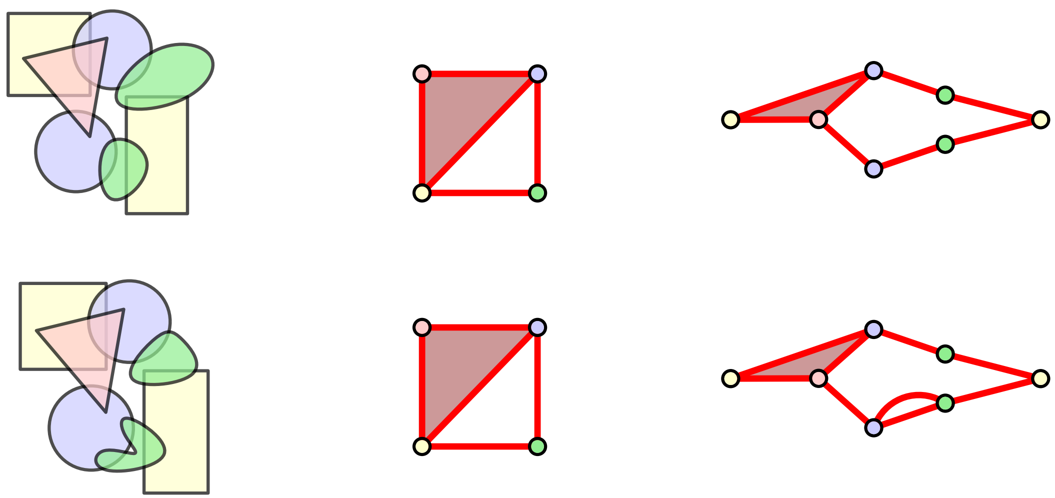

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Open: Is it enough if every $X \in \mathcal{F}^{\cap}$ has **bounded** $\beta_0, \beta_1, \ldots$?

Zooming in...

Sharp conditions

using some Ramsey theory

A classic: Helly from Radon...

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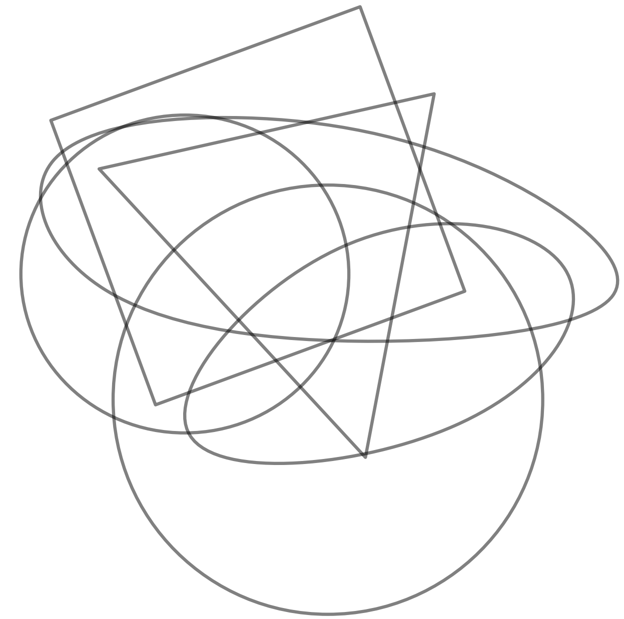
$(d = 2)$



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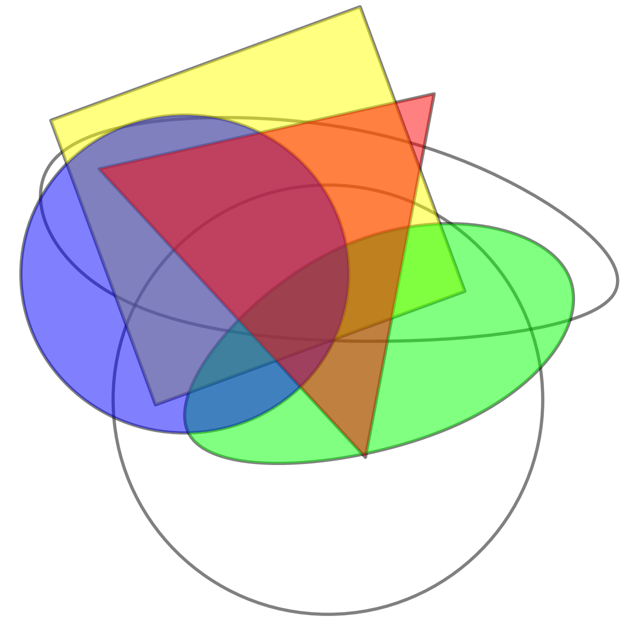

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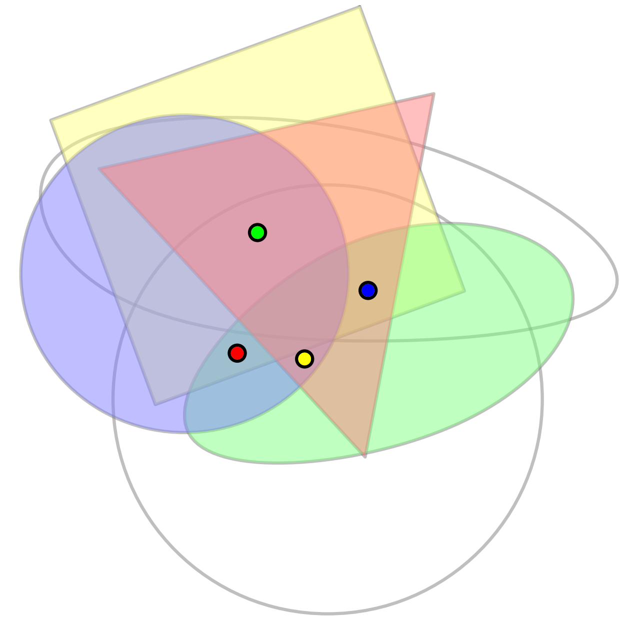

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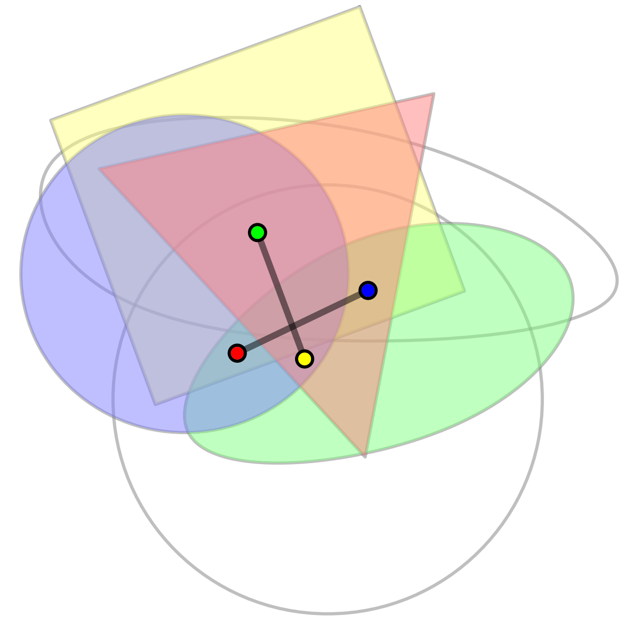

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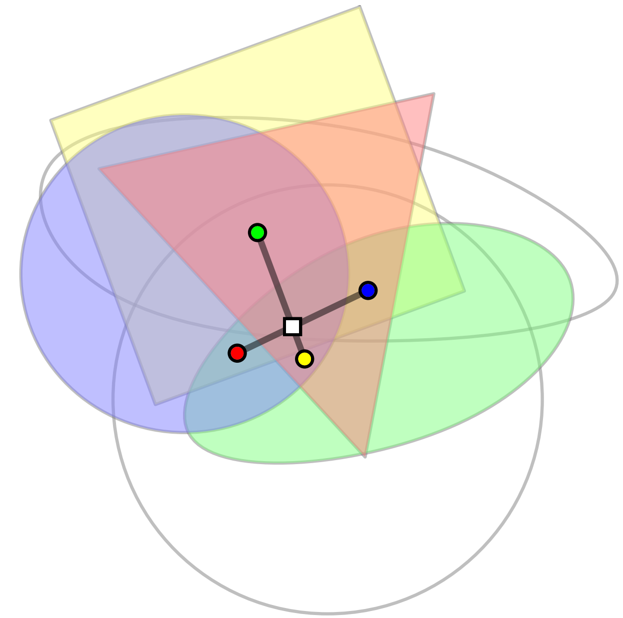

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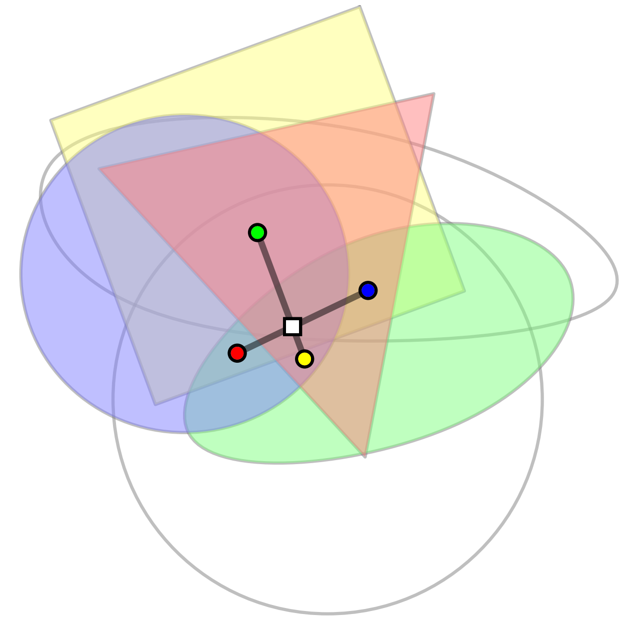

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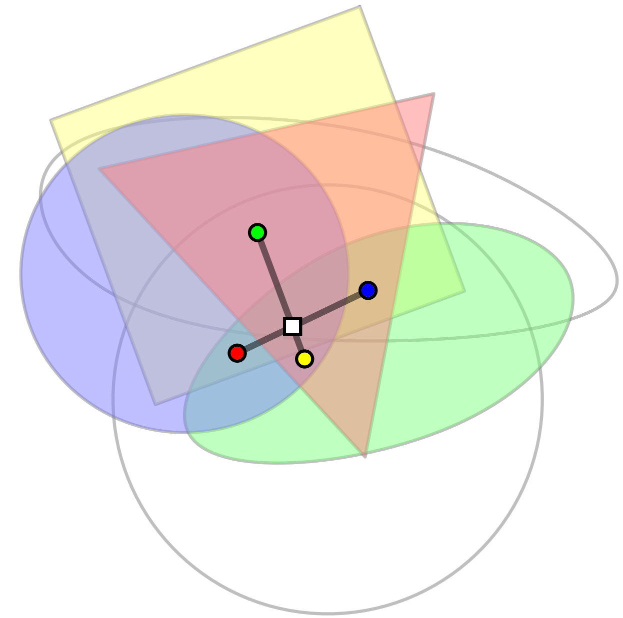

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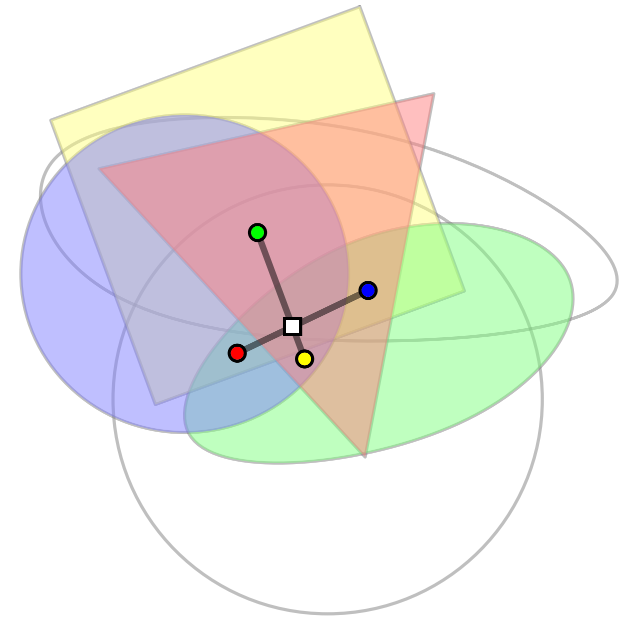

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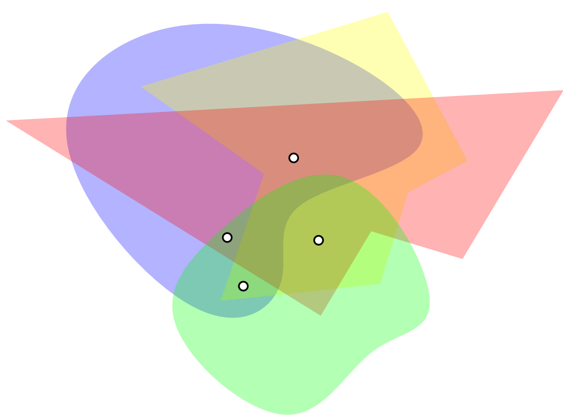

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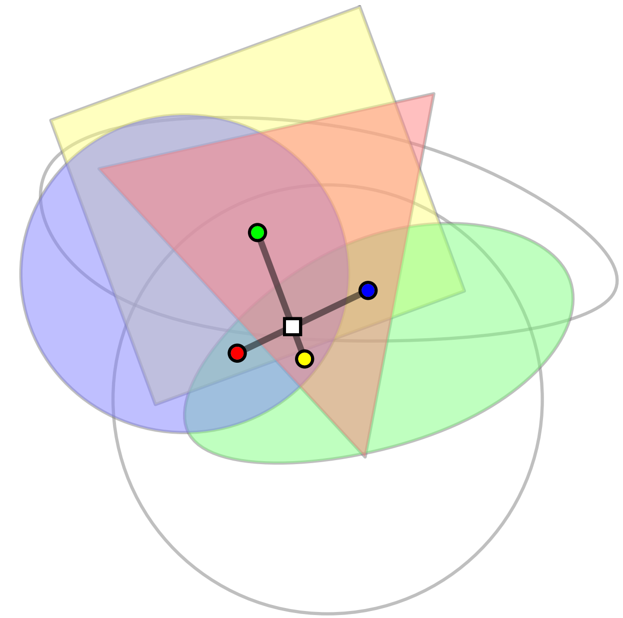

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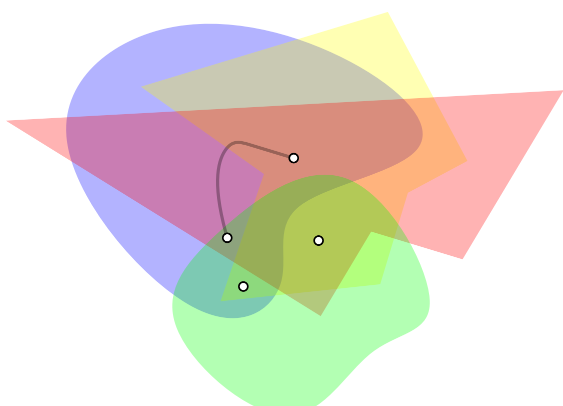

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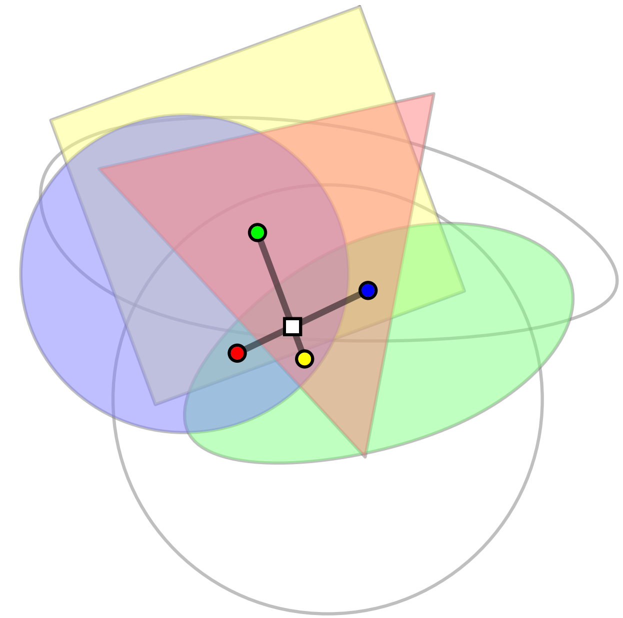

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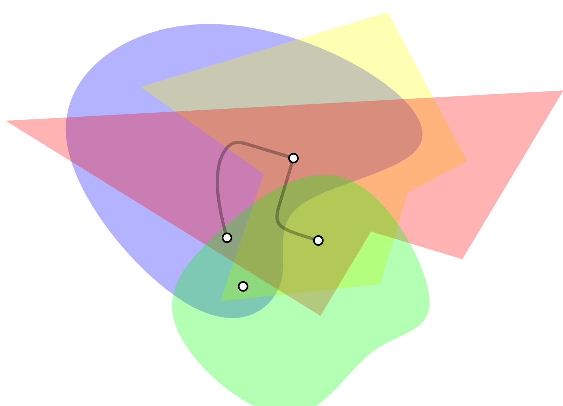

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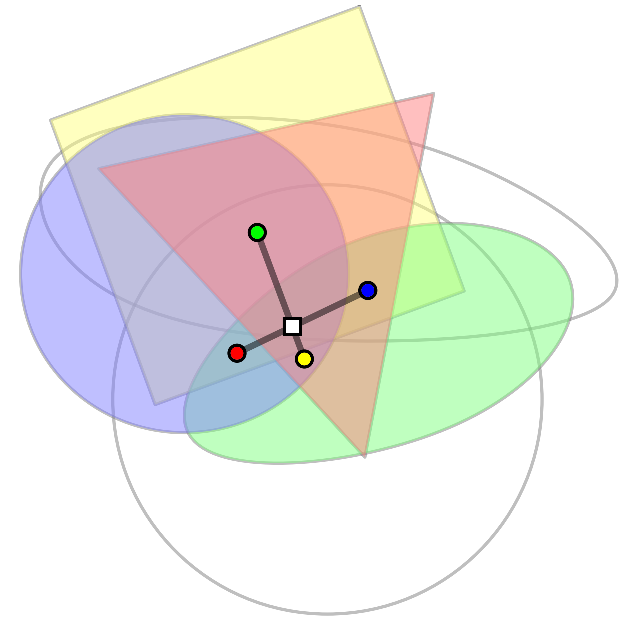

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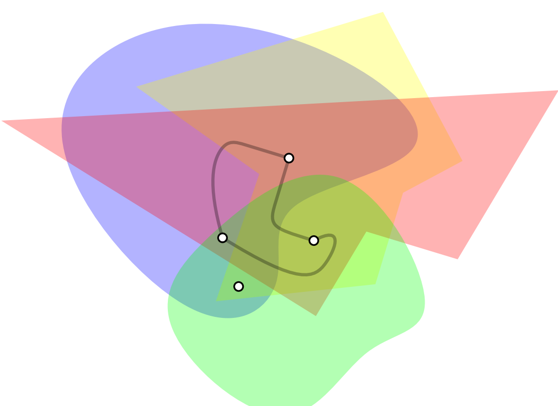

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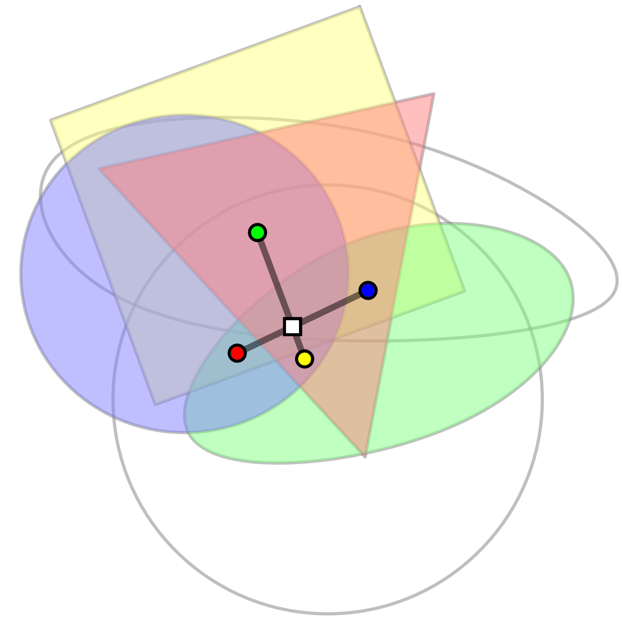

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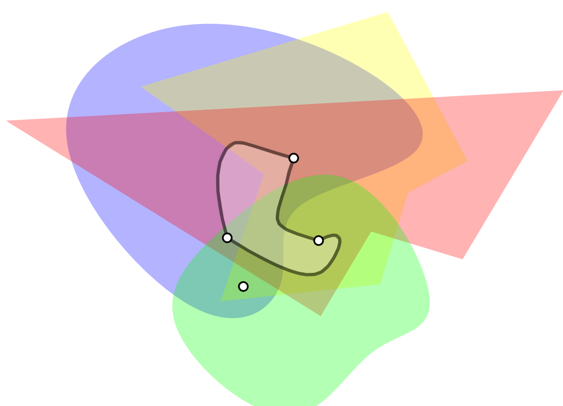

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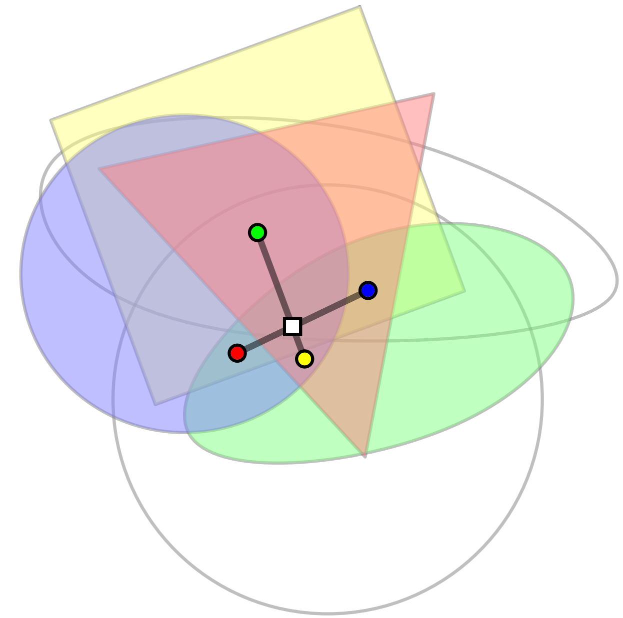

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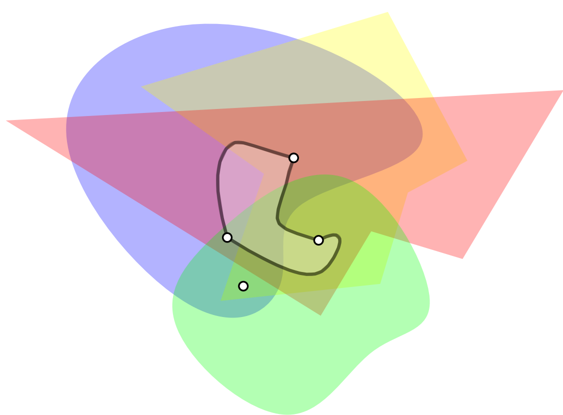

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Idea: Analyze intersection patterns of **topological** set systems by drawing **non-embeddable** complexes inside!

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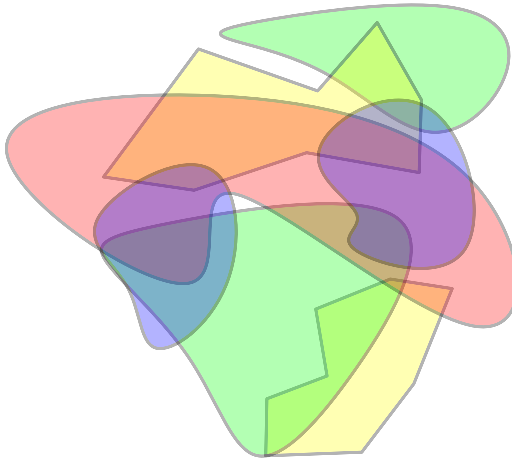

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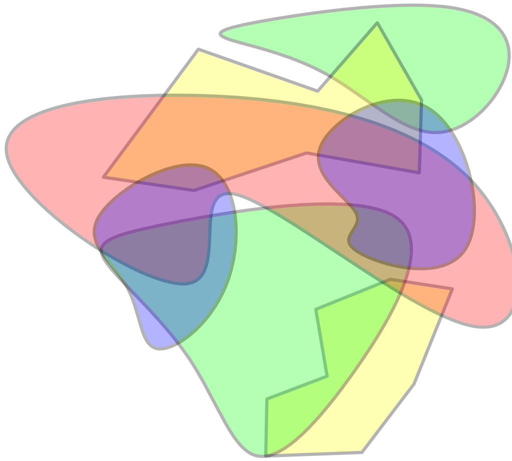

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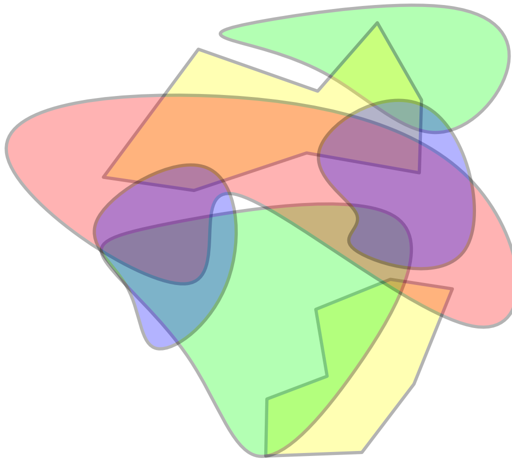

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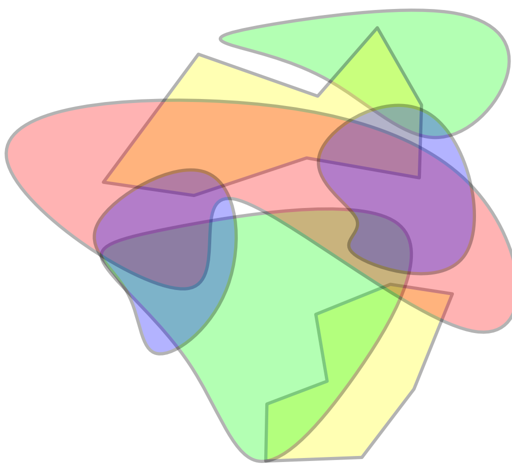

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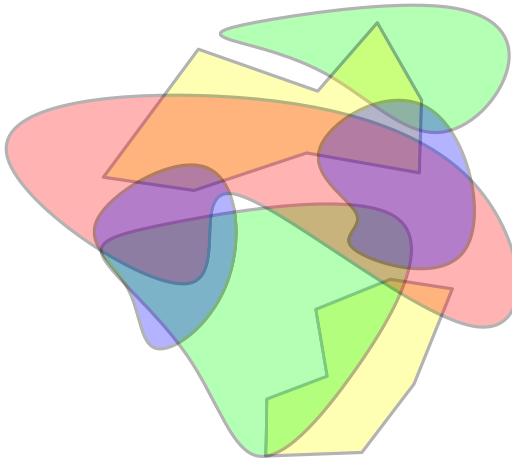
For every family $G \subset \mathcal{F}$ of size $|\mathcal{F}| - (b + 1)$.

Two points can be connected **inside** $\cap G$. *Label the edge with $\mathcal{F} \setminus G$.*

Convexity $\rightsquigarrow$ **bounded topological complexity**.



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Suppose $\mathcal{F}$ has empty intersection and is **minimal** *for that.* $(d = 2)$

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Ramsey $\Rightarrow$ if $\mathcal{F}$ is large enough, some K_5 has disjoint edges with disjoint labels.

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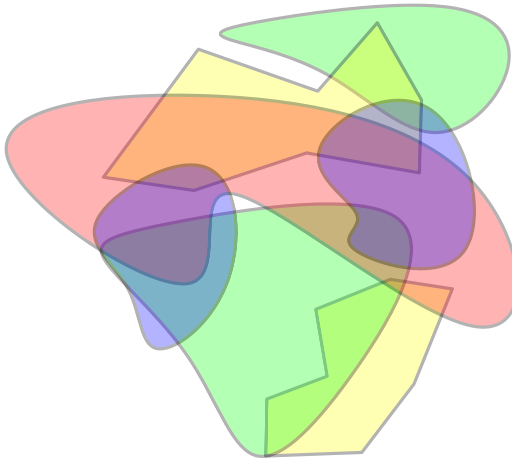

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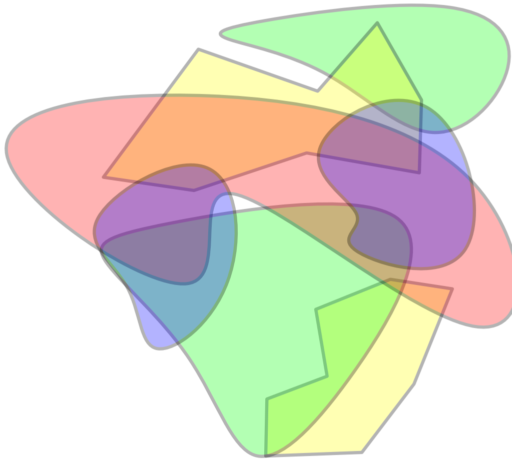

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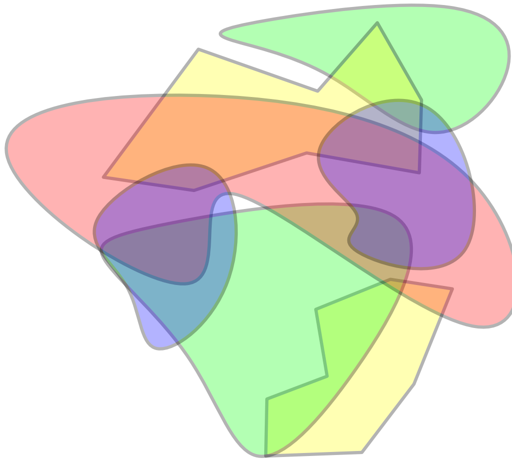

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Try to continue: fill triangles within intersections.

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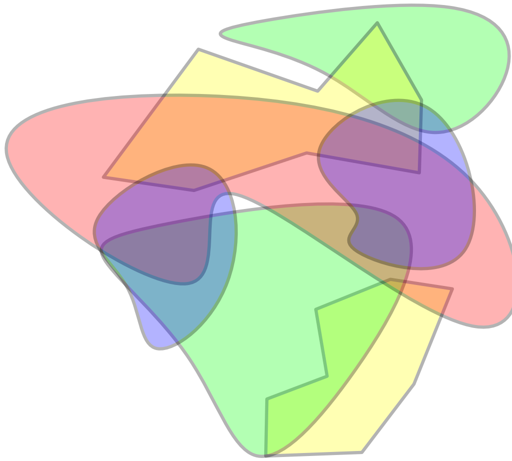

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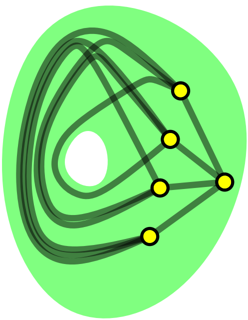

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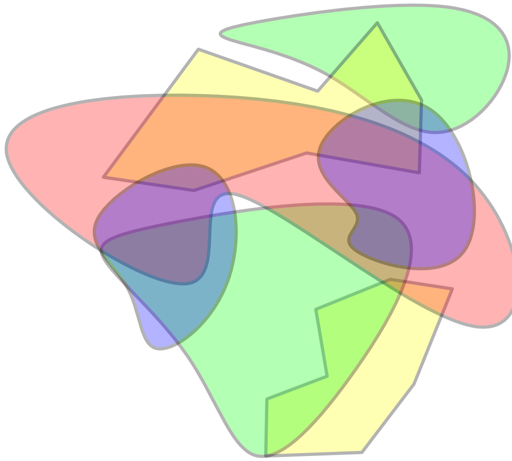

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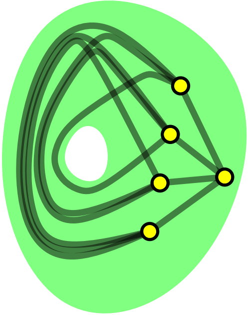
Work with $\mathbb{Z}_2$-homology.



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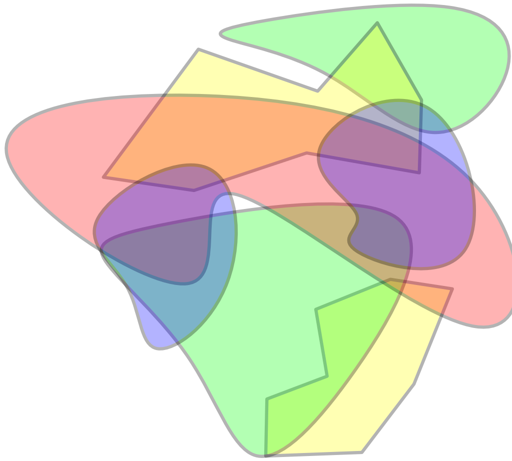

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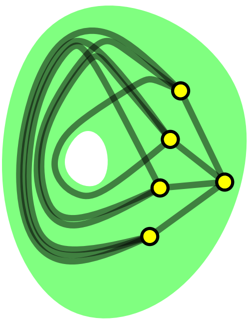
Build **homological minors***.* [Wagner 2011]



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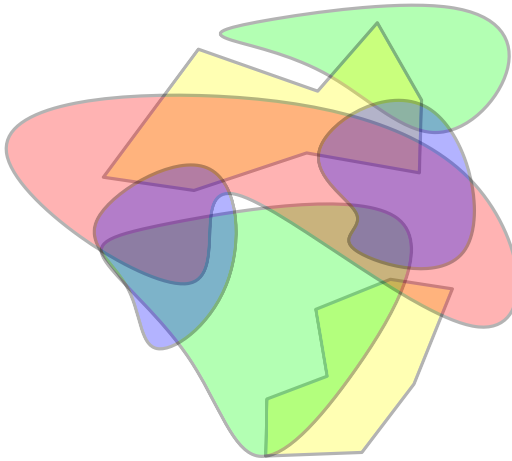

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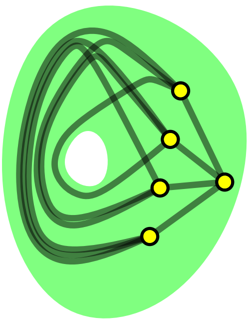

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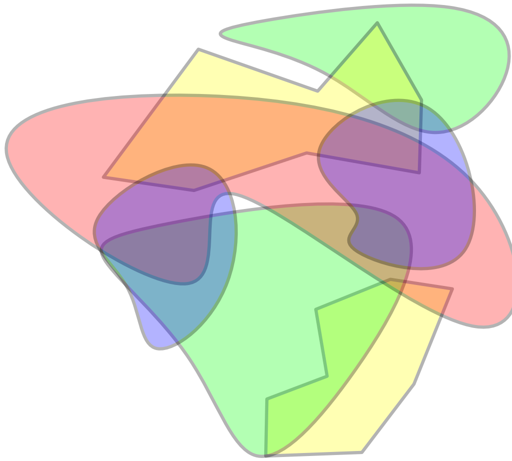

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$\Rightarrow$ **Fractional Helly**, (p, q), **weak ϵ-nets, ...**

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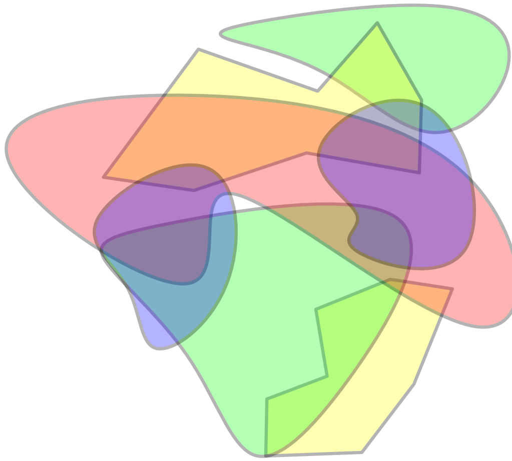

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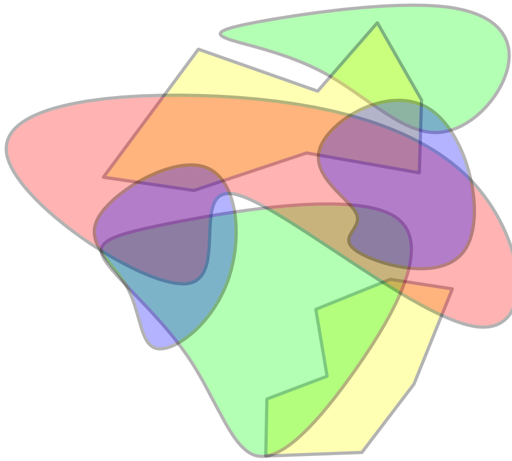

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▷ The fractional Helly number is always $d + 1$. [G-Holmsen-Patáková 2021]

Wrapping up!

Convexity **reveals** much more general properties.

- ▷ overlap properties of maps from simplicial complexes,
- ▷ properties of hypergraphs with certain forbidden patterns.
- ▷ consequences of properties of nerves,

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Effective use-cases? More applications?

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More interplay of geometry, combinatorics, topology and algorithms?

Many active research directions...

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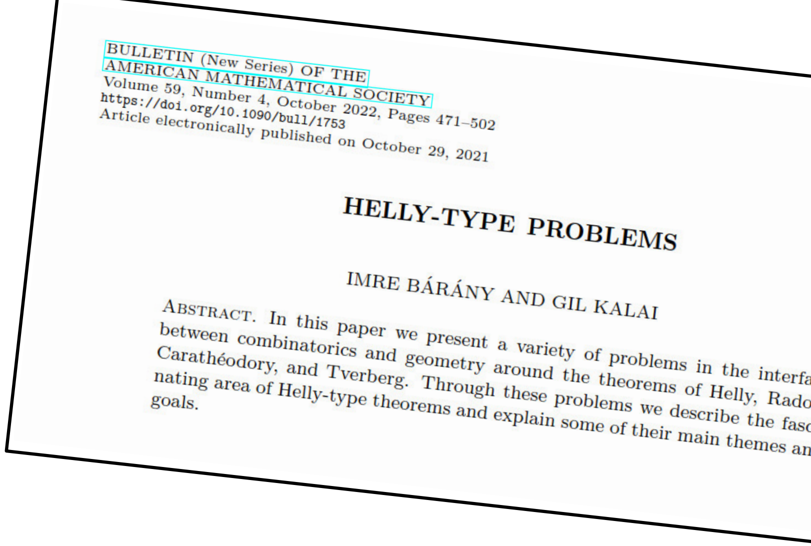

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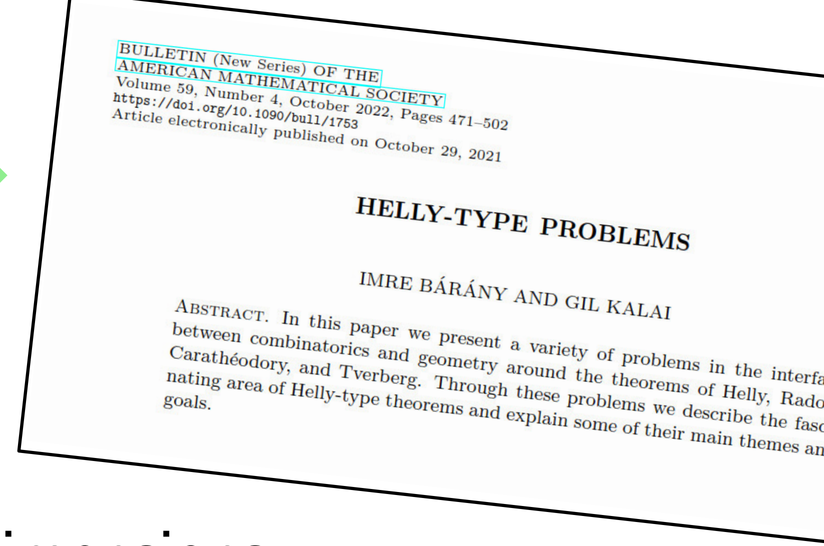
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Question. Suppose a family of red/blue convex sets in $\mathbb{R}^d$ are such that any red/blue pair intersect. Can a positive fraction of one color be pierced by a single line?

[Martinez-Roldán-Rubin 2020]

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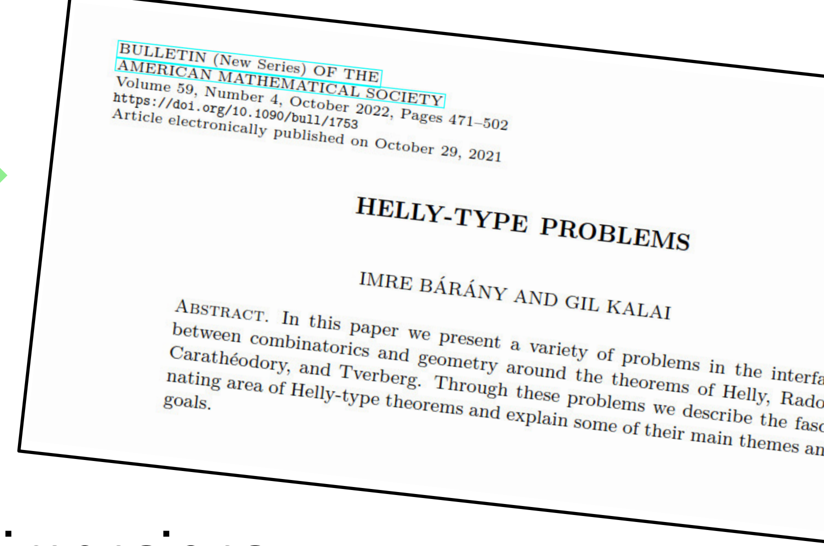
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▷ A "Homological VC dimension?"

Conjecture. For any $\gamma > 0$, if $\mathcal{F}$ is a set system in $\mathbb{R}^d$ such that for any $m \geq 1$, for any intersection of m sets from $\mathcal{F}$, the Betti numbers sum to at most γm^{d+1}, then $\mathcal{F}$ satisfies a fractional Helly theorem.

[Kalai-Meshulam 2004]

Thank you for your attention!