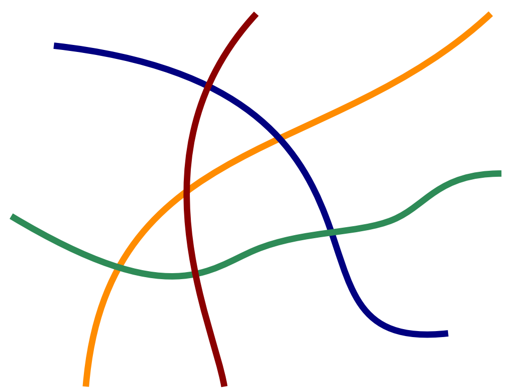
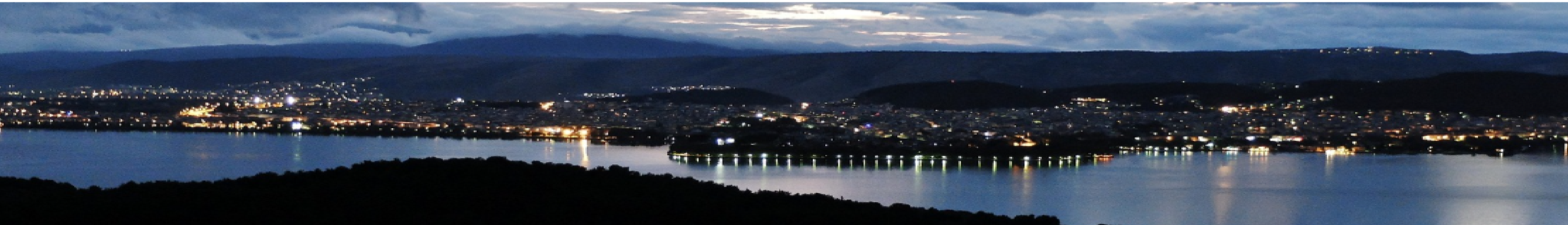
40th European Workshop on Computational Geometry

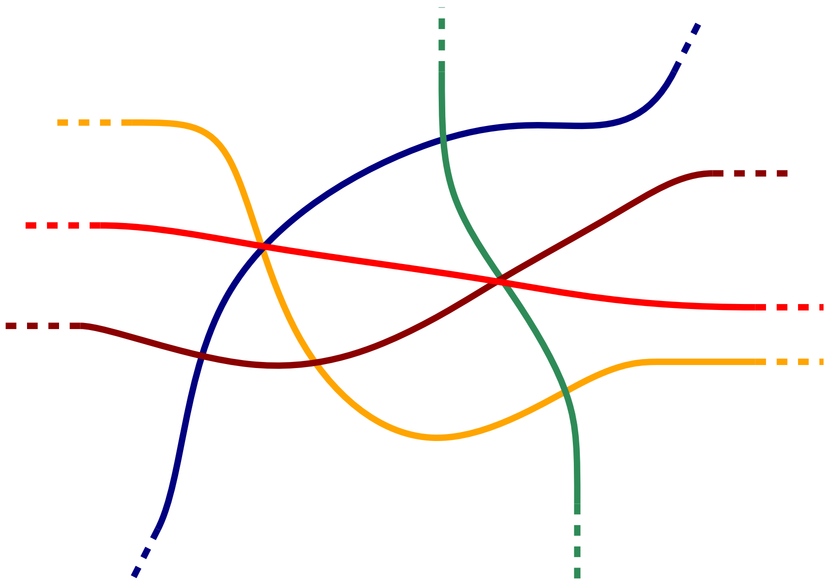
Coloring problems on arrangements of pseudolines



Sandro M. Roch



pseudoline arrangements

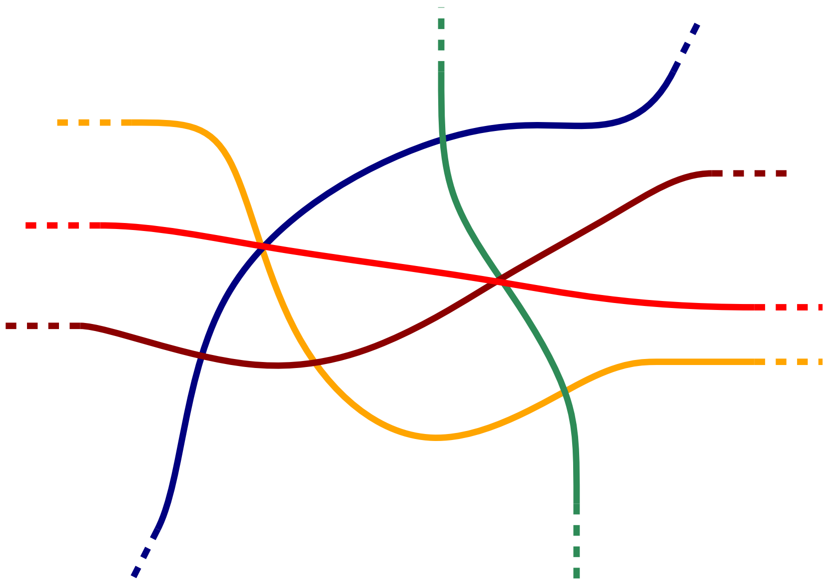


pseudoline arrangements

pseudoline arrangement:
continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ with

$$\lim_{t\to\infty} \|f_i(t)\| = \lim_{t\to-\infty} \|f_i(t)\| = \infty,$$

each two cross in exactly one point.



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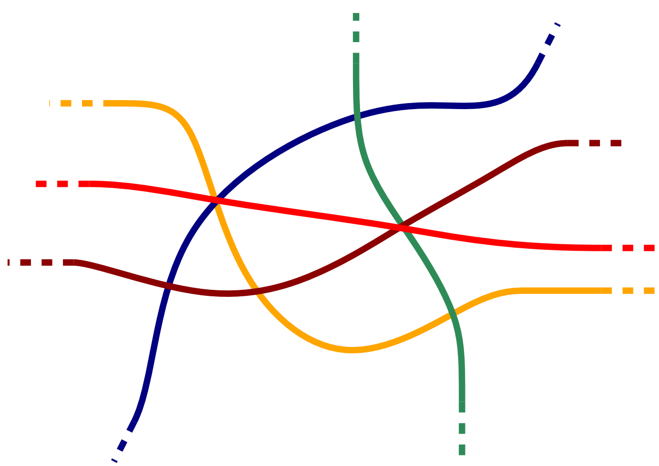
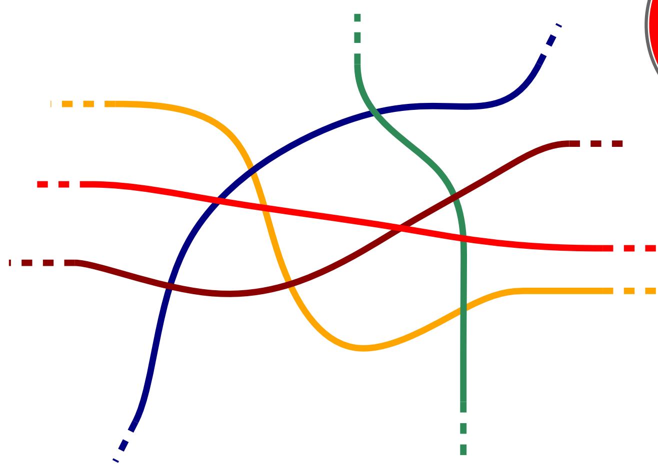
Ex: *nonsimple* arrangement:



Ex: *simple* arrangement:



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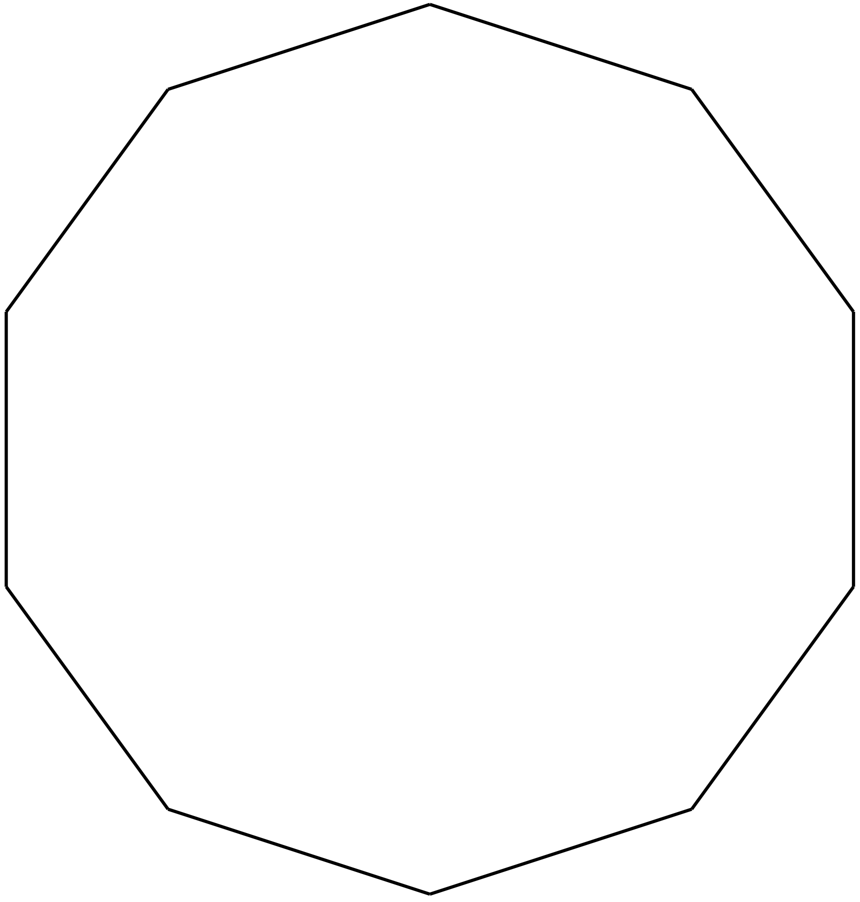

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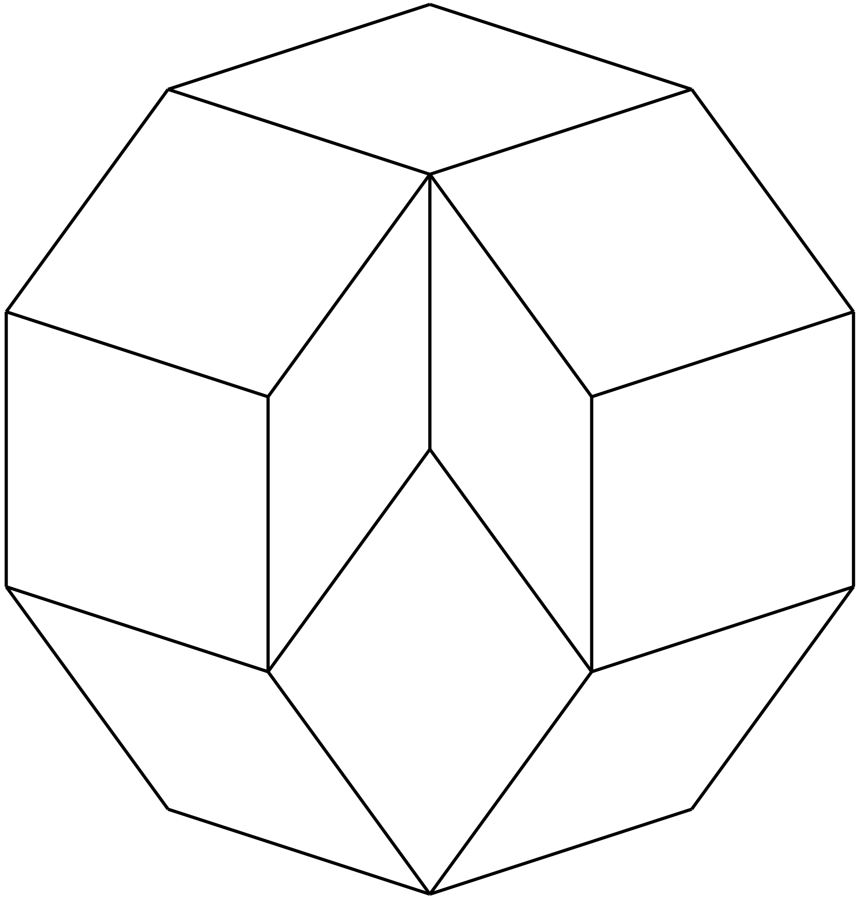

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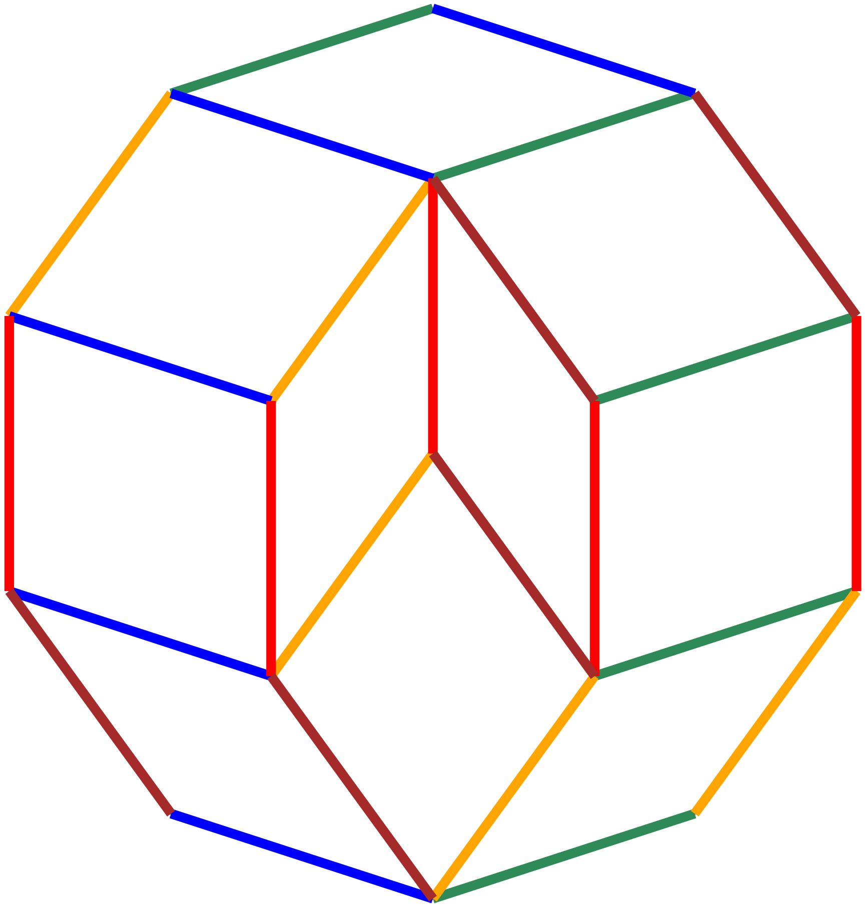

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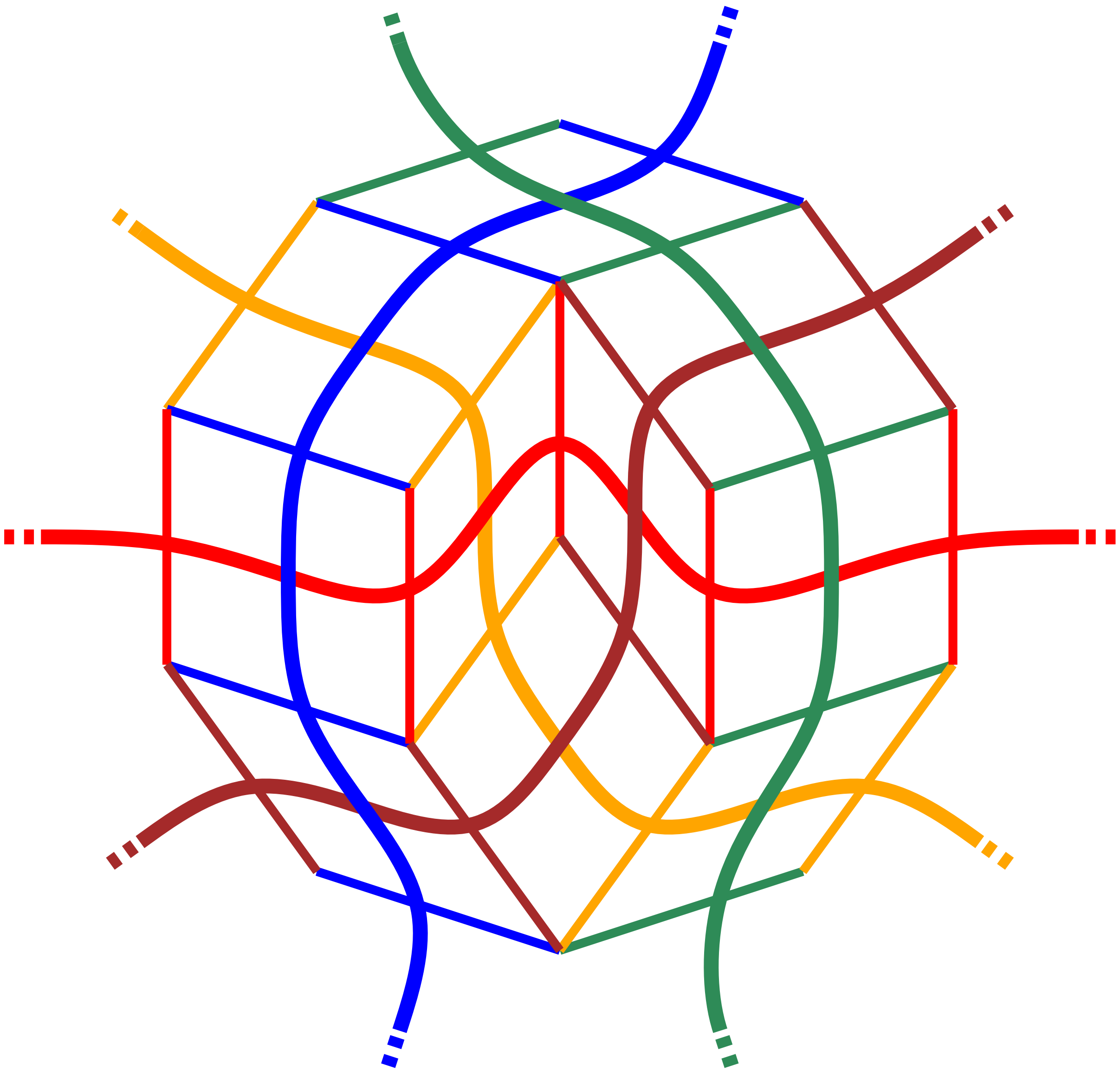

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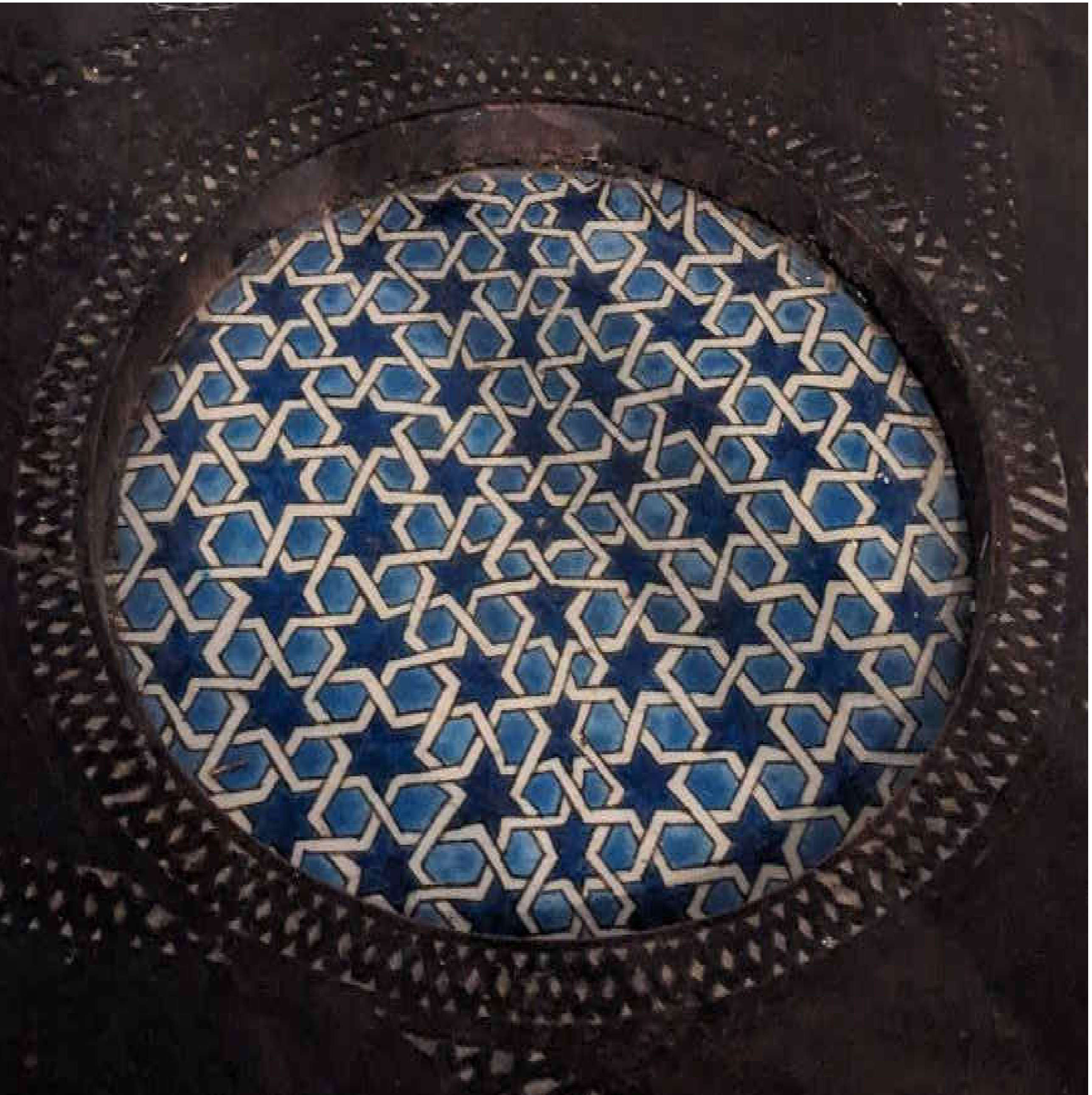
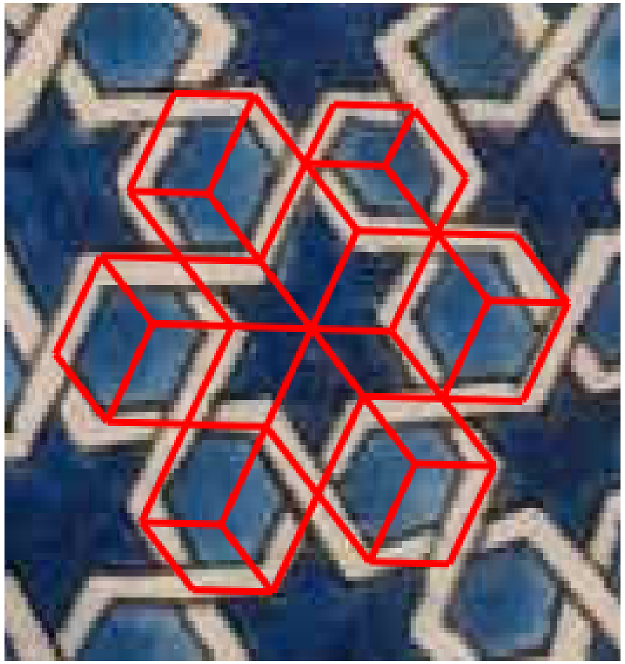
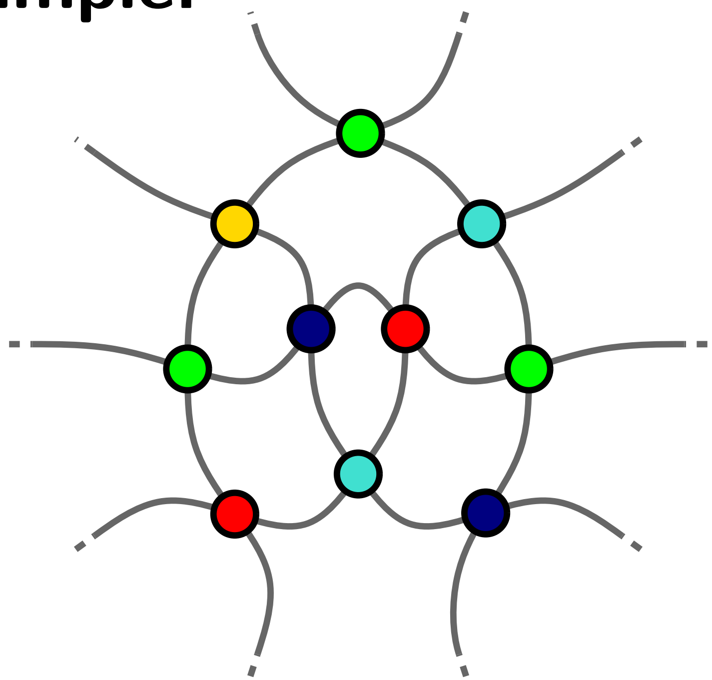




face respecting colorings

Theorem I: Let $\mathscr{A}$ be an arrangement of n pseudolines.
The crossings of $\mathscr{A}$ can be colored using n colors so that no color appears twice **on the boundary of any cell**.

Example:



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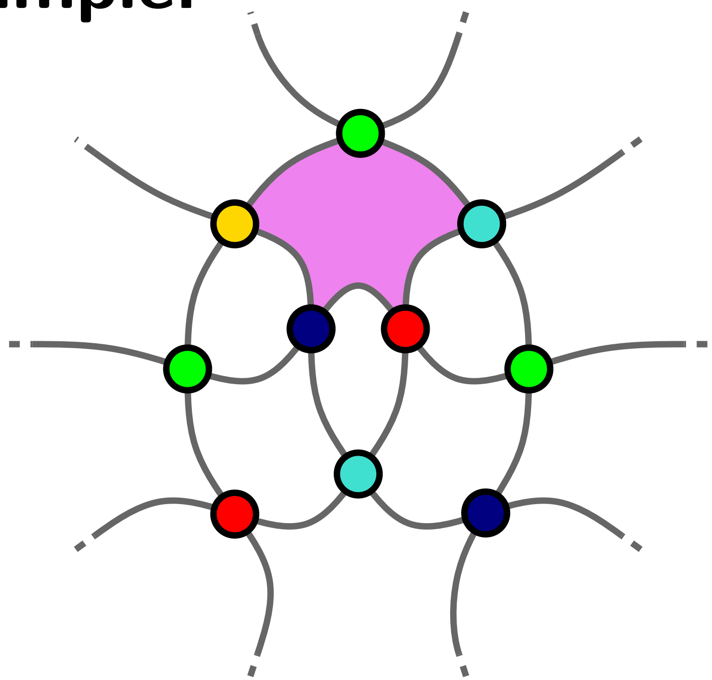

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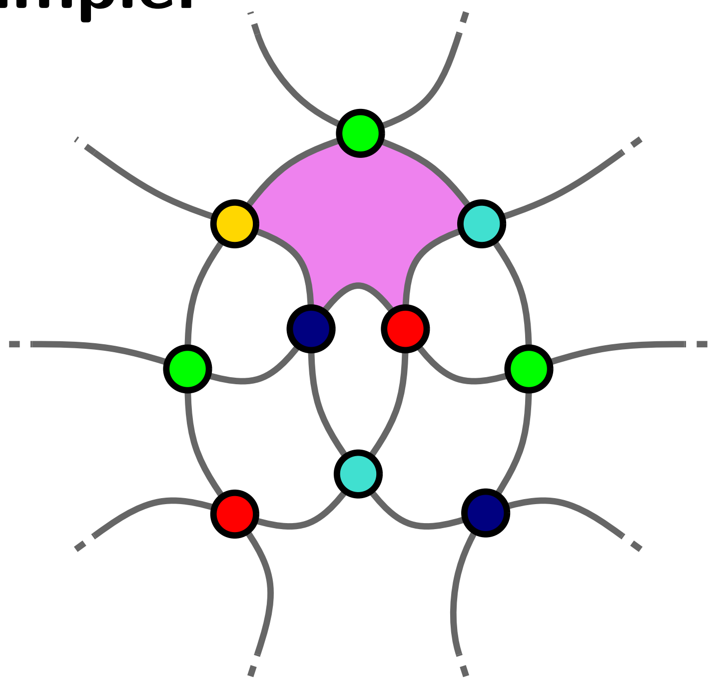
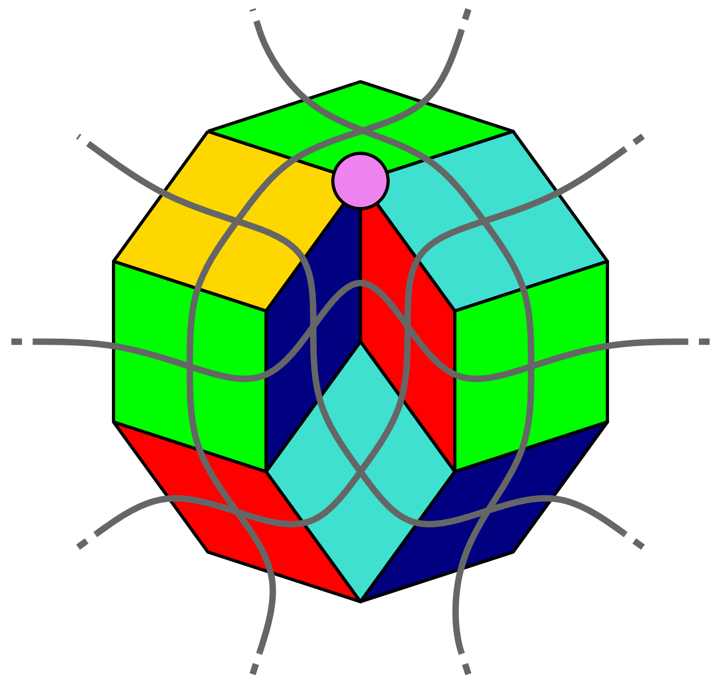
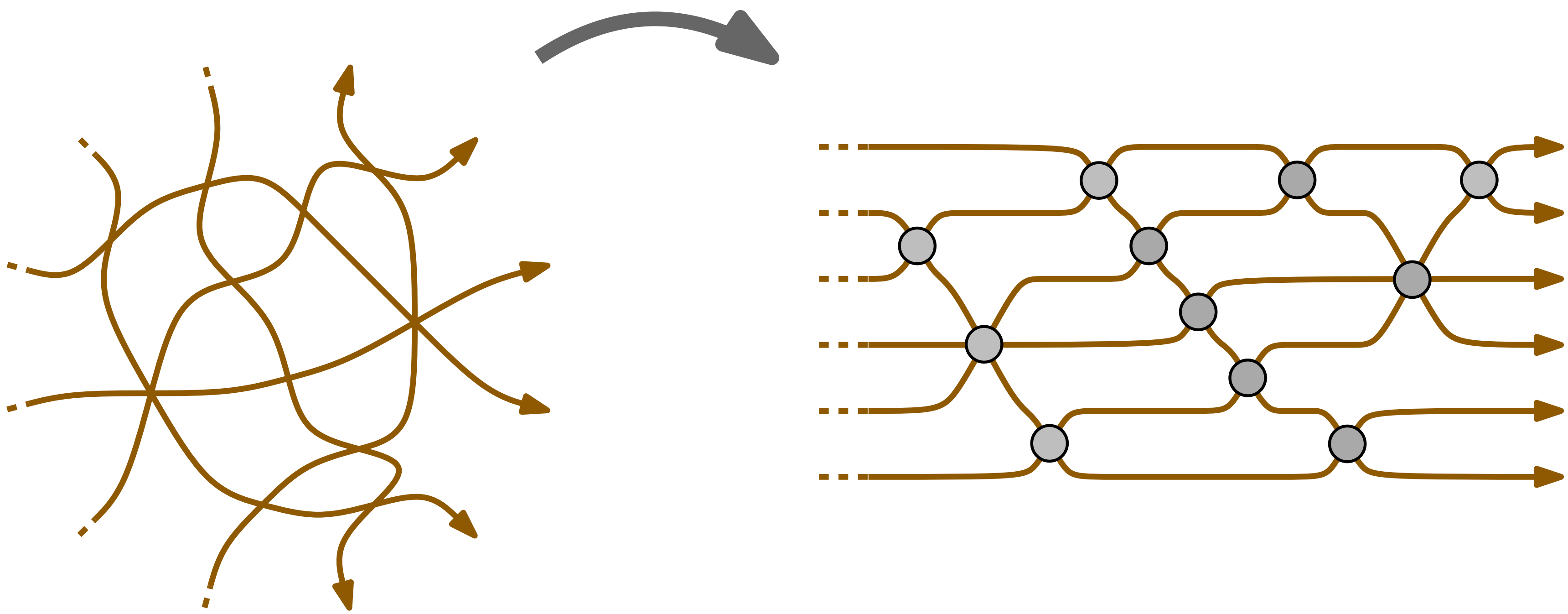




face respecting colorings

Proof idea: Greedily color the wiring diagram!



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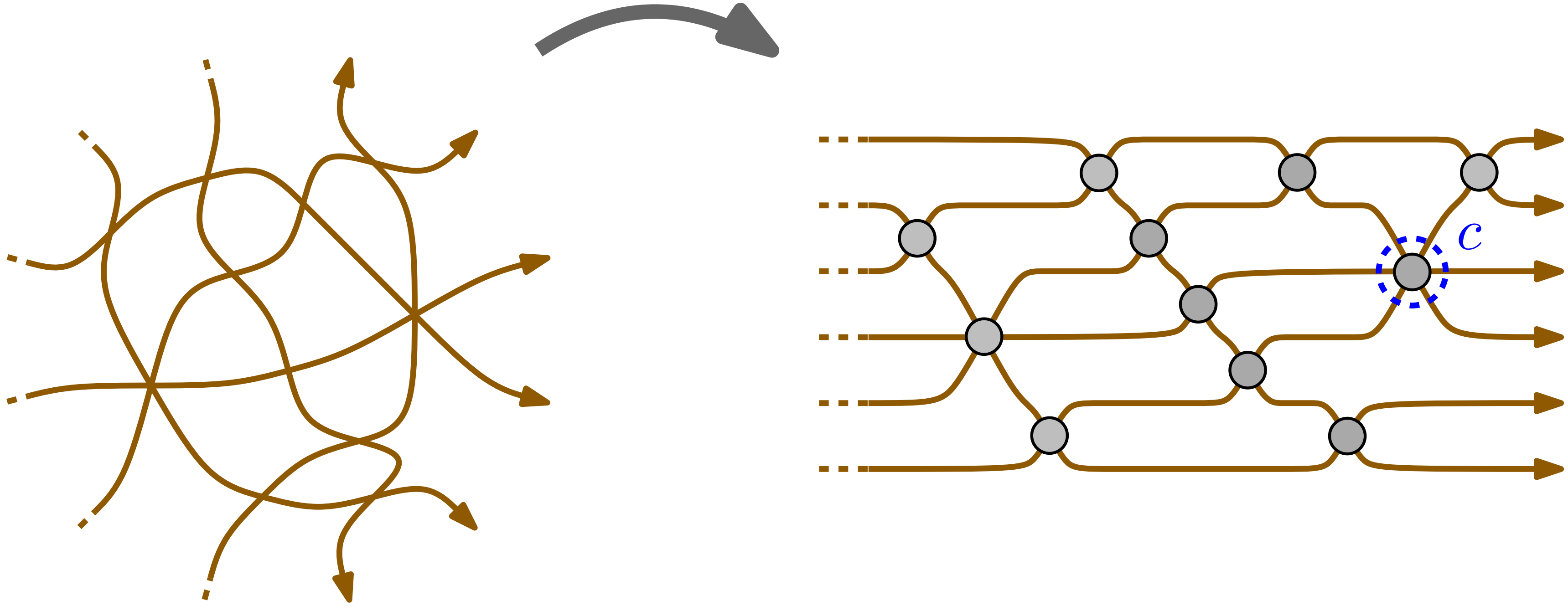

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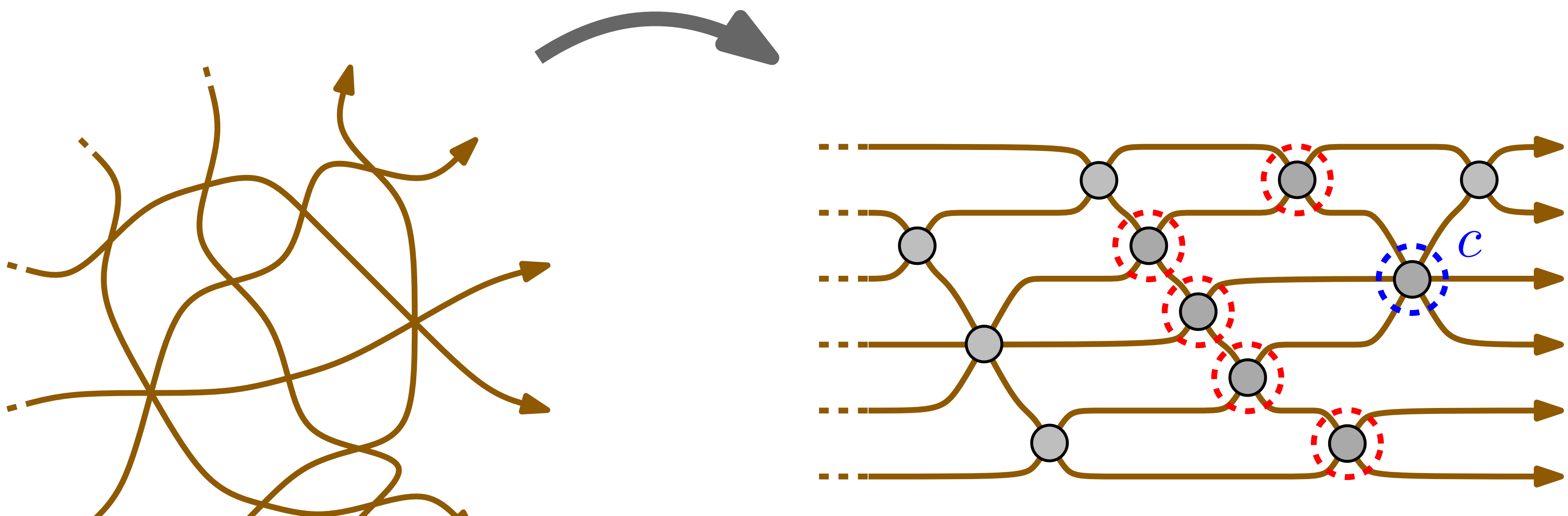
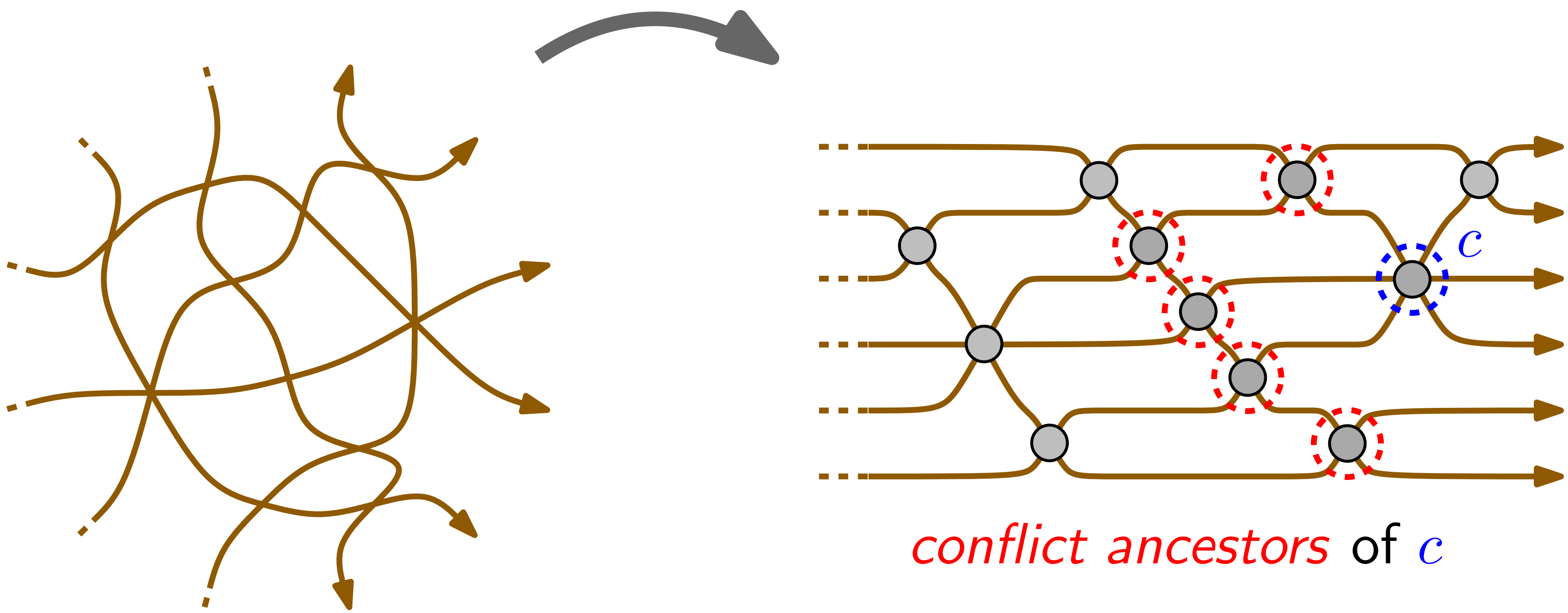


conflict ancestors of c

face respecting colorings

Proof idea: Greedily color the wiring diagram!



Claim: Every crossing has at most $n-1$ conflict ancestors.

$\square$

line respecting colorings

Theorem II: Let $\mathscr{A}$ be an arrangement of n pseudolines. The crossings of $\mathscr{A}$ can be colored using n colors so that no color appears twice **along any pseudoline**.

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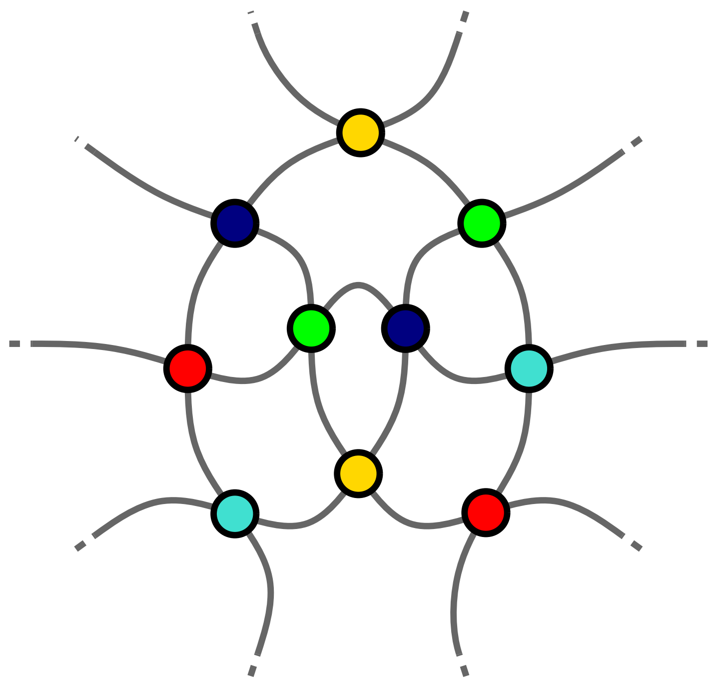

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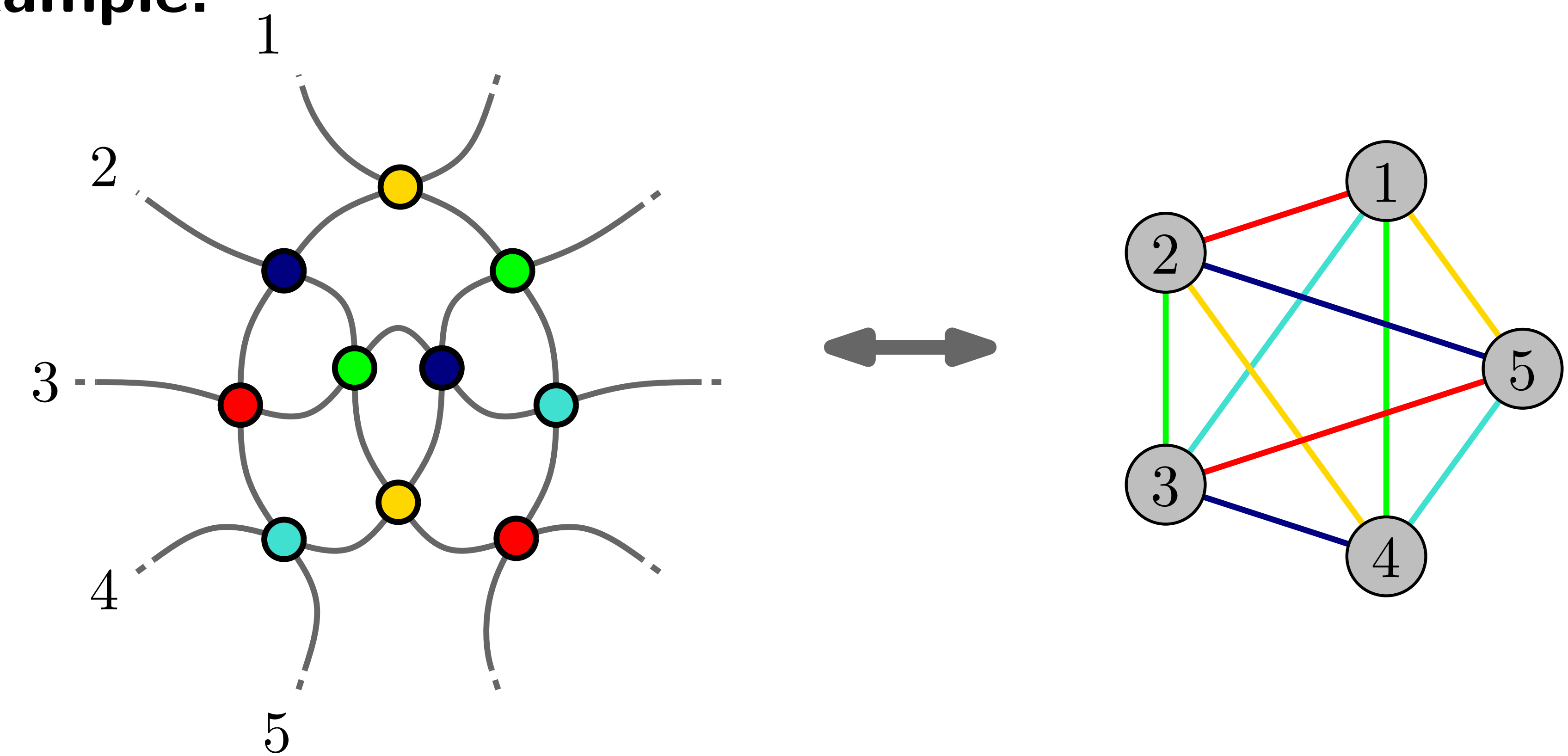

line respecting colorings

proof:

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Hypergraph $\mathscr{H}(\mathscr{A})$:

- vertices $\sim$ pseudolines
- hyperedges $\sim$ crossings

line respecting colorings

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Hypergraph $\mathscr{H}(\mathscr{A})$:

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- hyperedges $\sim$ crossings

Theorem (Kang, Kelly, Kühn, Methuku, Osthus, 2023)
Every simple hypergraph on n vertices can be edge-colored using n colors.

Recent breakthrough in hypergraph coloring!!!

$\square$

line respecting colorings

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Hypergraph $\mathscr{H}(\mathscr{A})$:

- vertices $\sim$ pseudolines
- hyperedges $\sim$ crossings

direct proof?

deterministic algorithm?

Theorem (Kang, Kelly, Kü… Methuku, Osthus, 2023)
Every simple hypergraph … n vertices can be edge-colored using n colors.

Recent breakthrough in hypergraph coloring!!!

□

line respecting colorings

Def:

$\mathrm{mx}(\mathscr{A}) :=$ max. number of crossings per pseudoline in $\mathscr{A}$

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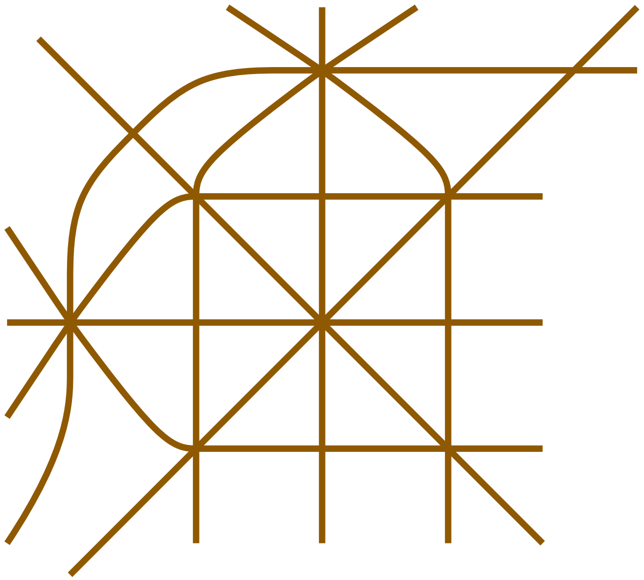

$\mathrm{mx}(\mathscr{A}) = 4$

line respecting colorings

Def:

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Fact: number of pseudolines $n \leq 845 \cdot \mathrm{mx}(\mathscr{A})$

(Dumitrescu, 2023)

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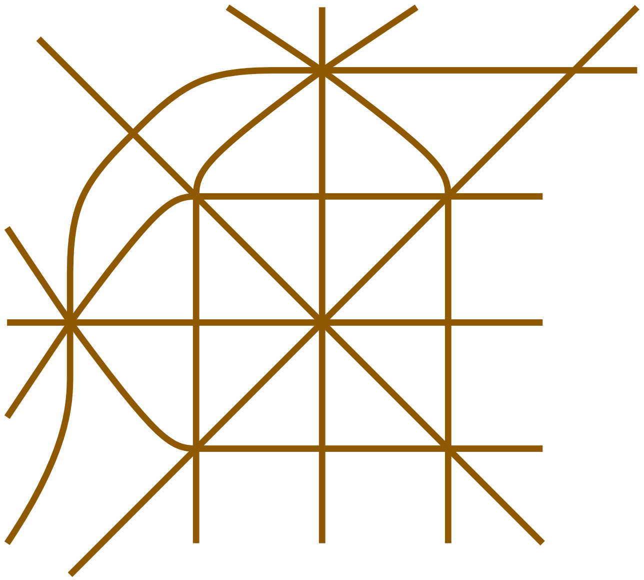

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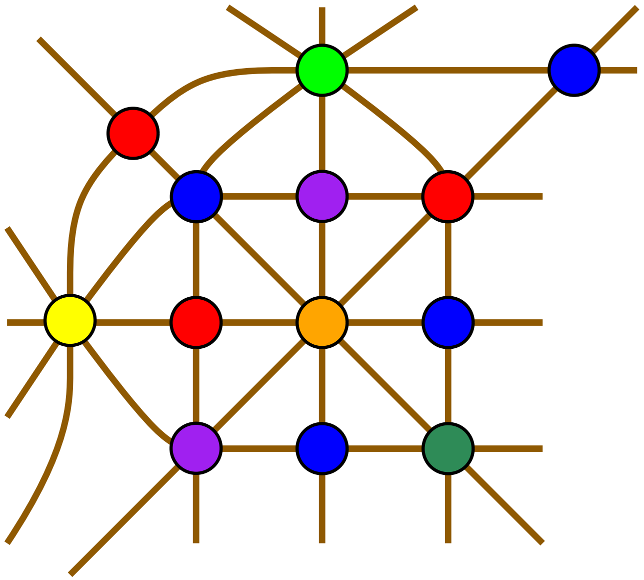

$\mathrm{mx}(\mathscr{A}) = 4$

need $\mathrm{mx}(\mathscr{A}) + 3 = 7$ colors

line respecting colorings

Conjecture:
There exists some constant c so that one can color the crossings of every arrangement using $\mathrm{mx}(\mathscr{A}) + c$ colors.

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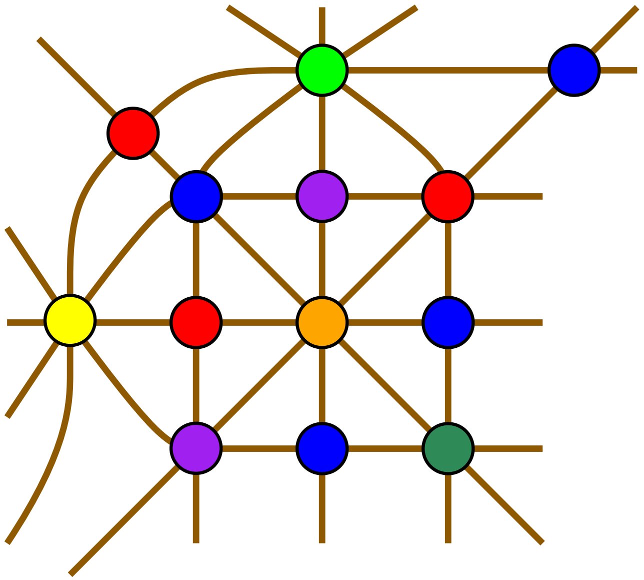

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pseudoline coloring

Def: *pseudoline coloring* of arrangement $\mathscr{A}$:

- color the pseudolines of $\mathscr{A}$
- avoiding monochromatic crossings

$\chi_{pl}(\mathscr{A})$: minimal number of colors in pseudoline coloring

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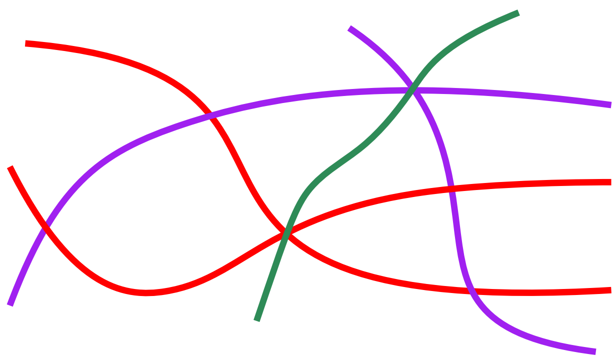

$$\chi_{pl}(\mathscr{A}) = 3$$

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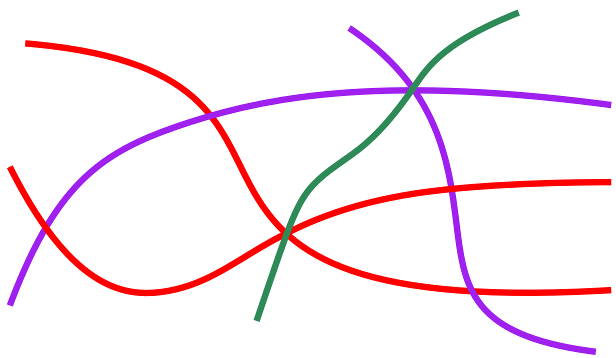

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First observations:

- $2 \leq \chi_{pl}(\mathscr{A}) \leq n$ (unless $n < 2$)

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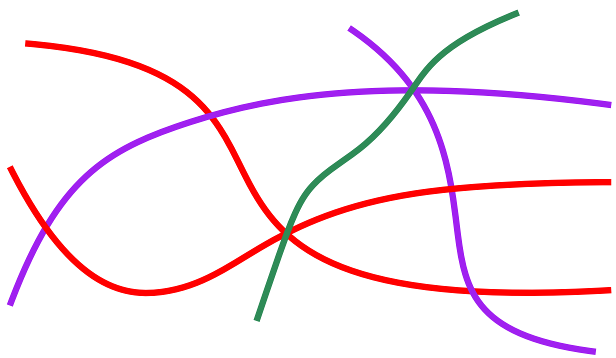

$\chi_{pl}(\mathscr{A}) = 3$

First observations:

- $2 \leq \chi_{pl}(\mathscr{A}) \leq n$ (unless $n < 2$)
- $\mathscr{A}$ simple $\Leftrightarrow \chi_{pl}(\mathscr{A}) = n$

pseudoline coloring

Theorem III:

Let $\mathscr{A}$ be an arrangement of n pseudolines.

The pseudolines of $\mathscr{A}$ can be colored using $\mathcal{O}(\sqrt{n})$ colors avoiding monochromatic crossings of degree at least 4.

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Proposition:

Given an arrangement $\mathscr{A}$ of n pseudolines, it is NP-hard to compute $\chi_{pl}(\mathscr{A})$.

Questions?