Faces in Rectilinear Drawings of Complete Graphs

Martin Balko, Anna Brötzner, Fabian Klute, and Josef Tkadlec

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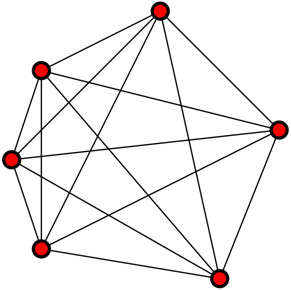

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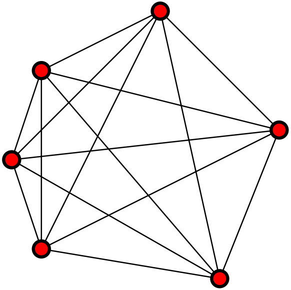

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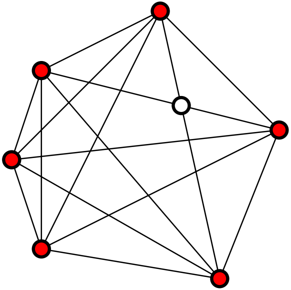

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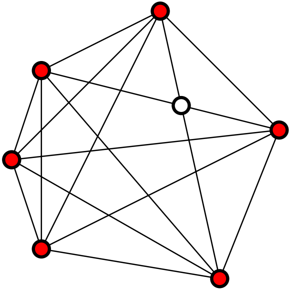

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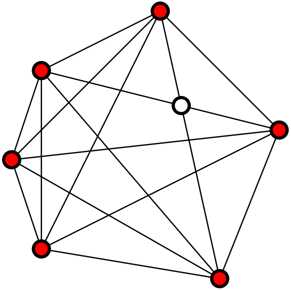

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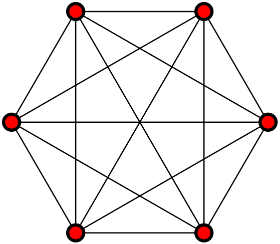

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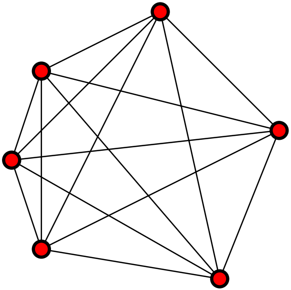

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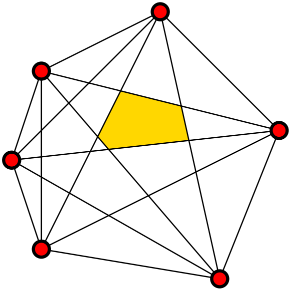

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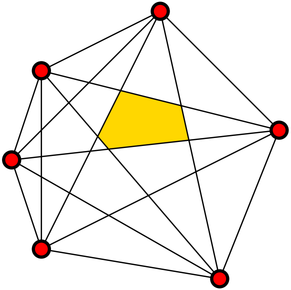

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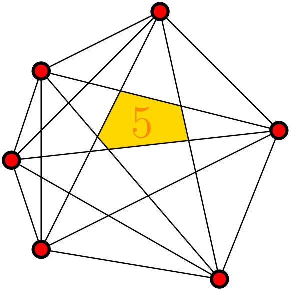

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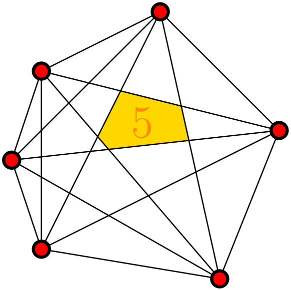

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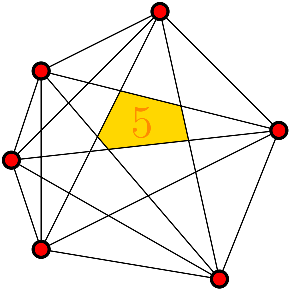

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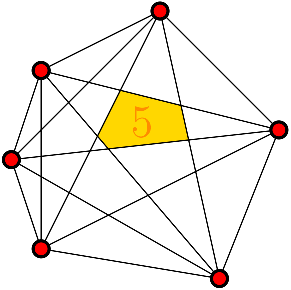

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- How about generic convex drawings? Or regular drawings?

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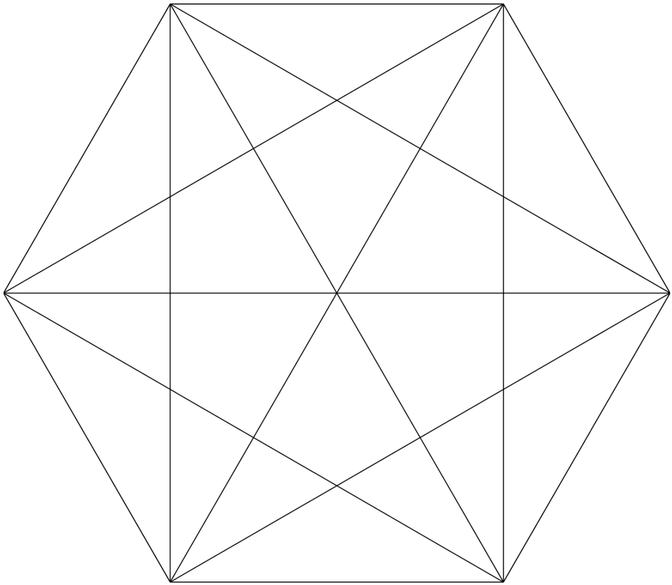

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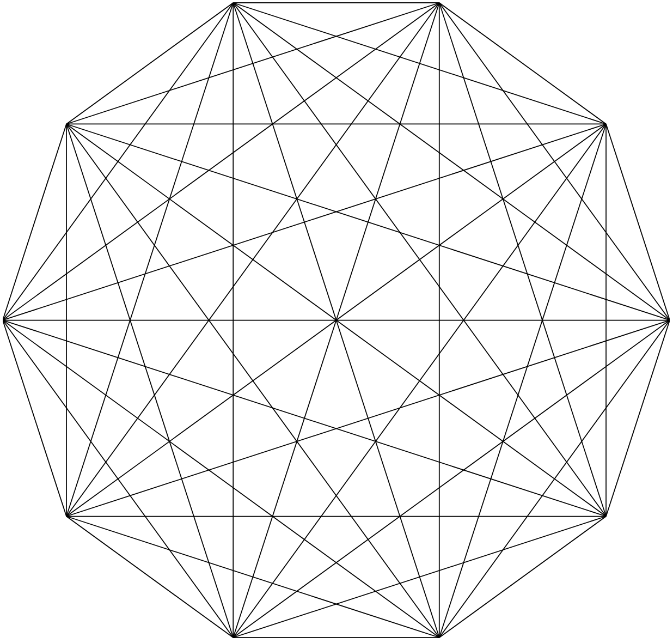

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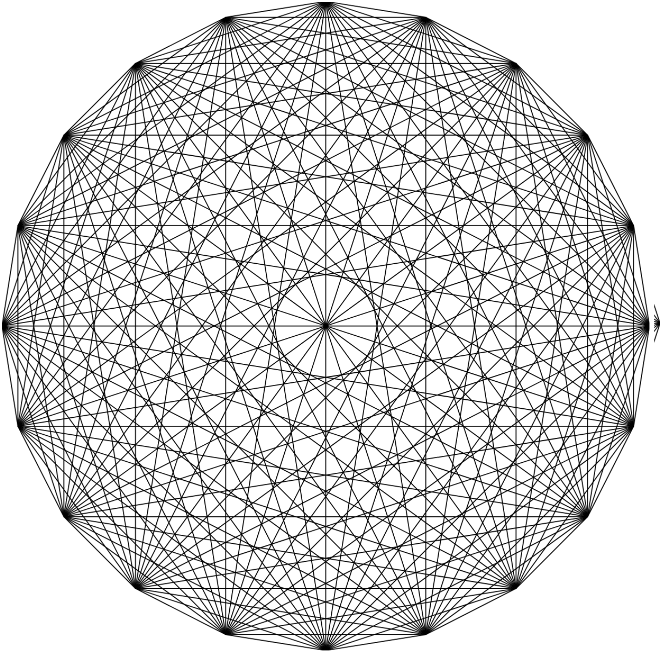

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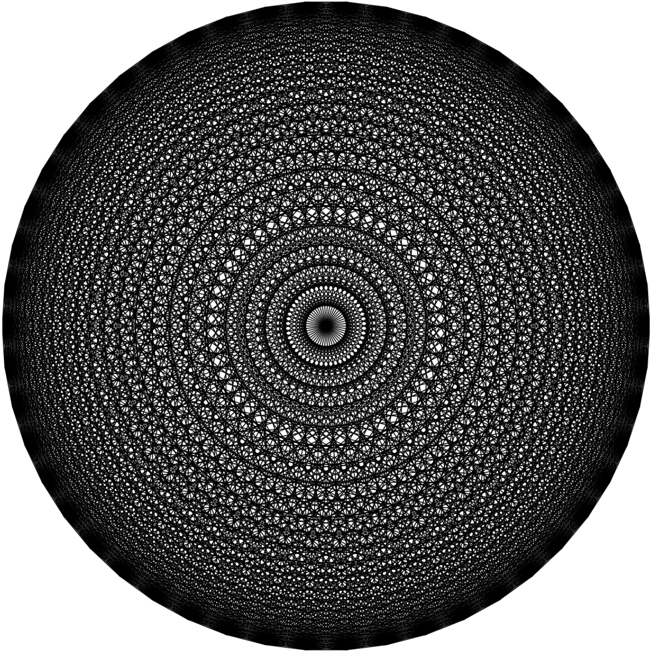

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- Closely related is the work of Poonen and Rubinstein who gave a formula for the number of crossings in regular drawings of K_n.
- It follows from their formula that all regular drawings of K_n with odd n are generic. Also that, apart from the center, no crossing is the intersection of more than 7 edges of a regular drawing of K_n for any n.

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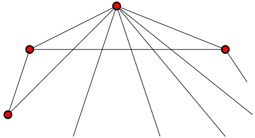

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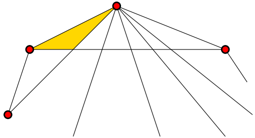

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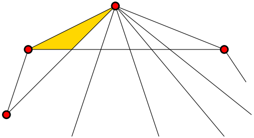

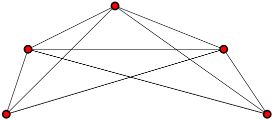

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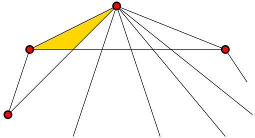

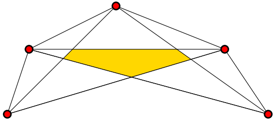

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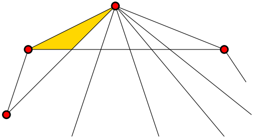

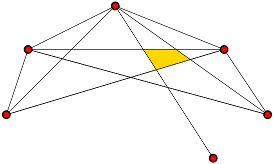

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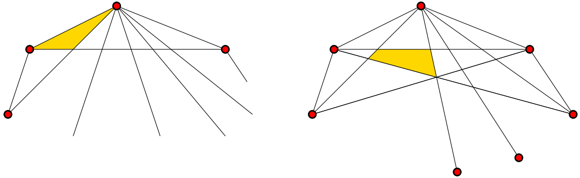

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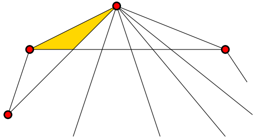

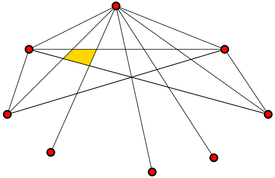

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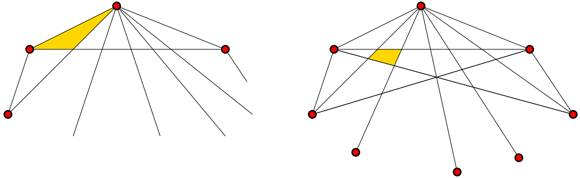

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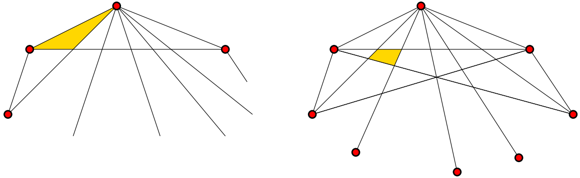

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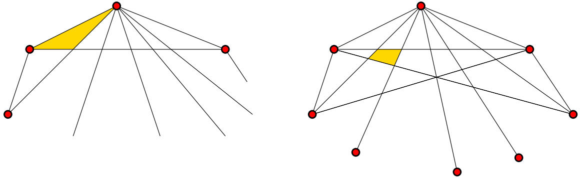

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- We do not know if every convex drawing of large K_n contains a 5-face.

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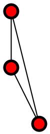

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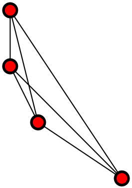

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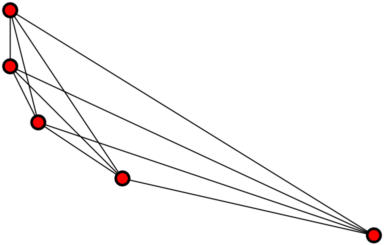

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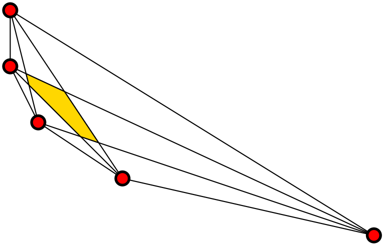

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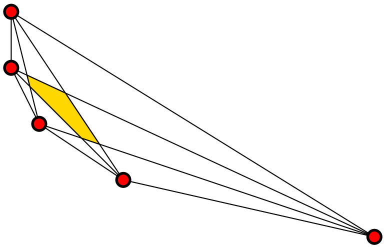

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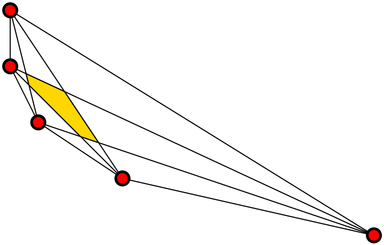

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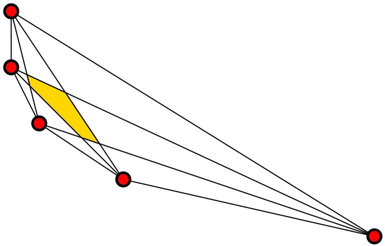

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- The problem of finding 5-faces is difficult if we allow crossings of more than two edges. We know that every convex drawing of K_7 has a 5-face.

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Theorem 3

For $n \in \mathbb{N}$, a regular drawing of K_n contains a 5-face if and only if

$$n \notin \{1, 2, 3, 4, 6, 8, 12\}.$$

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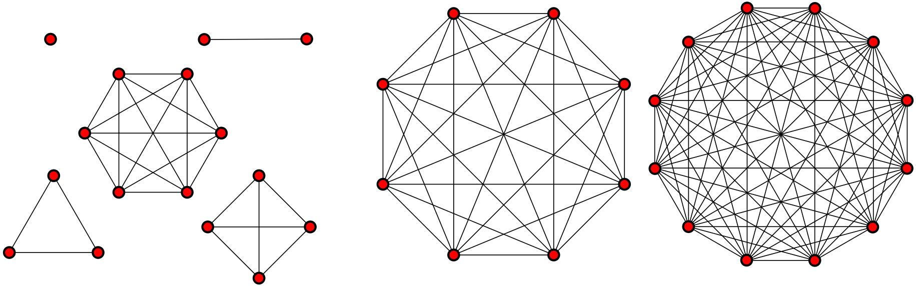

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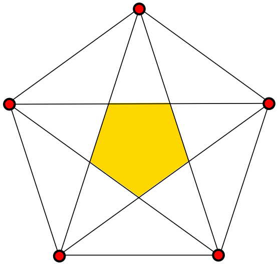

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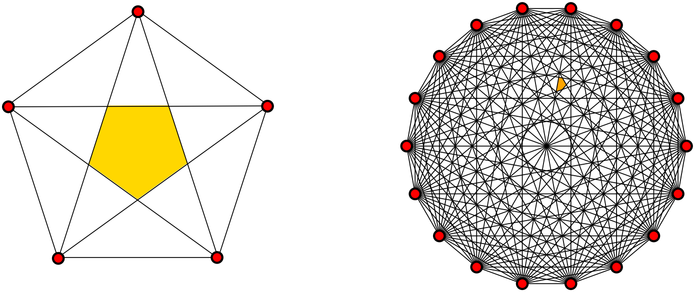

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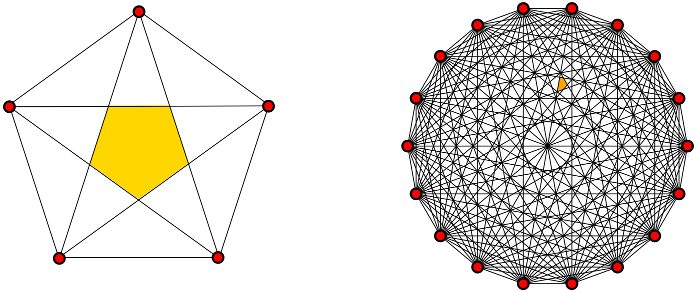

- The proof is quite involved and uses results of Poonen and Rubinstein.

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Thank you for your attention.