Counting Pseudoline Arrangements

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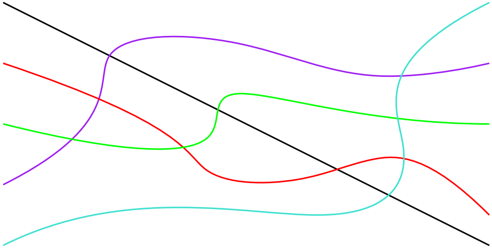
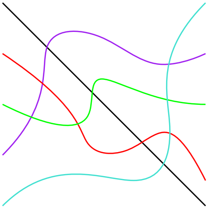
Definition

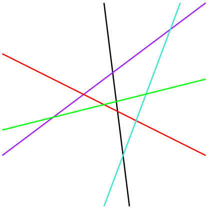
Arrangement of pseudolines: finite family of simple curves s.t.

- each curve approaches infinity in both directions
- every pair intersects in one point where the two curves cross
- *simple*: no three pseudolines intersect in a common point



Isomorphism Classes





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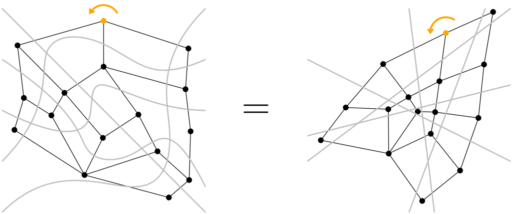

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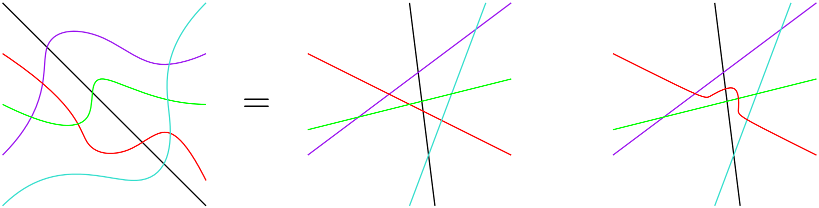

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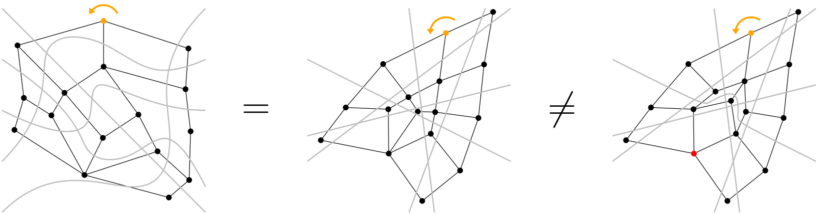

We Are Interested In

$B_n := \#\{\text{marked, simple arrangements of } n \text{ pseudolines}\}$

Small Numbers

n	B_n	
3	2	
4	8	
5	62	
6	908	
7	24698	
8	1232944	
9	112018190	Knuth '92
10	18410581880	Widom et al. '98
11	5449192389984	Yamanaka et al. '10
12	2894710651370536	Samuel '11
13	2752596959306389652	Kawahara et al. '11
14	4675651520558571537540	Tanaka '13
15	14163808995580022218786390	
16	76413073725772593230461936736	Rote '21

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$c^+ < 0.6974$ [Felsner '97]

$c^- > 0.1887$ and $c^+ < 0.6571$ [Felsner & Valtr '11]

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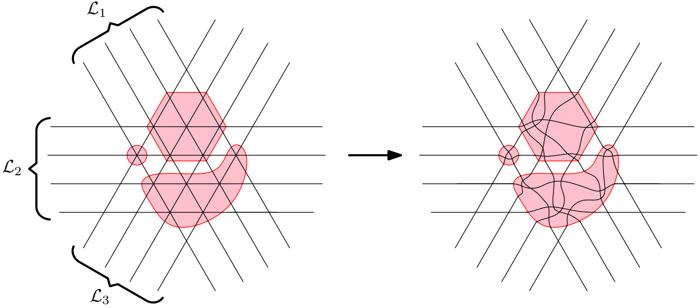
(recently $c^- > 0.27$ in collaboration with Justin Dallant)

General Strategy: Step 1

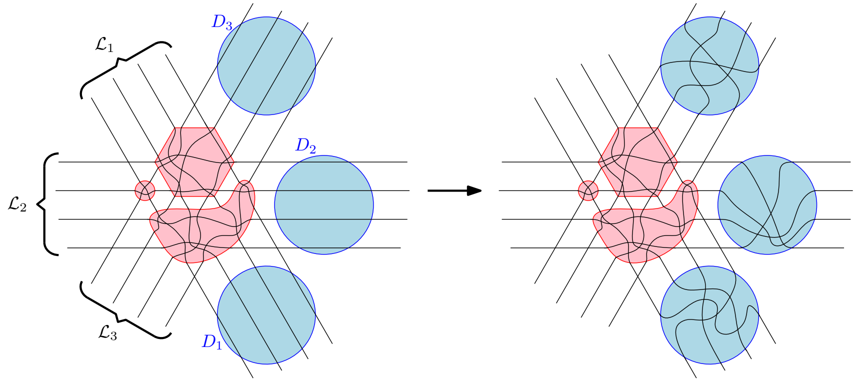
k bundles of lines

bound from below number $F_k(n)$ of consistent partial arrangements



General Strategy: Step 2

$$B_n \geq \underbrace{F_k(n)}_{\text{Step 1}} \cdot \underbrace{\left(B_{\lfloor \frac{n}{k} \rfloor}\right)^k}_{\text{Step 2}}$$

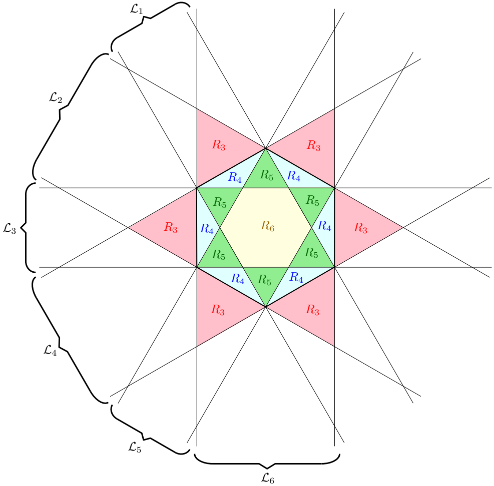


General Strategy: Step 2

Lemma

If $F_k(n) \geq 2^{cn^2 - O(n)}$ *for some* $c > 0$ *then* $B_n \geq 2^{\frac{k}{k-1}cn^2 - O(n \log n)}$.

Precise Setup



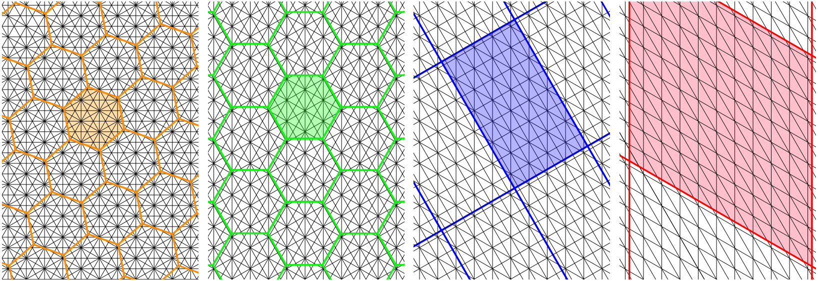
Precise Setup

compute numbers $F(P_i)$ of reroutings within patch P_i via DP

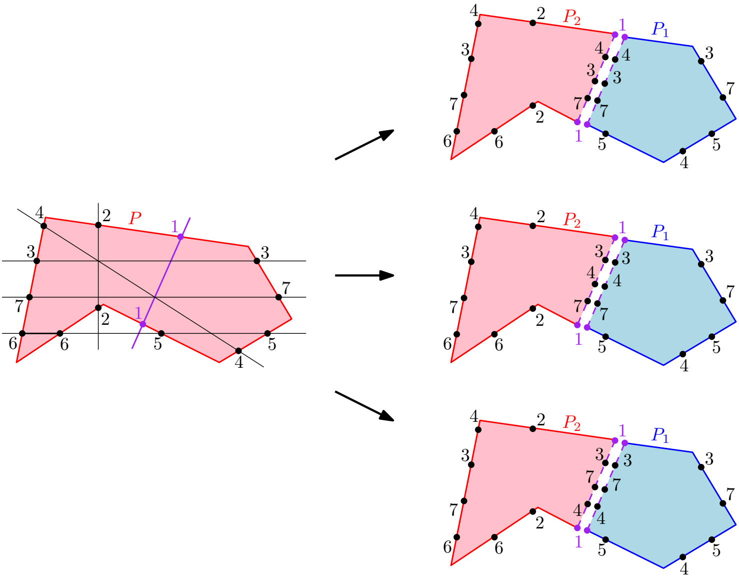
there are $\mu_i(n)$ copies of patch P_i

therefore $F_6(n) \geq \prod_{i=3}^{6} F(P_i)^{\mu_i(n)} \geq 2^{0.2105n^2 - O(n)}$

$c^- > 0.2526$ follows by step 2 (recursion)



Dynamic Program



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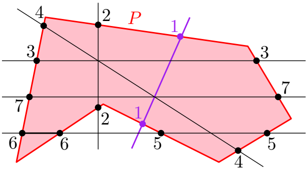

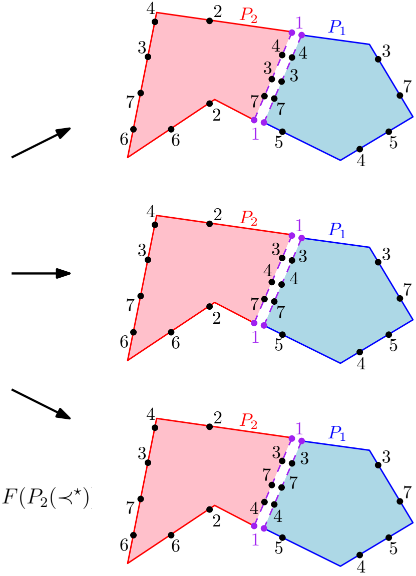

$$F(P) = \sum_{\prec^\star \text{ lin. ext. of } \prec} F(P_1(\prec^\star)) \cdot F(P_2(\prec^\star)$$

Thank you for listening!