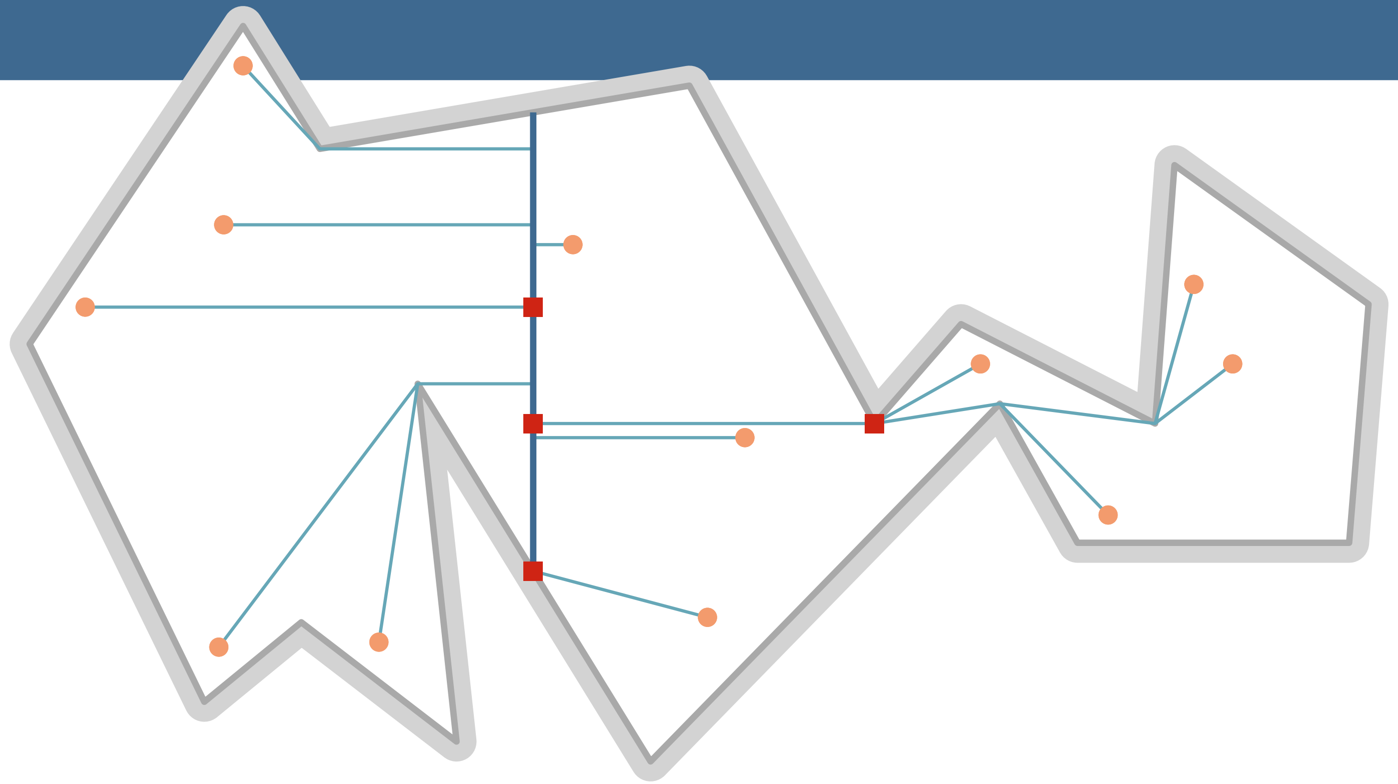
The Complexity of Geodesic Spanners using Steiner Points

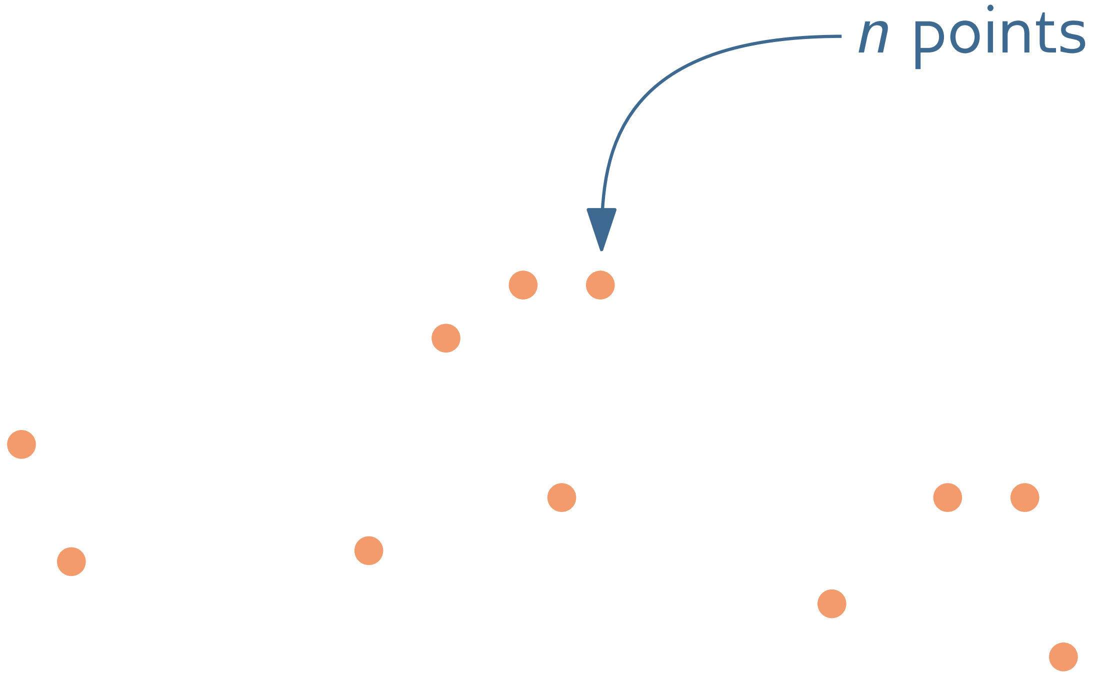
Sarita de Berg, Tim Ophelders, Irene Parada, Frank Staals, Jules Wulms



Geodesic spanners

A *geometric t-spanner* $\mathcal{G}$ connects the points in a set S using few *links* s.t.:

- Each link is a shortest path between two points in S
- The distance $d_{\mathcal{G}}(p, q)$ between two points p, q is at most $t \cdot d(p, q)$



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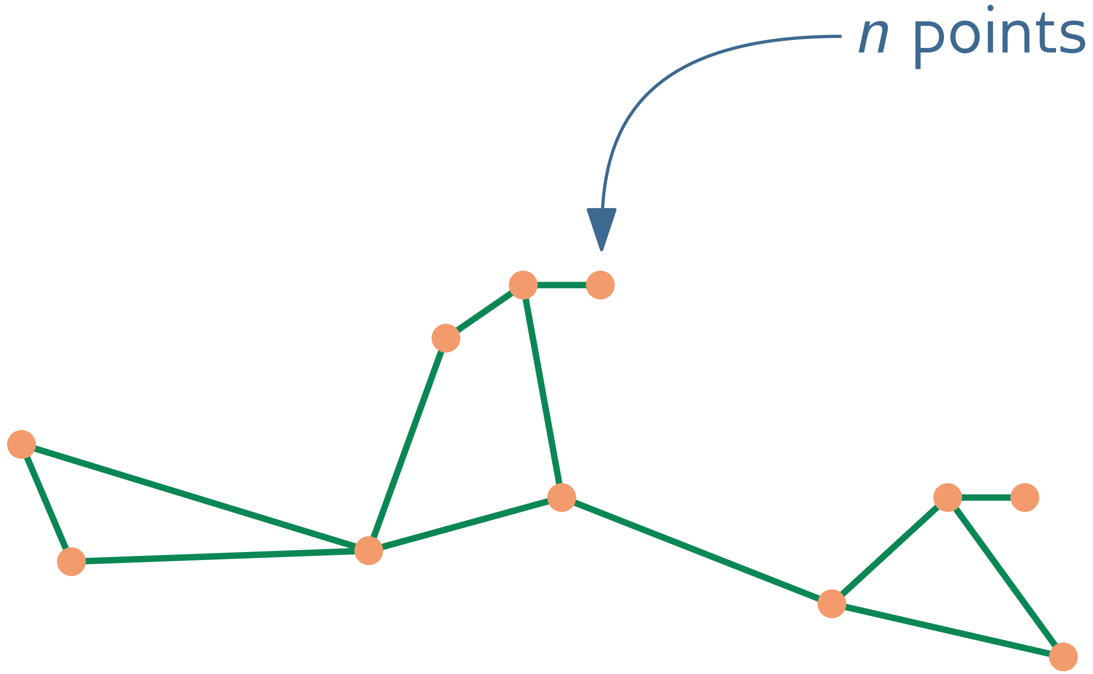

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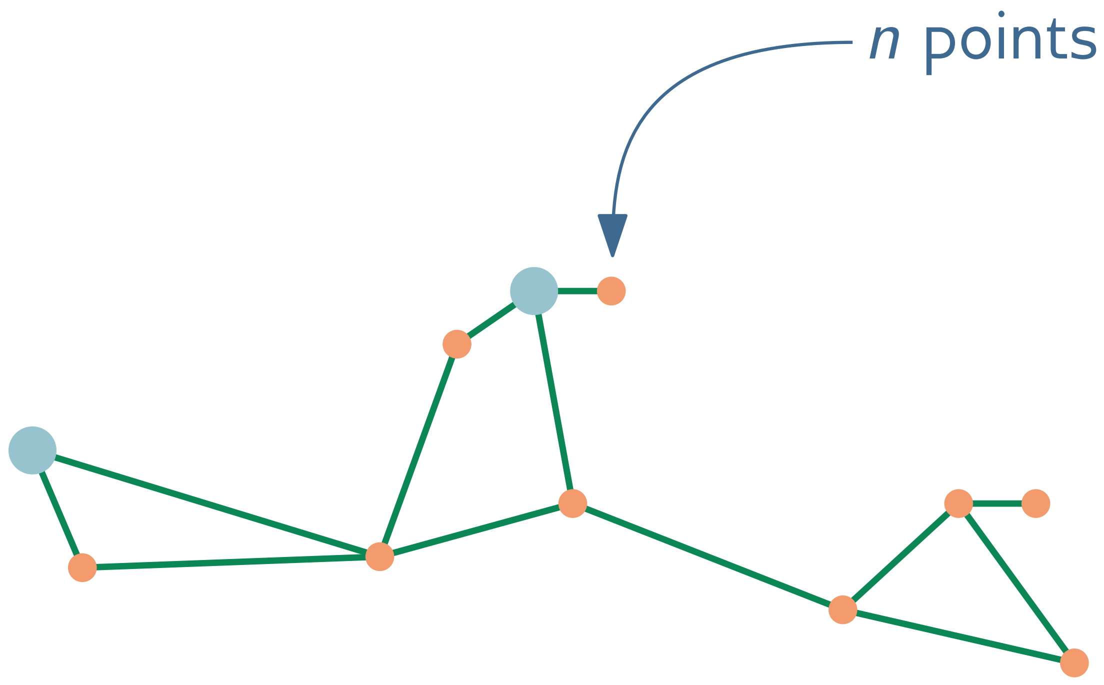

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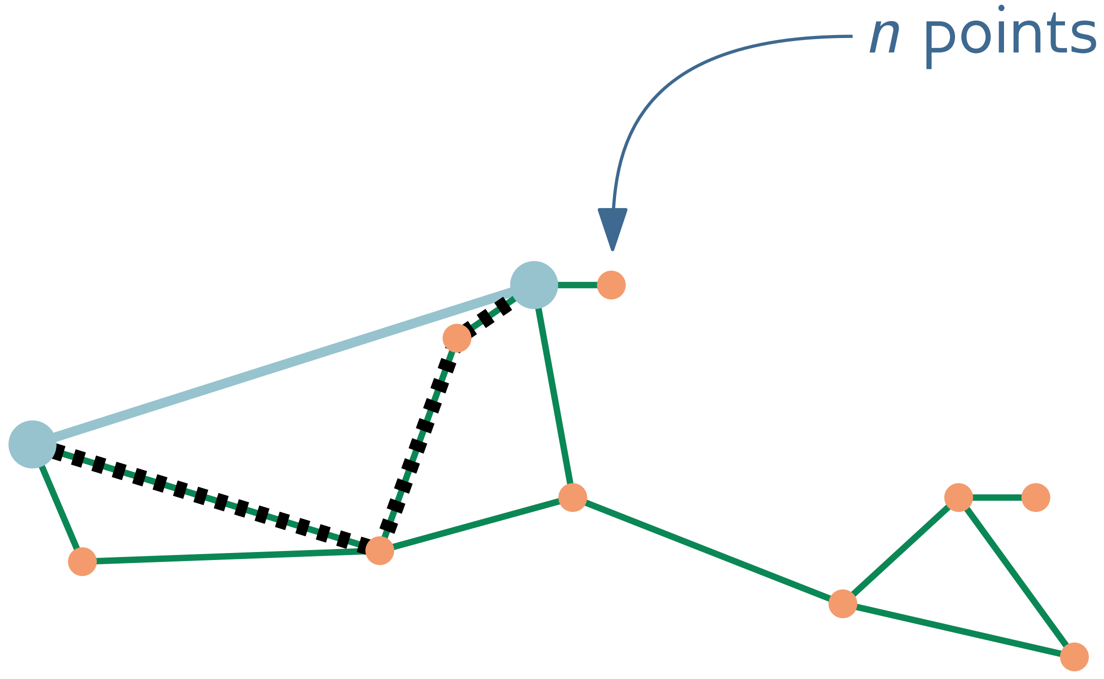

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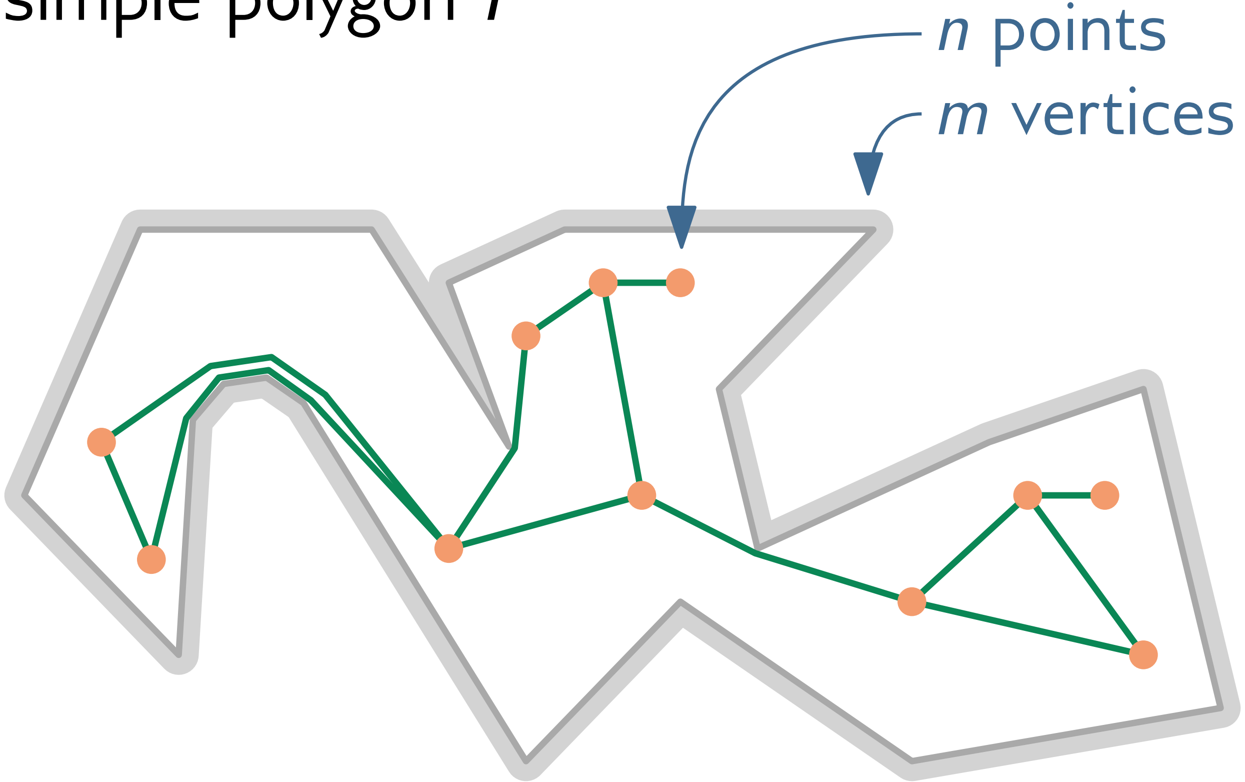
We study *geodesic* spanners for point sites in a simple polygon P



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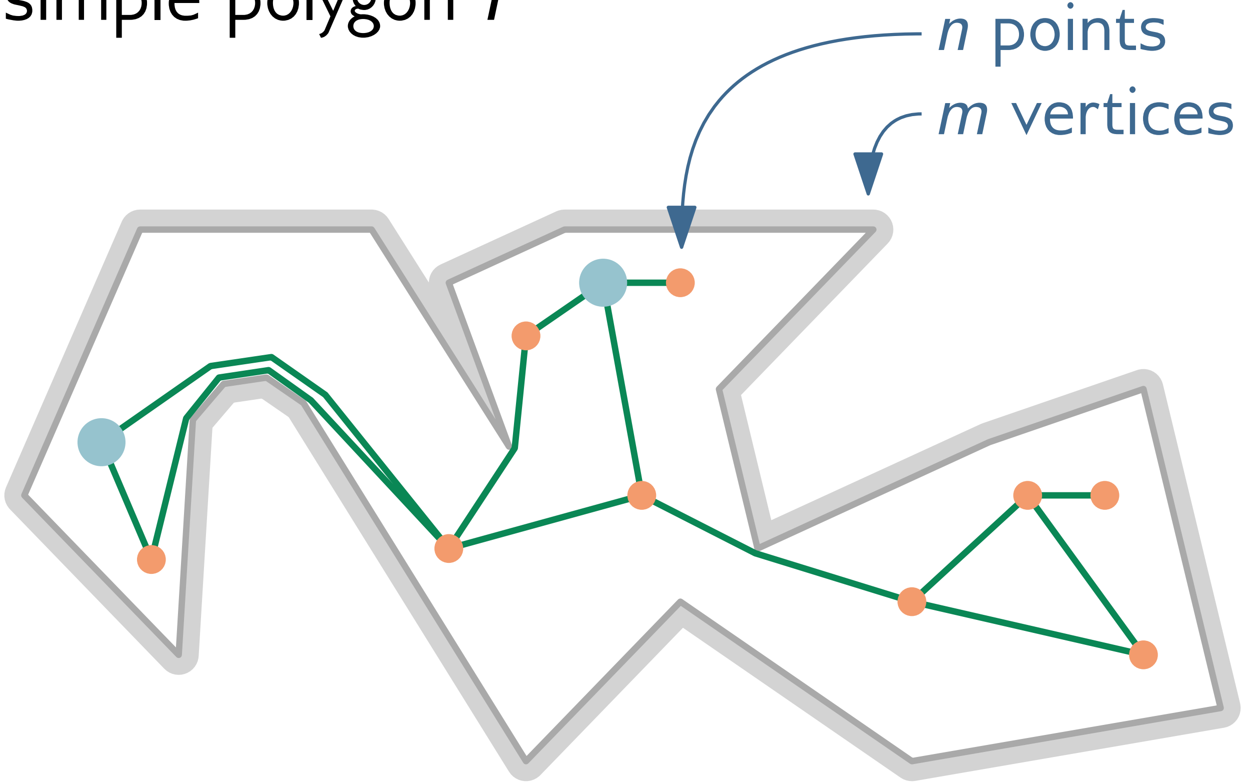

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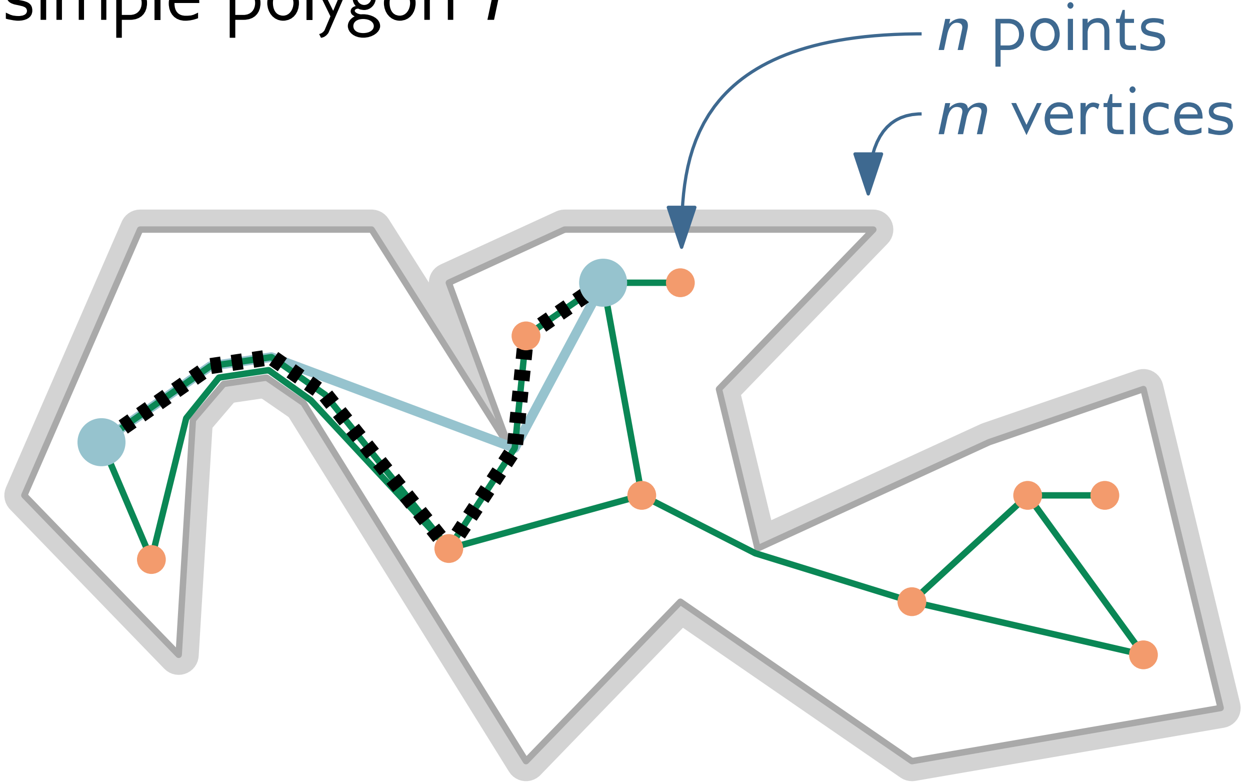

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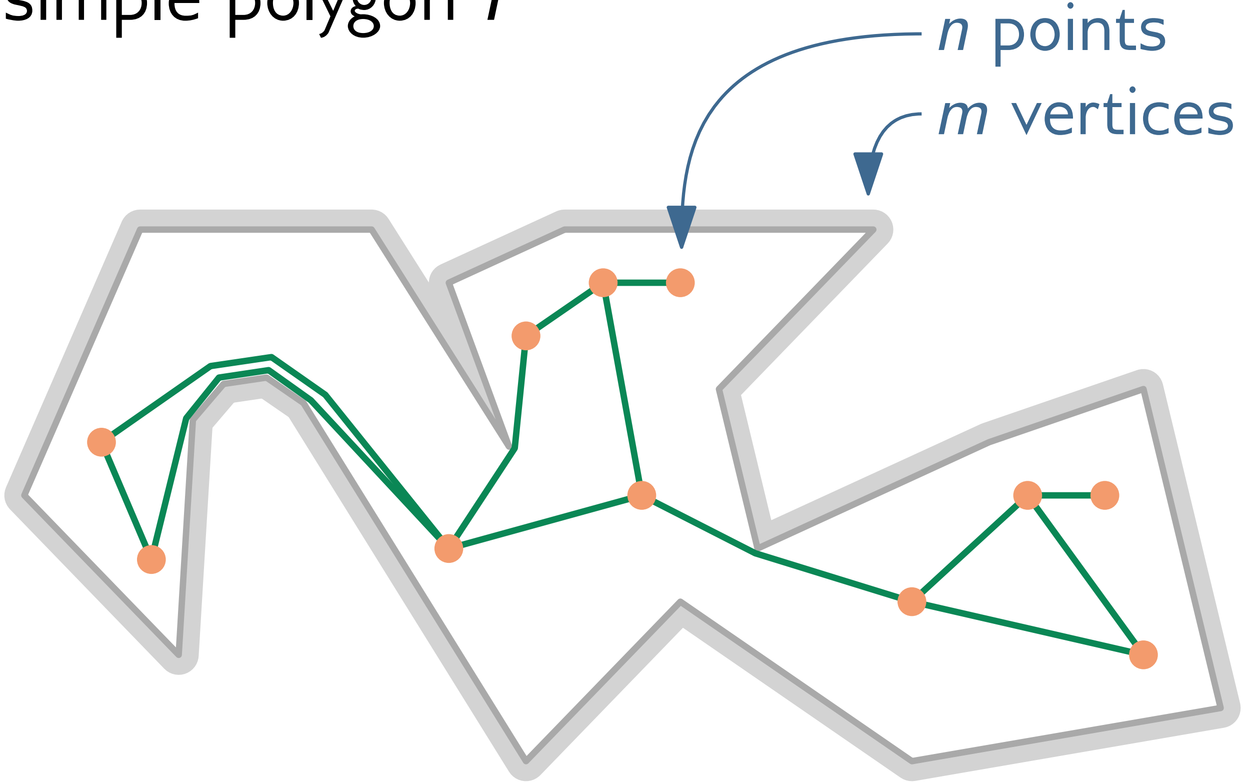
Measure "compactness" by: *spanner complexity*



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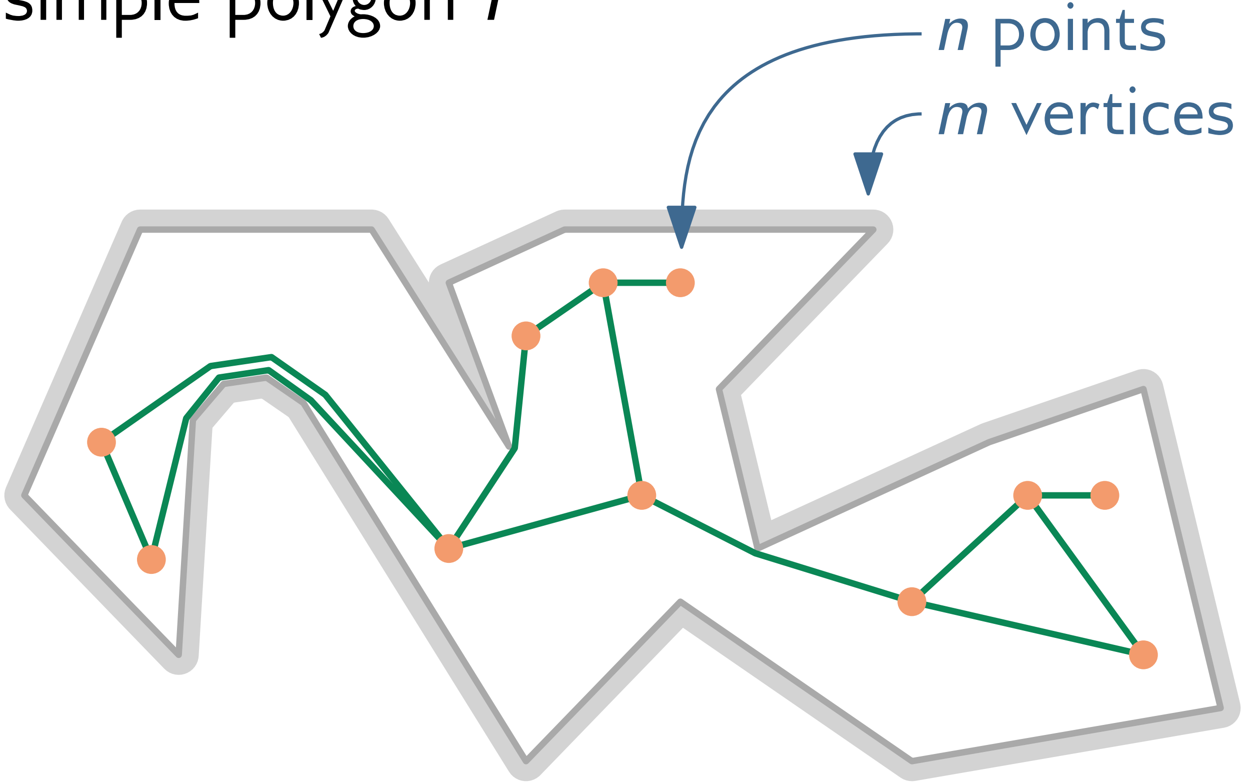

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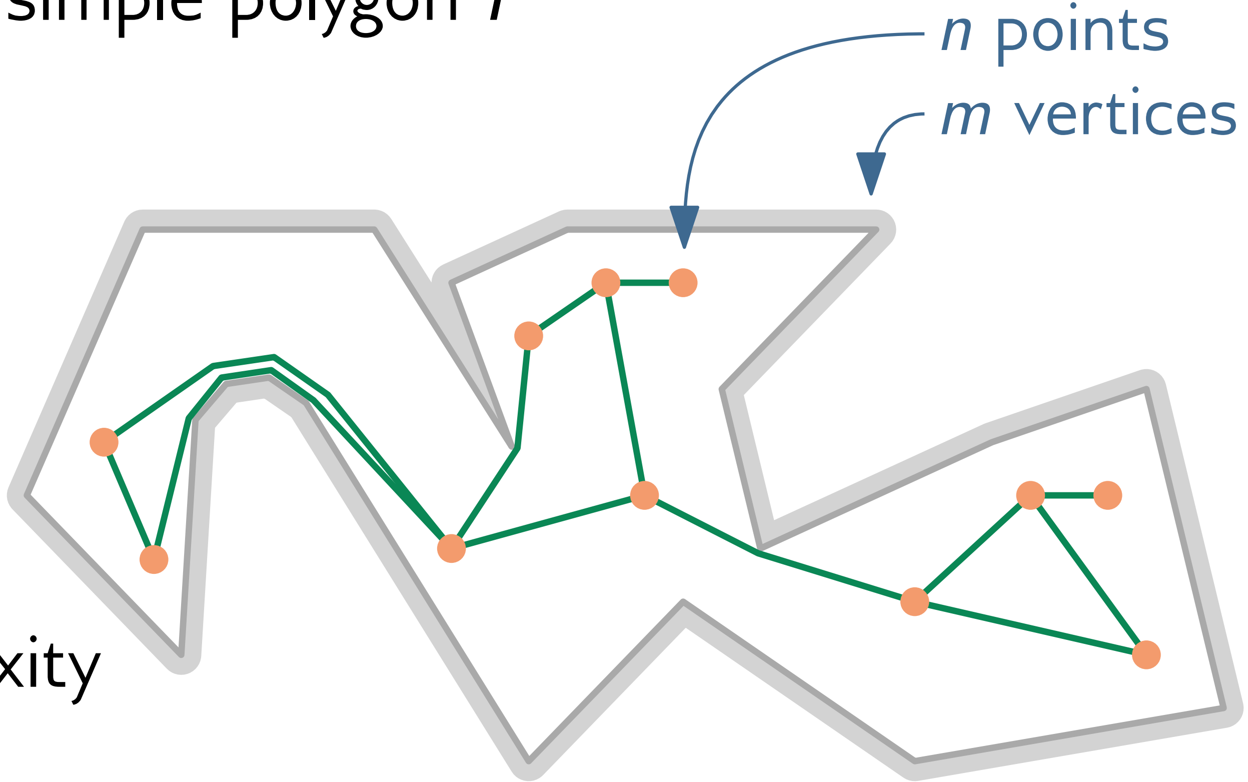


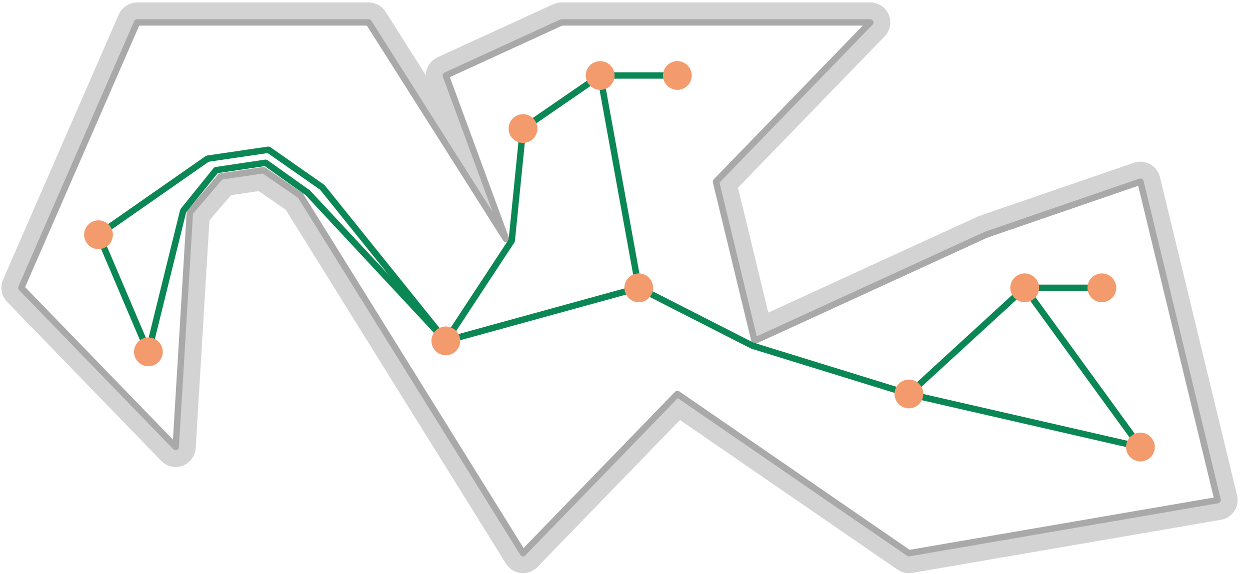
A $(3 - \varepsilon)$-spanner has $\Omega(mn)$ complexity

A $4\sqrt{2}$-spanner with $O(m\sqrt{n} + n \log^2 n)$ complexity

[dB, van Kreveld, Staals, 2023]

Introducing Steiner points

Can we do better?



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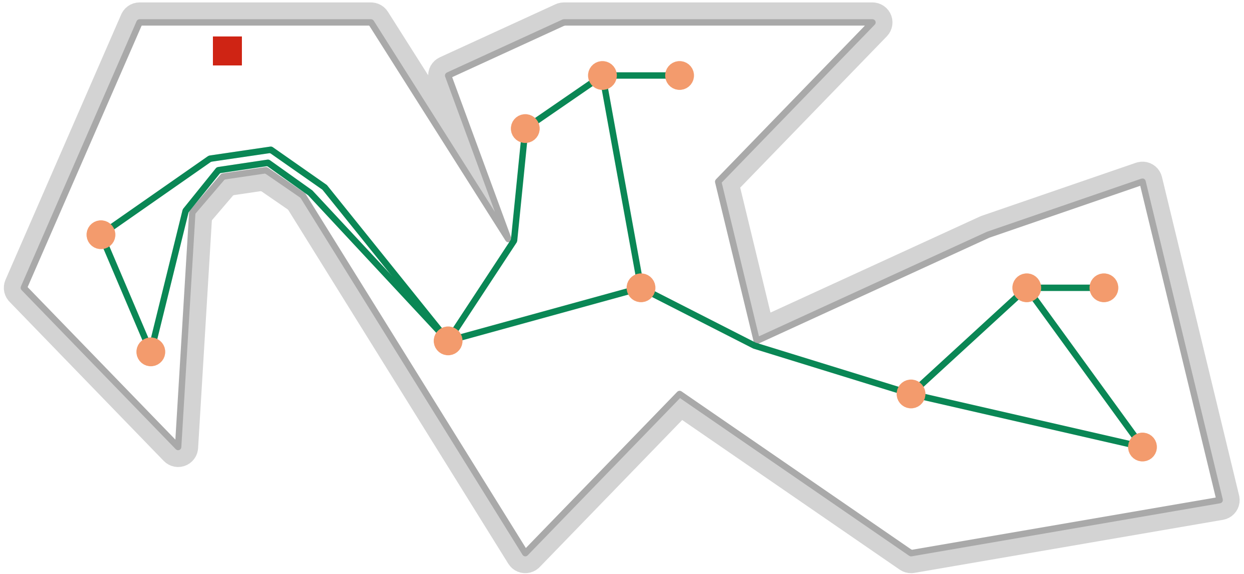
What if we are allowed to use Steiner points?



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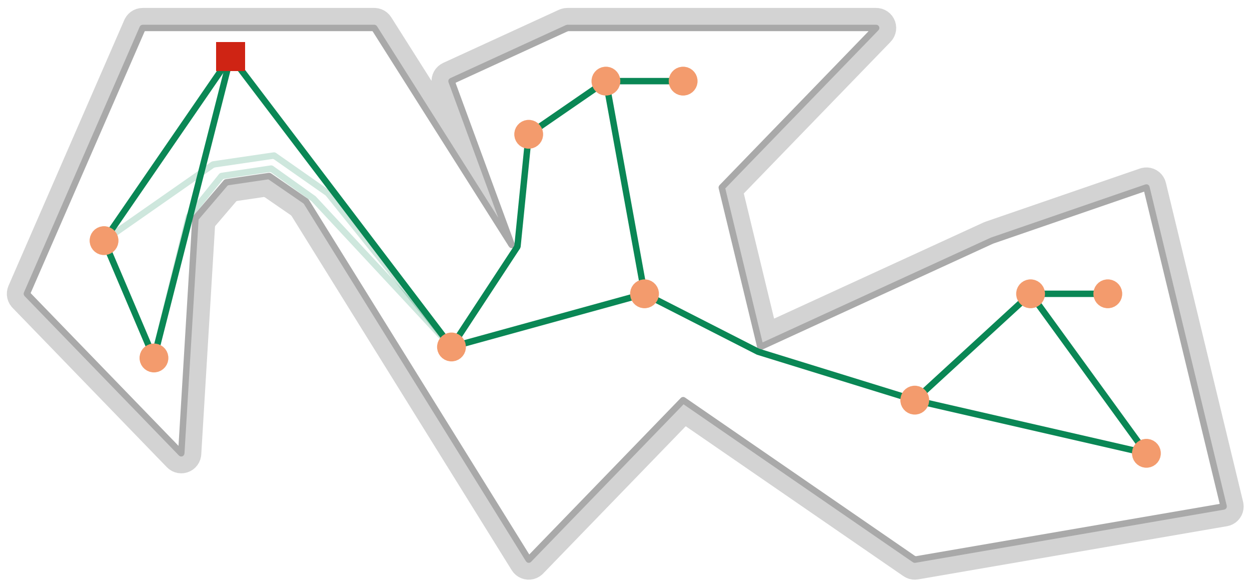

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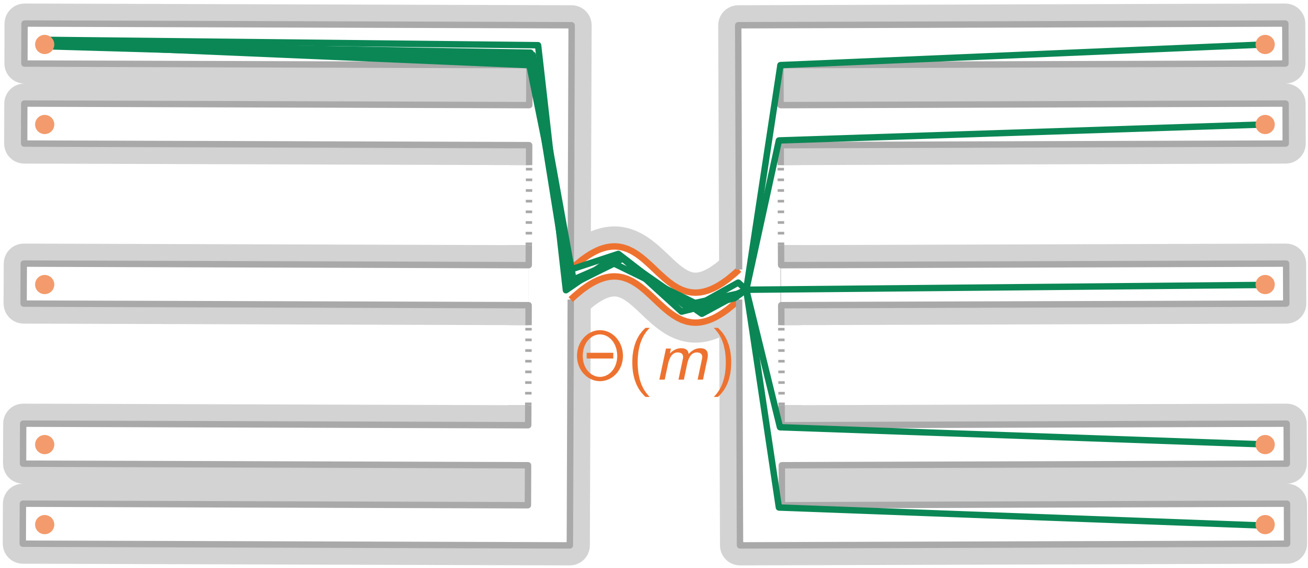

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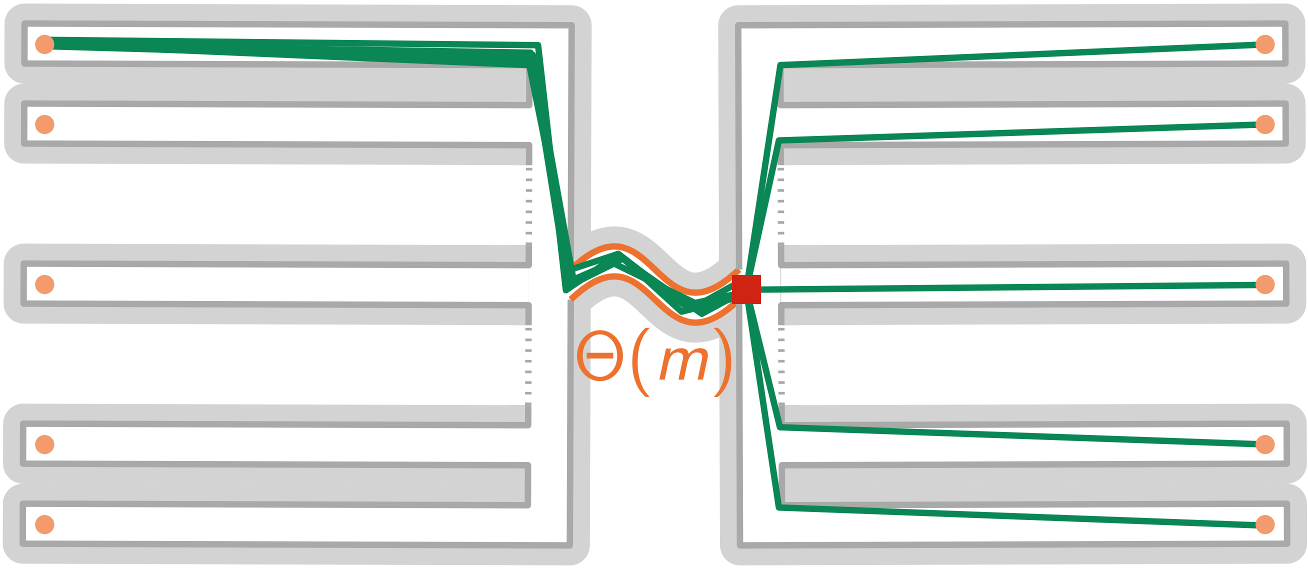

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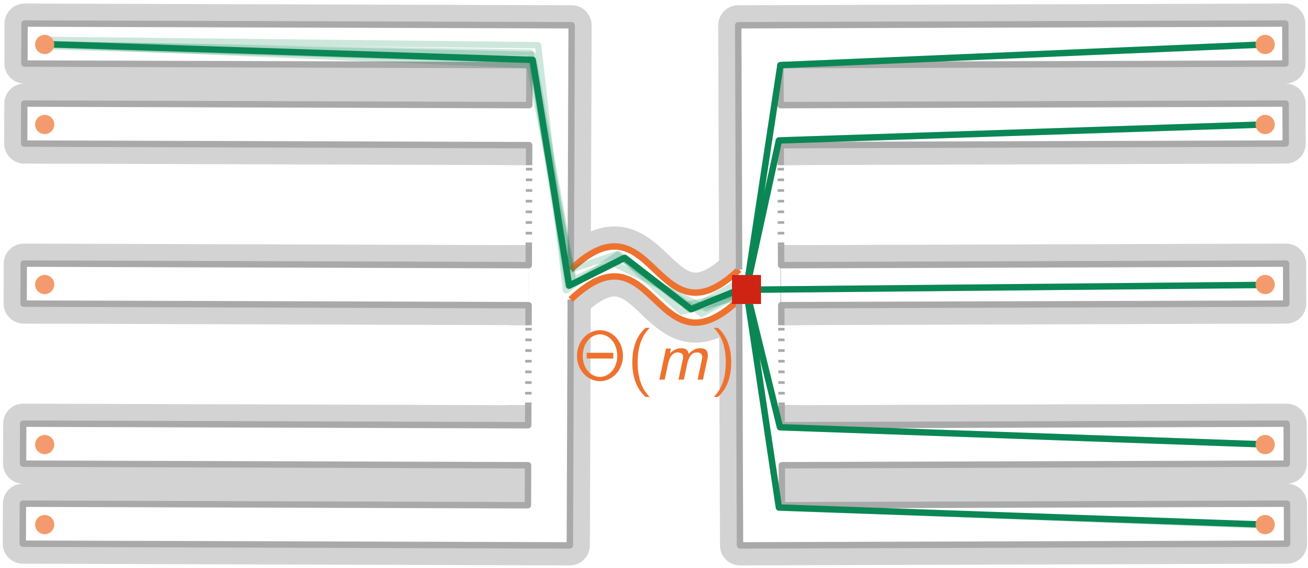
Complexity: $\Omega(mn) \to \Omega(m)$



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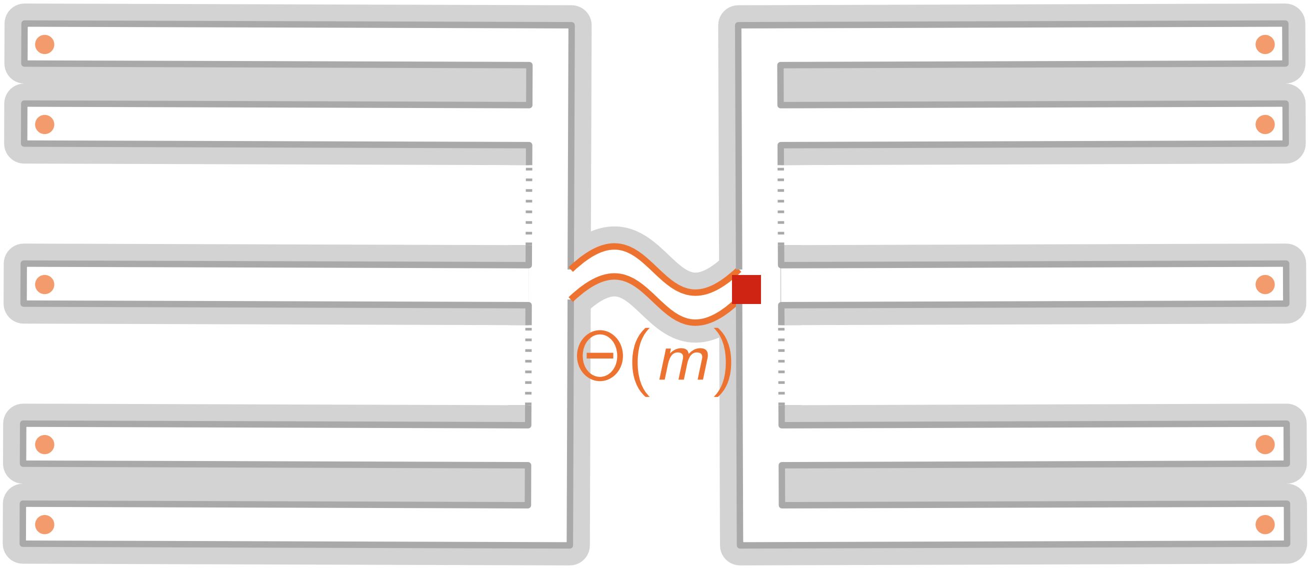
What if we are allowed to use Steiner points?

m Steiner points $\Rightarrow$ $(1+\varepsilon)$-spanner with $O((n+m)/\varepsilon)$ complexity [Clarkson, 1987]



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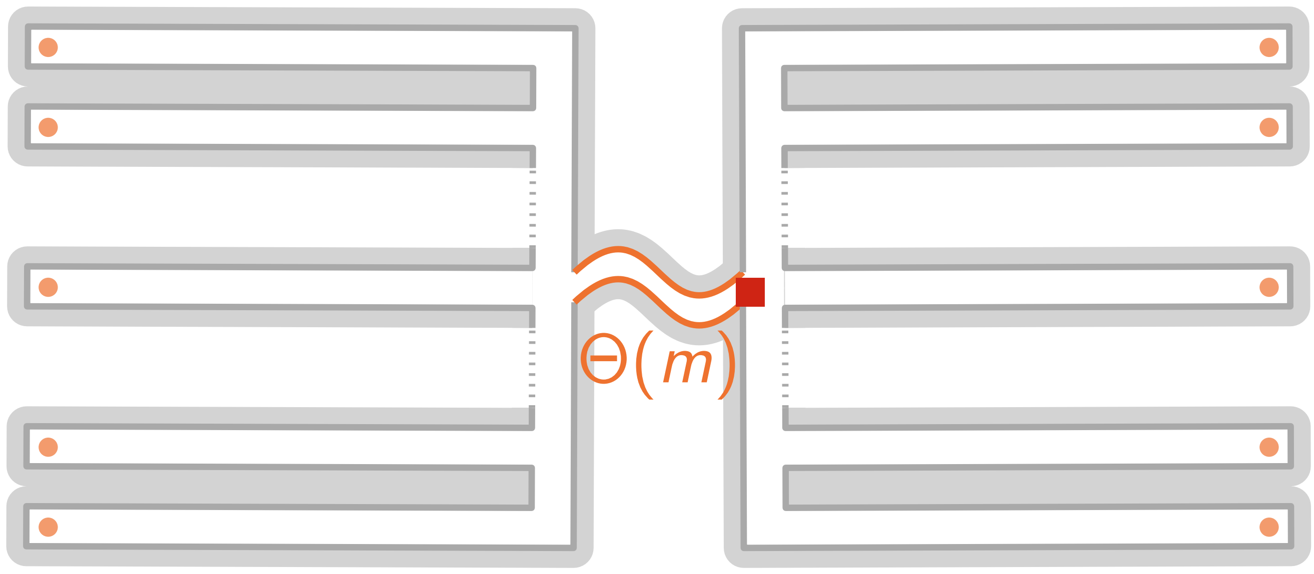

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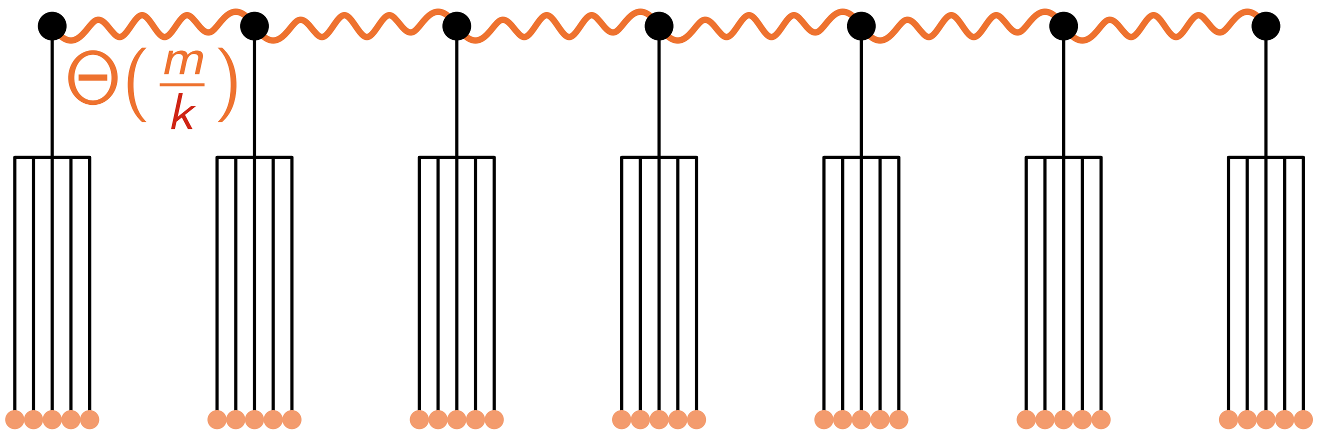
What if we are allowed to use *only* k Steiner points?

A $(3-\varepsilon)$-spanner has complexity $\Omega(mn/k)$



$2k+1$ combs

Can we do better?

Spanning ratio	Complexity	Complexity k Steiner
$2-\varepsilon$	$\Omega(mn^2)$	$\Omega(mn^2/k^2)$
$3-\varepsilon$	$\Omega(mn)$	$\Omega(mn/k)$
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Not so much :(

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Not so much :(

$2\sqrt{2}t$-spanner of complexity $\tilde{O}(mn^{\frac{1}{t}}/k^{\frac{1}{t}})$

Where to place Steiner points?

simple polygon

n points

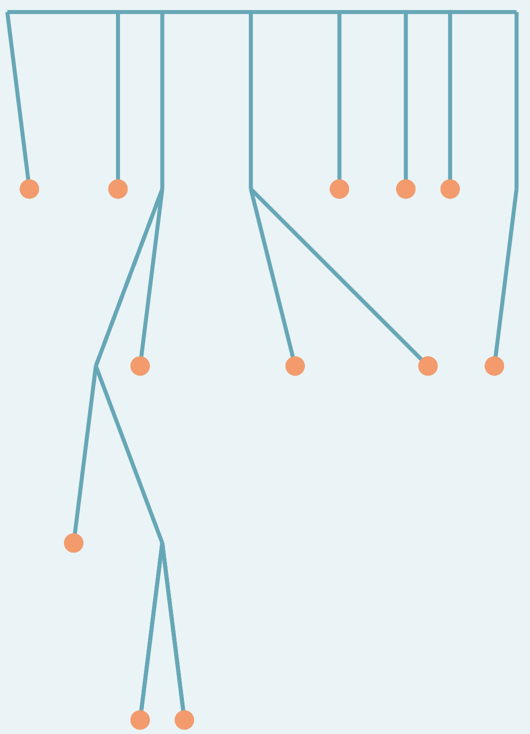
m vertices

Where to place Steiner points?

Look at easier setting: weighted tree

n leaves

m vertices



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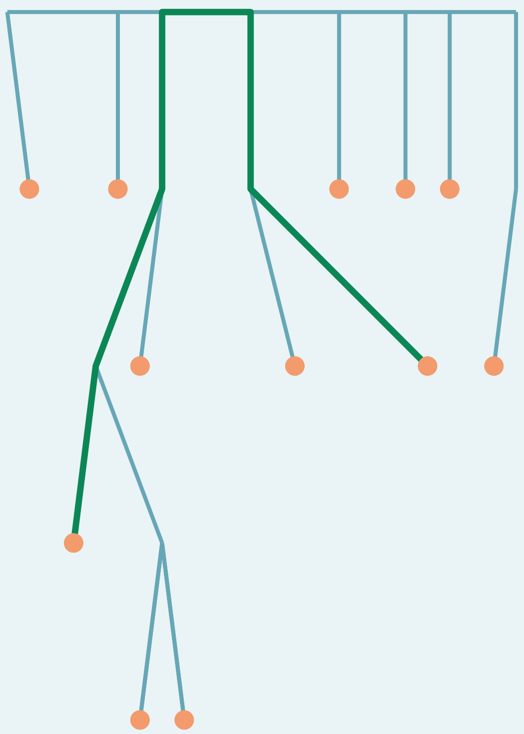

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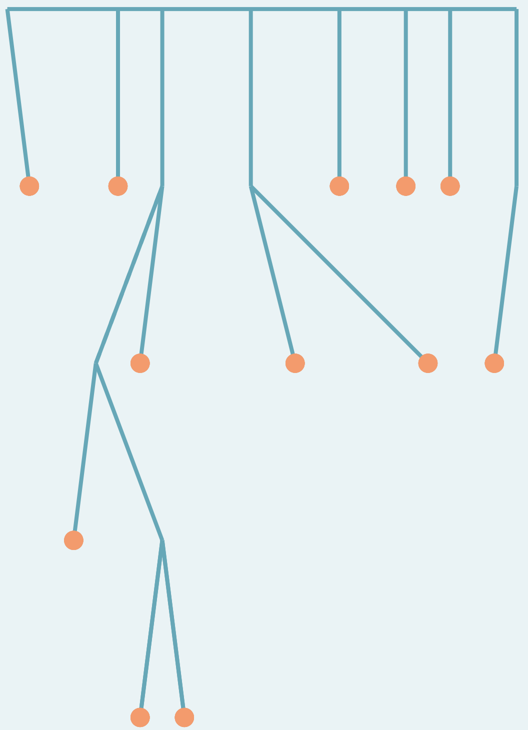

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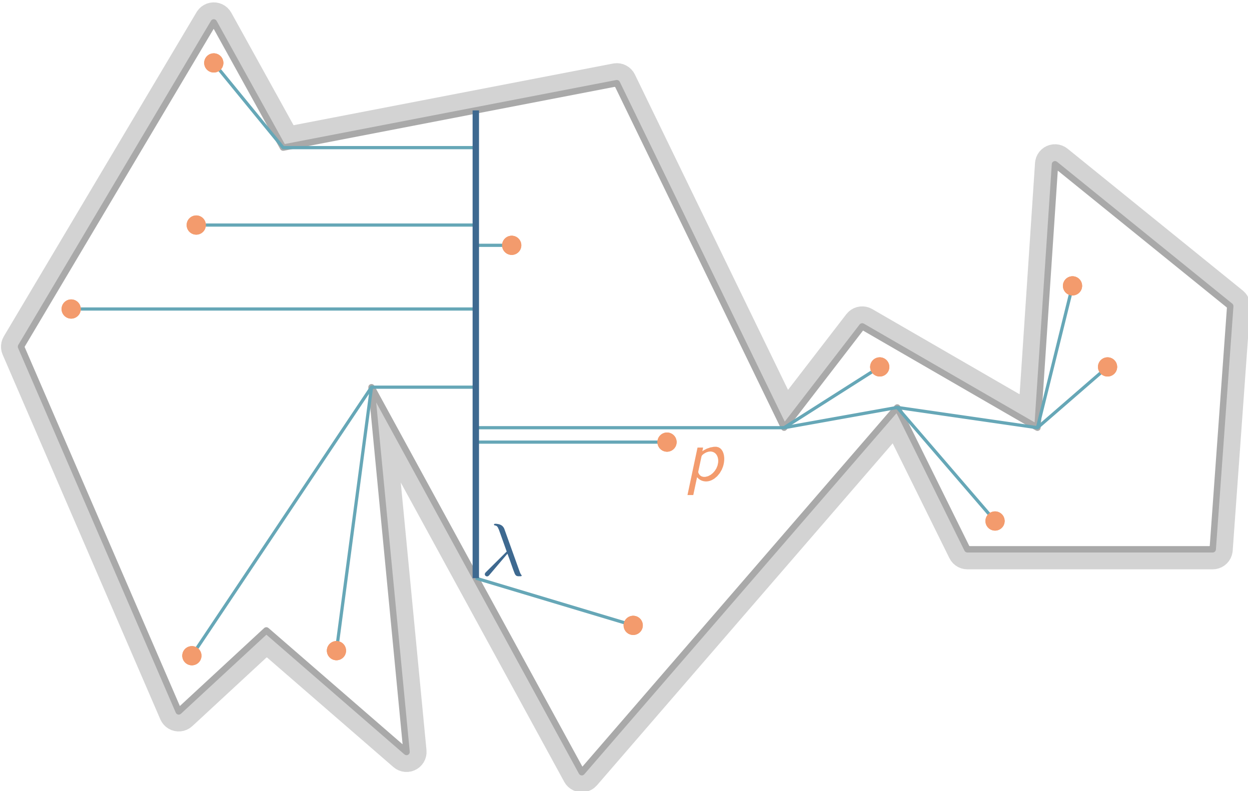

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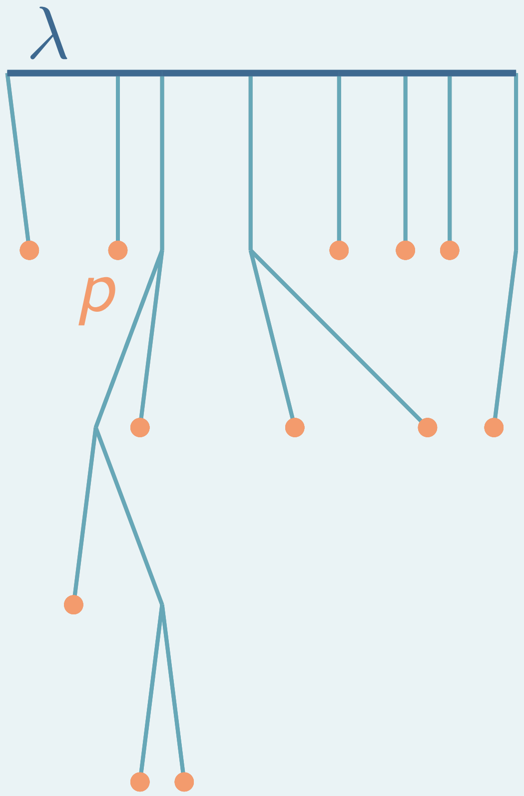

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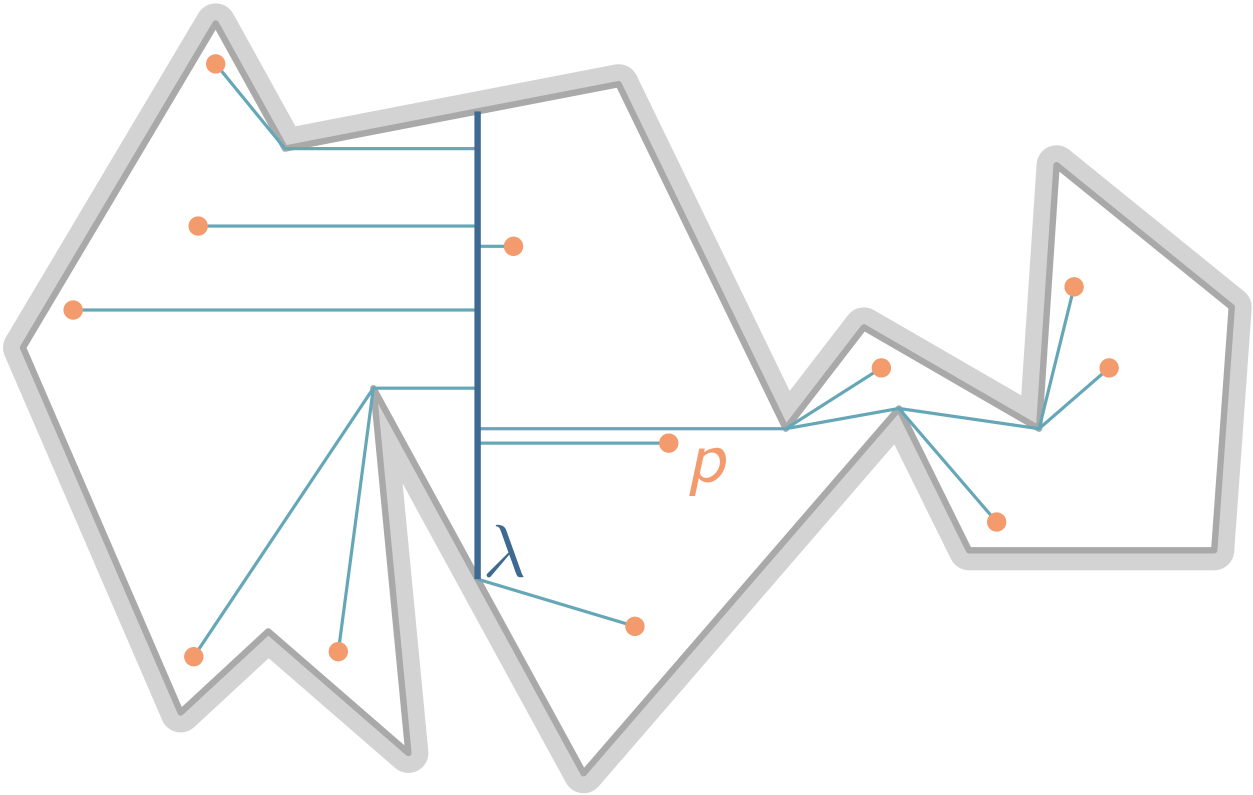

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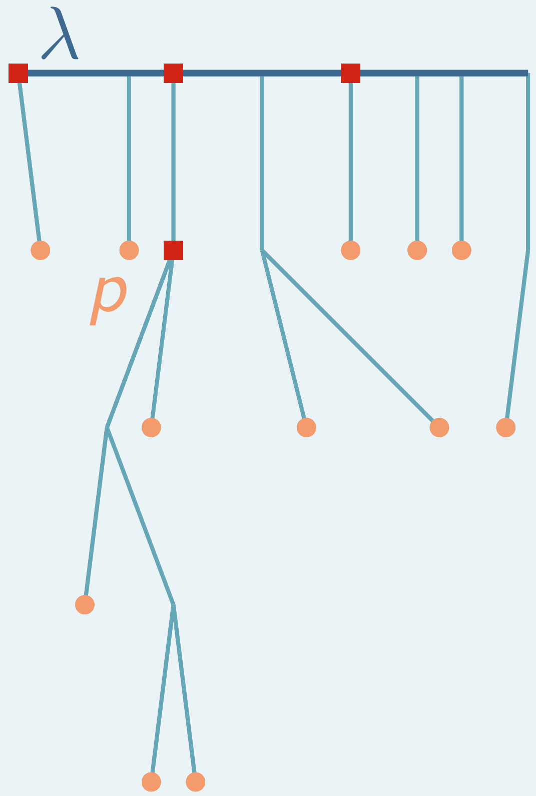

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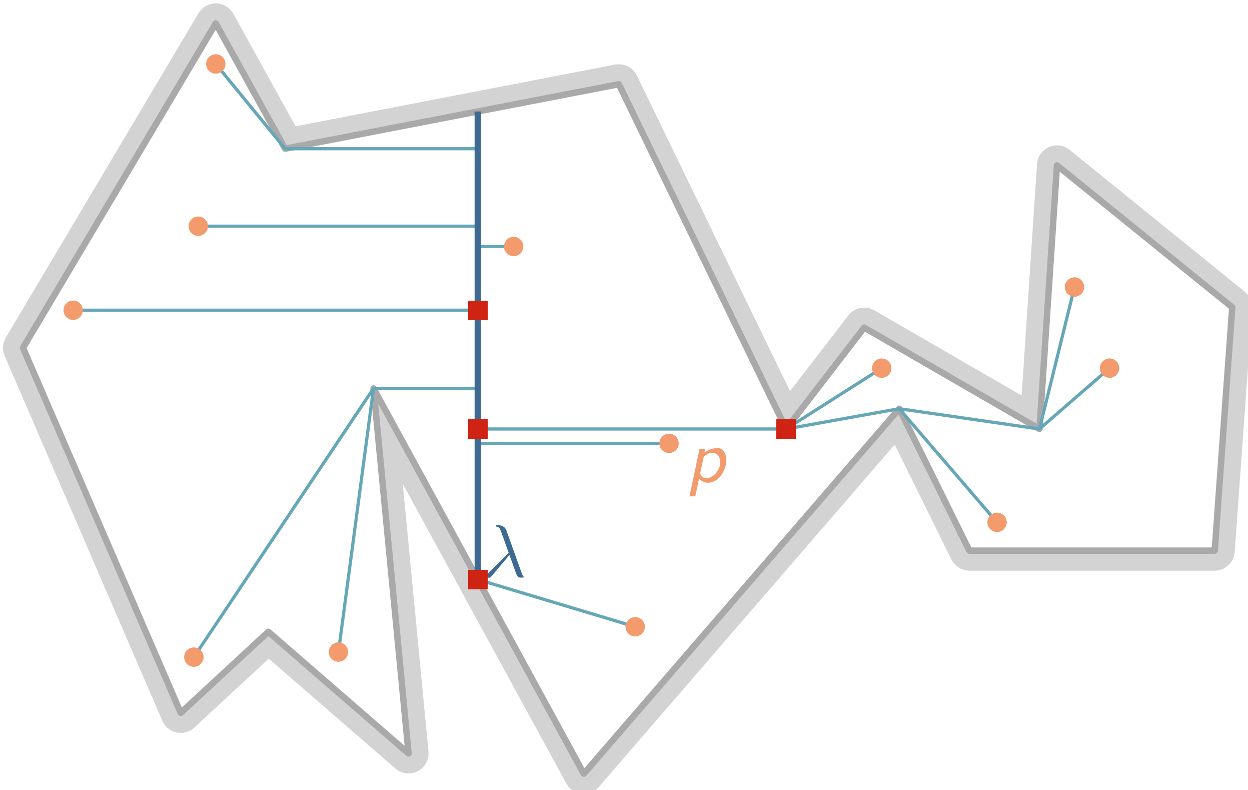

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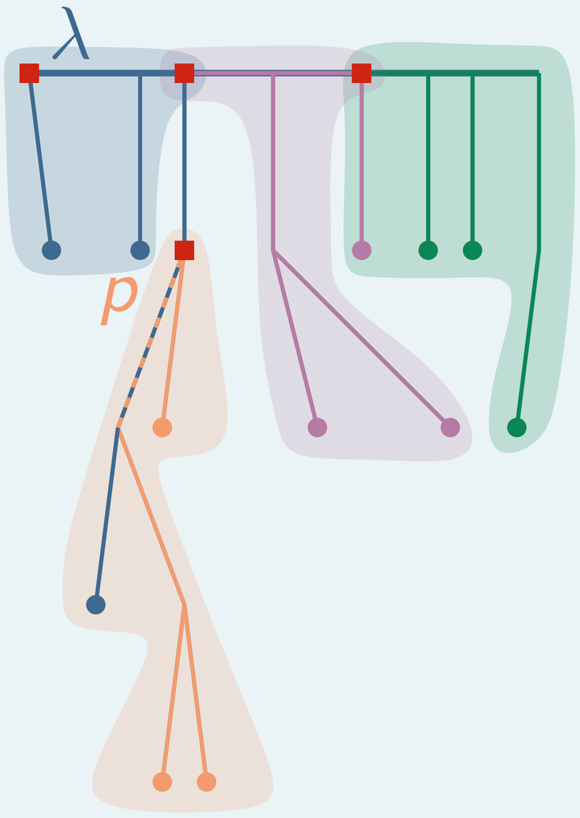

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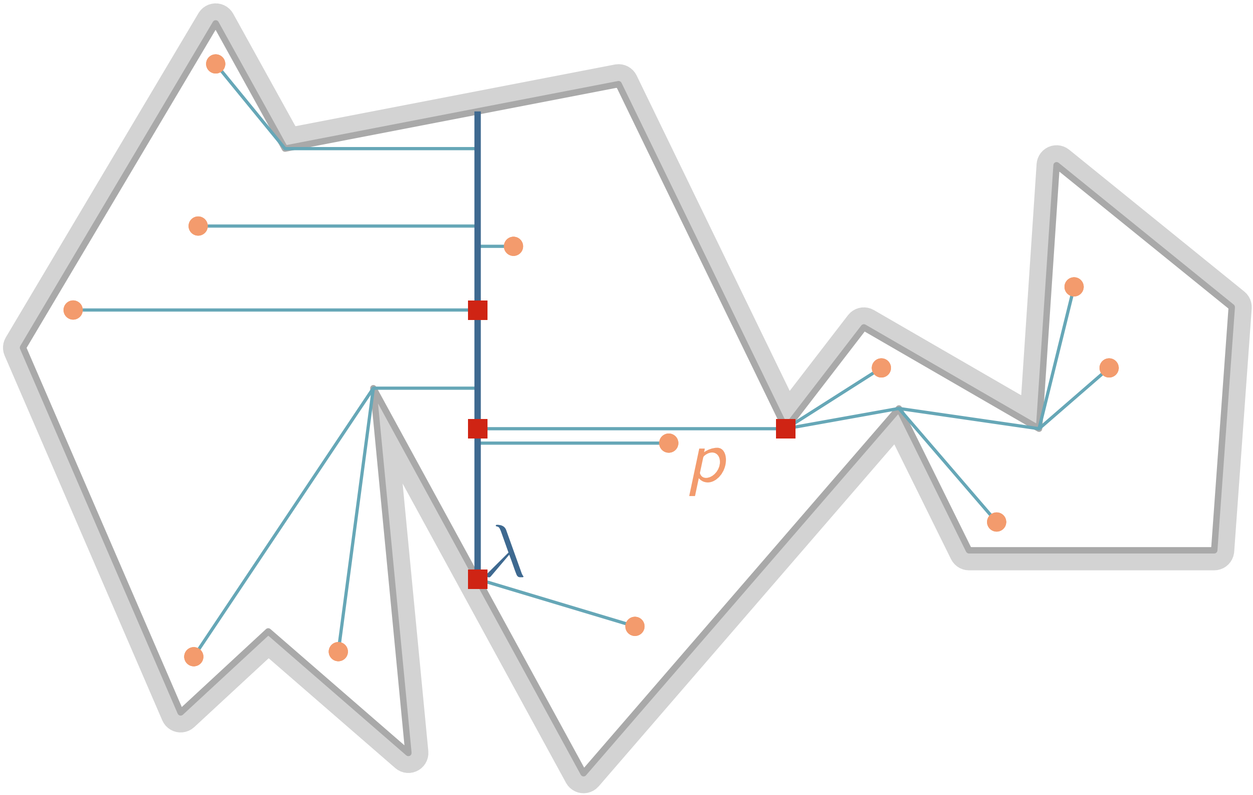

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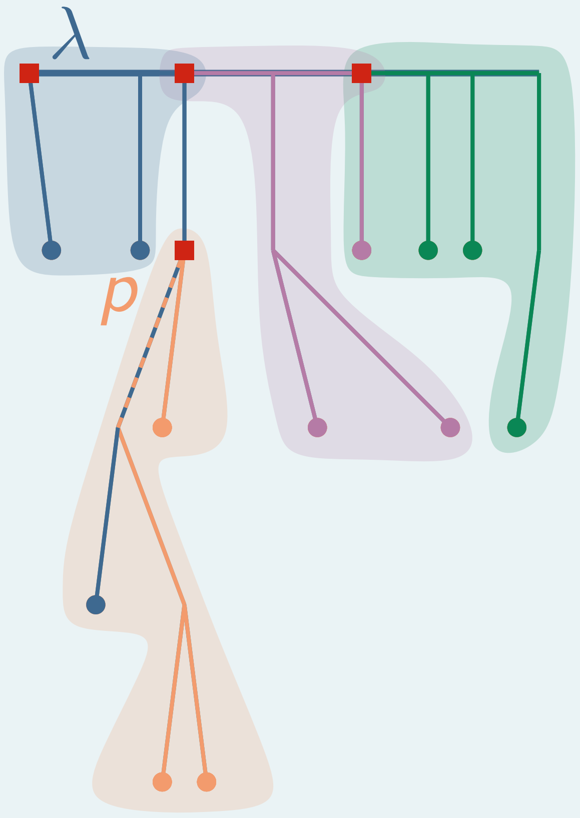

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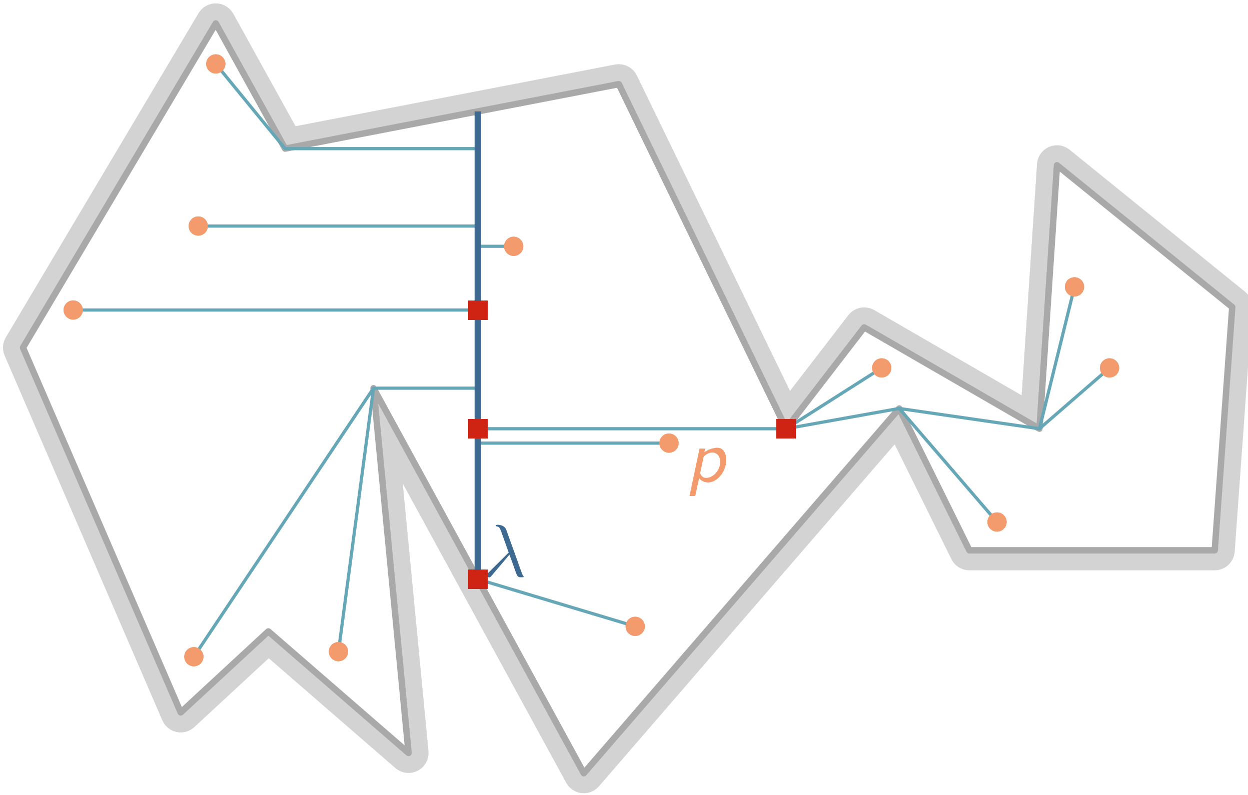

$2t$ **Spanning ratio**

$O(mn^{\frac{1}{t}}/k^{\frac{1}{t}})$ **Complexity**

Where to place Steiner points?

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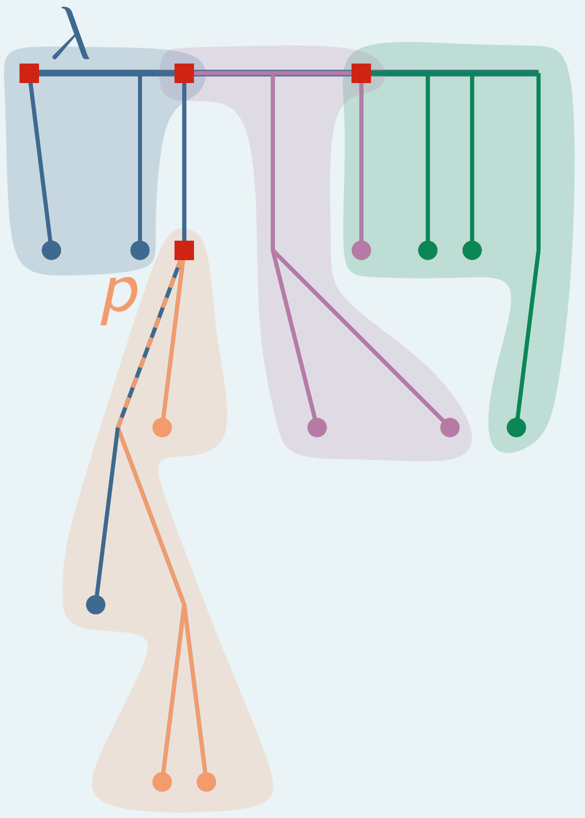

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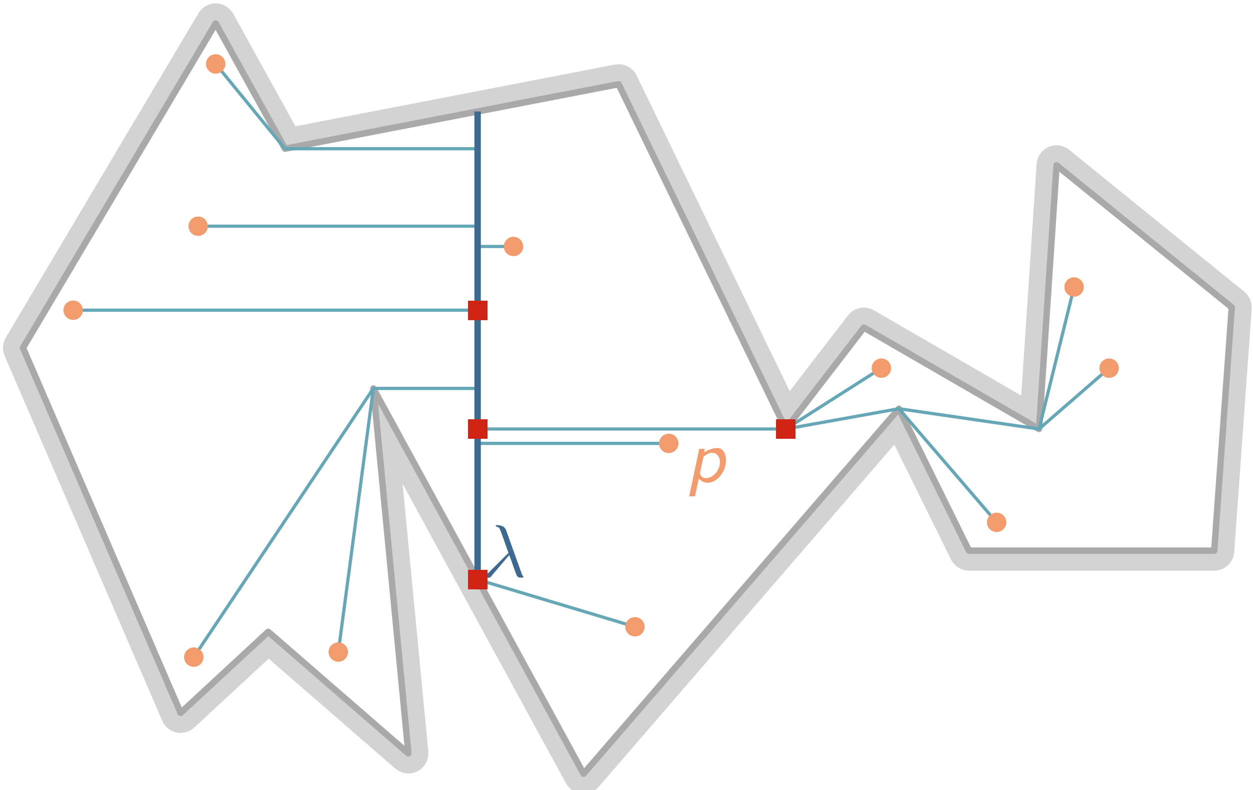

Weighted tree		Simple polygon
$2t$	**Spanning ratio**	$2\sqrt{2}t$
$O(mn^{\frac{1}{t}}/k^{\frac{1}{t}})$	**Complexity**	$O(mn^{\frac{1}{t}}/k^{\frac{1}{t}} \cdot (\log k)^{\frac{t+1}{t}})$

Results

Spanning ratio	Complexity k Steiner
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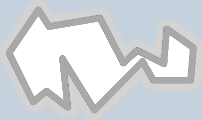
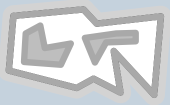
	$2\sqrt{2}t$	$O(mn^{\frac{1}{t}}/k^{\frac{1}{t}} \cdot (\log k)^{\frac{t+1}{t}})$
	$6t$	

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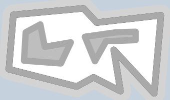
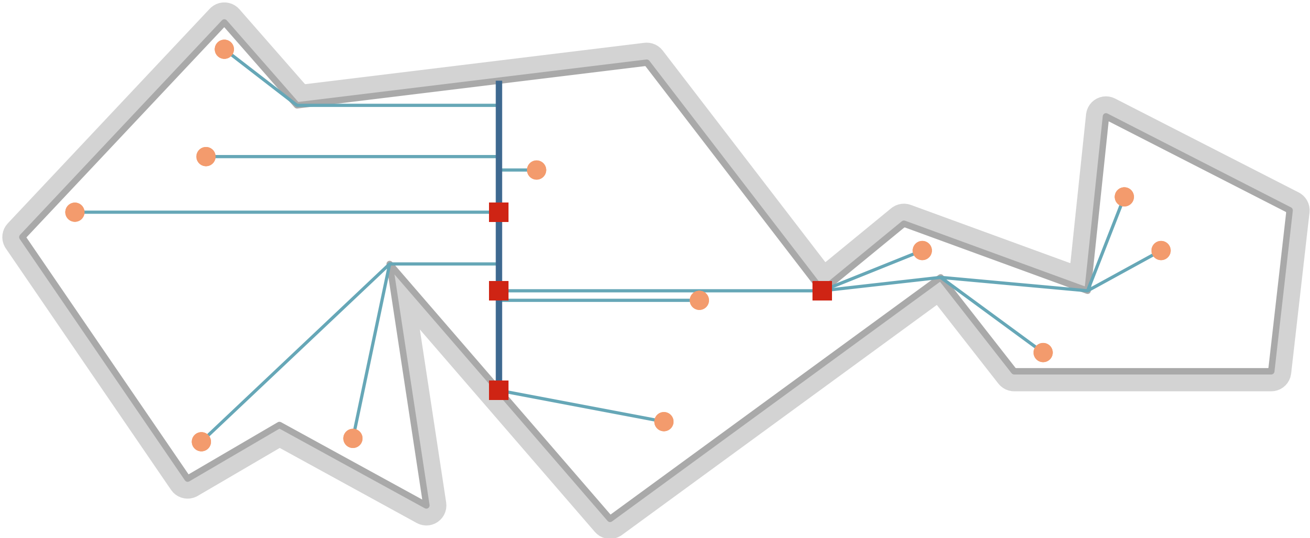
 $6t$	



Thank you

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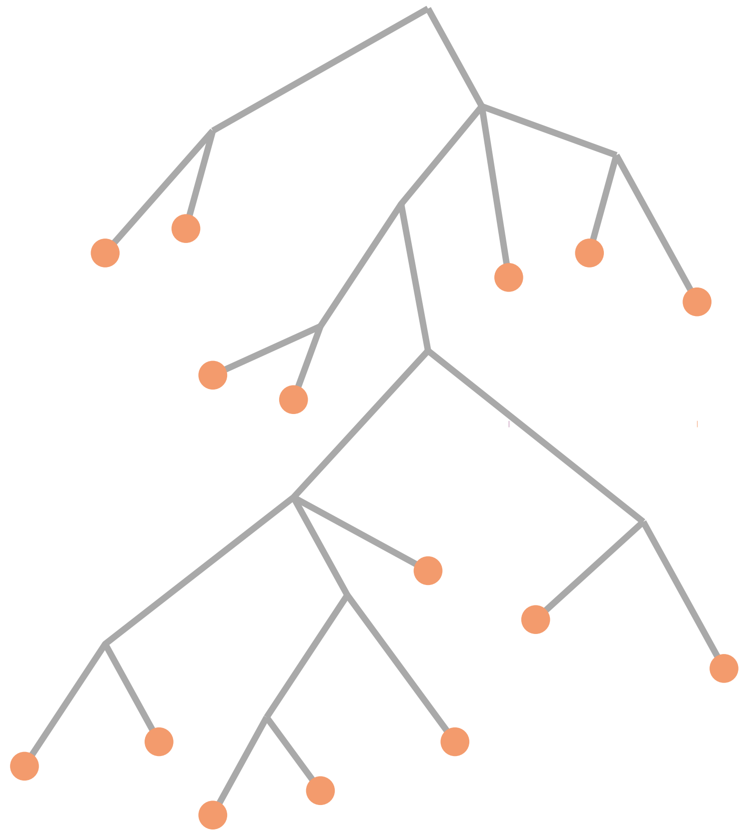
Look at a simpler setting: a tree



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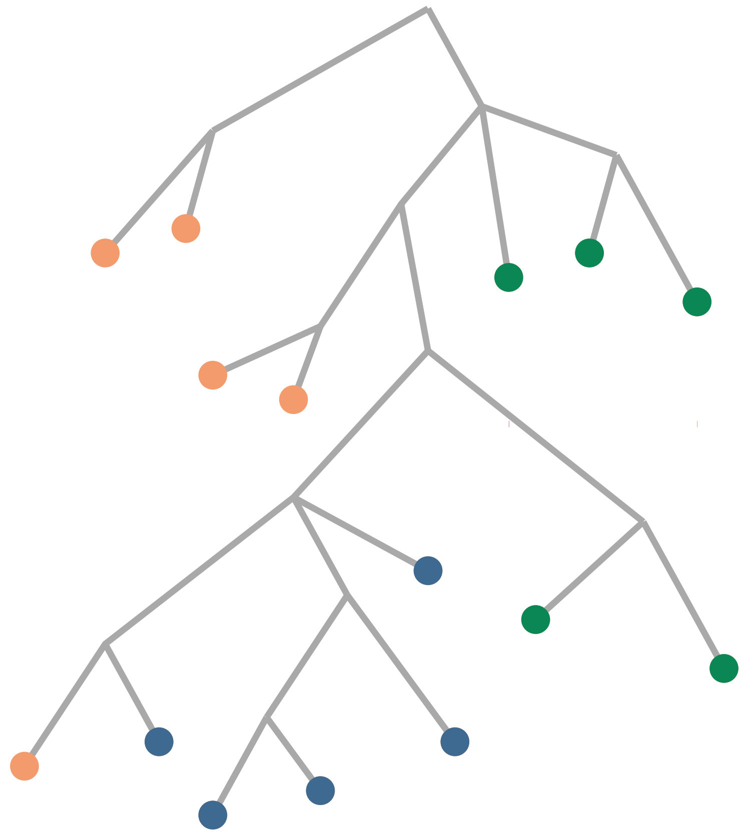
Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal

$k = 3$



Where to place Steiner points?

Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal
2. Color the tree

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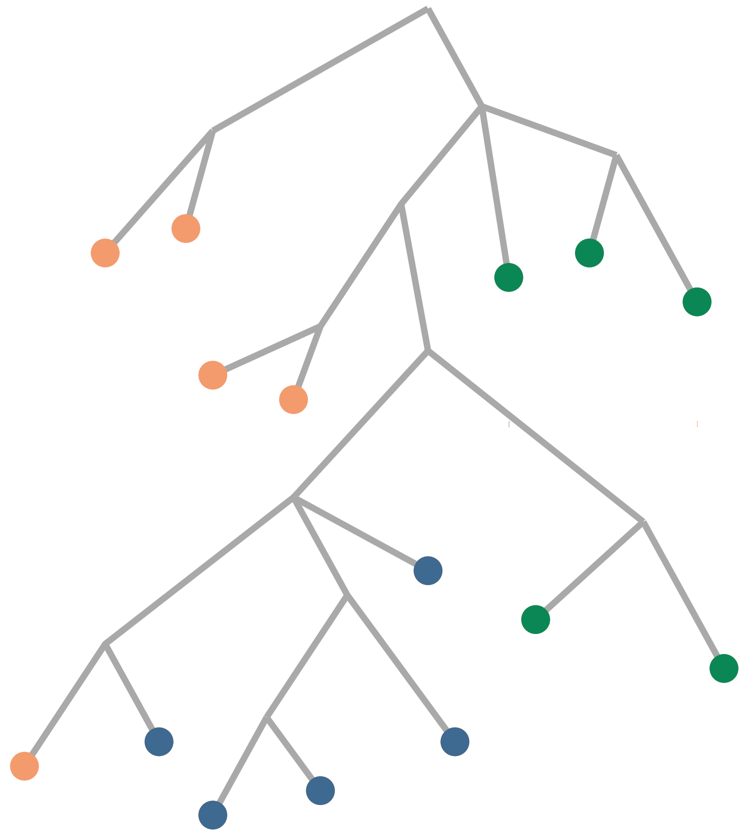

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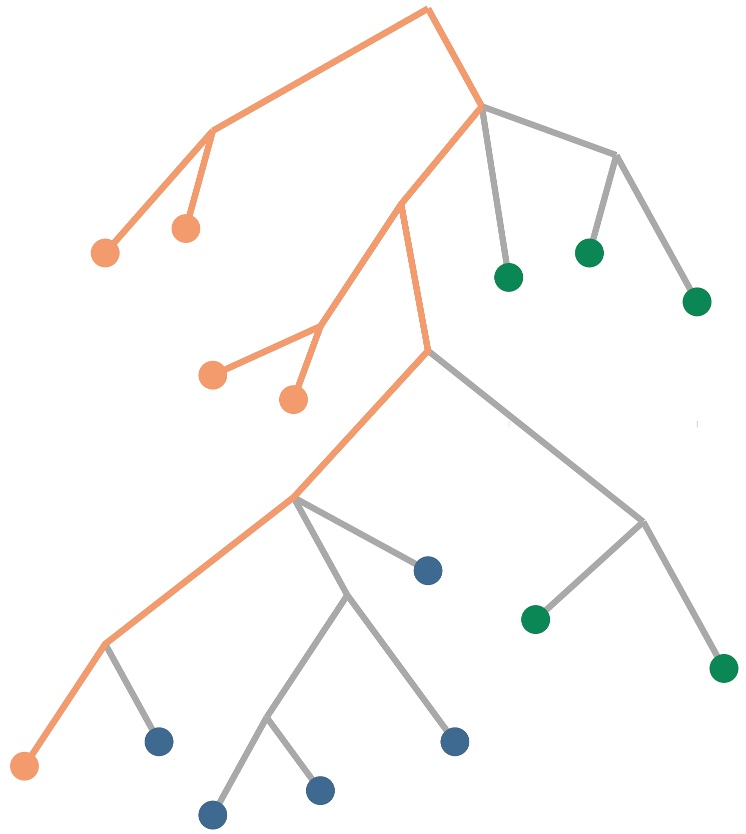

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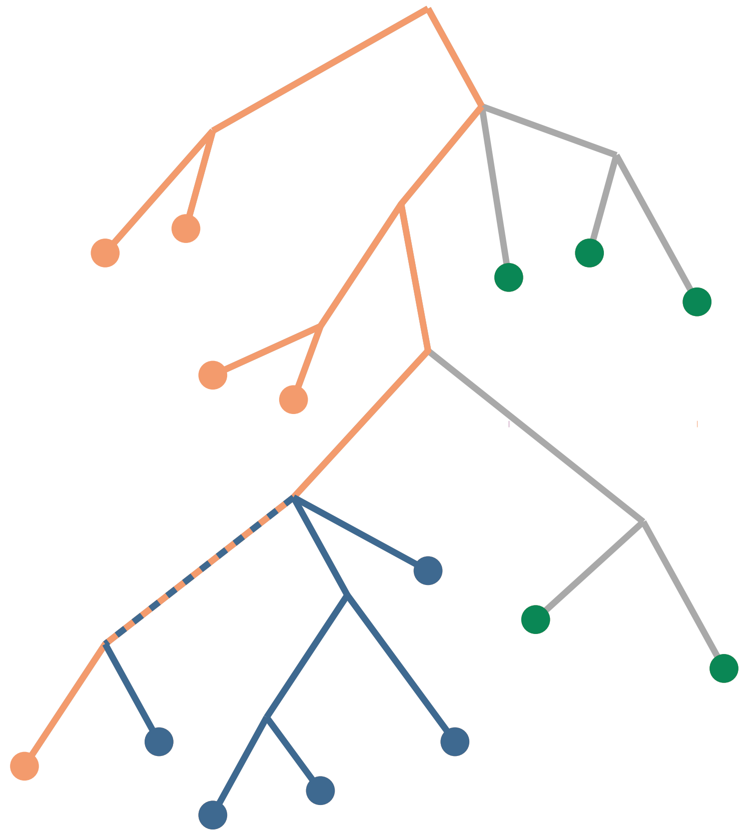

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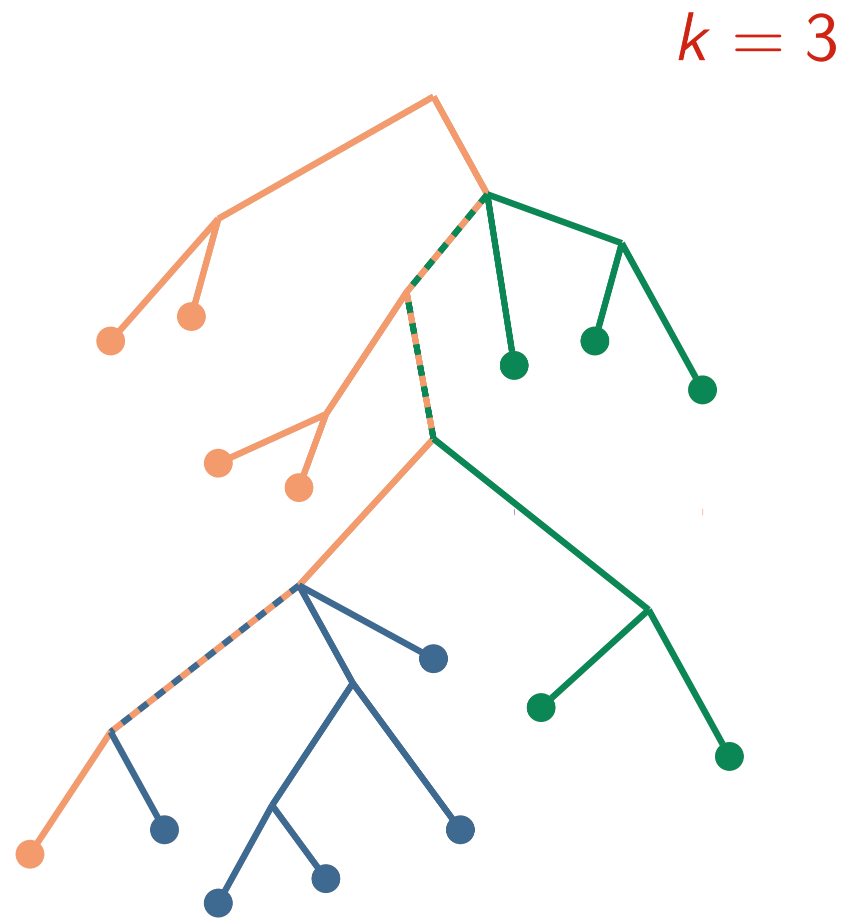

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3. Place Steiner points

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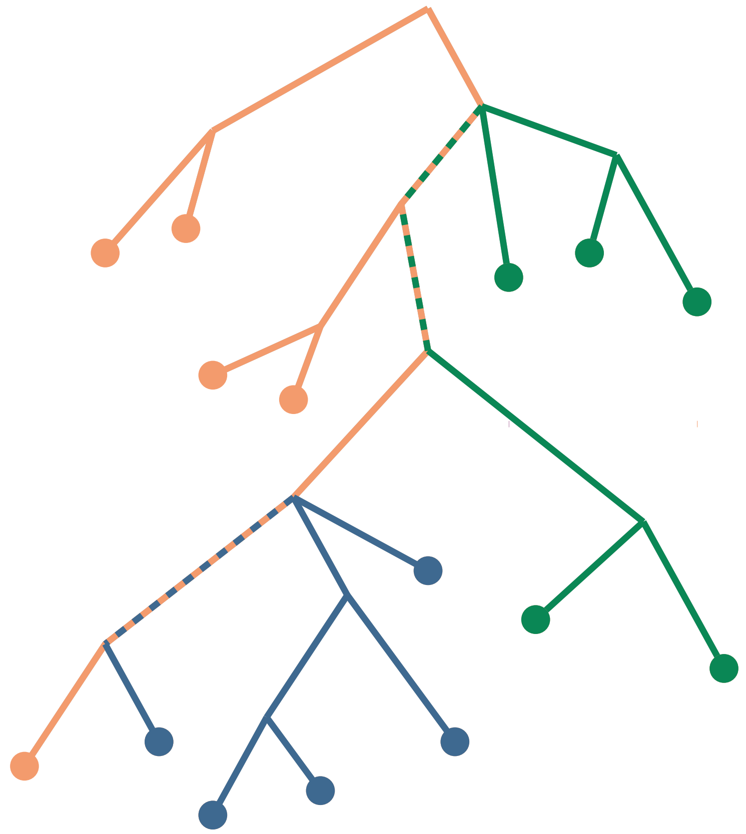

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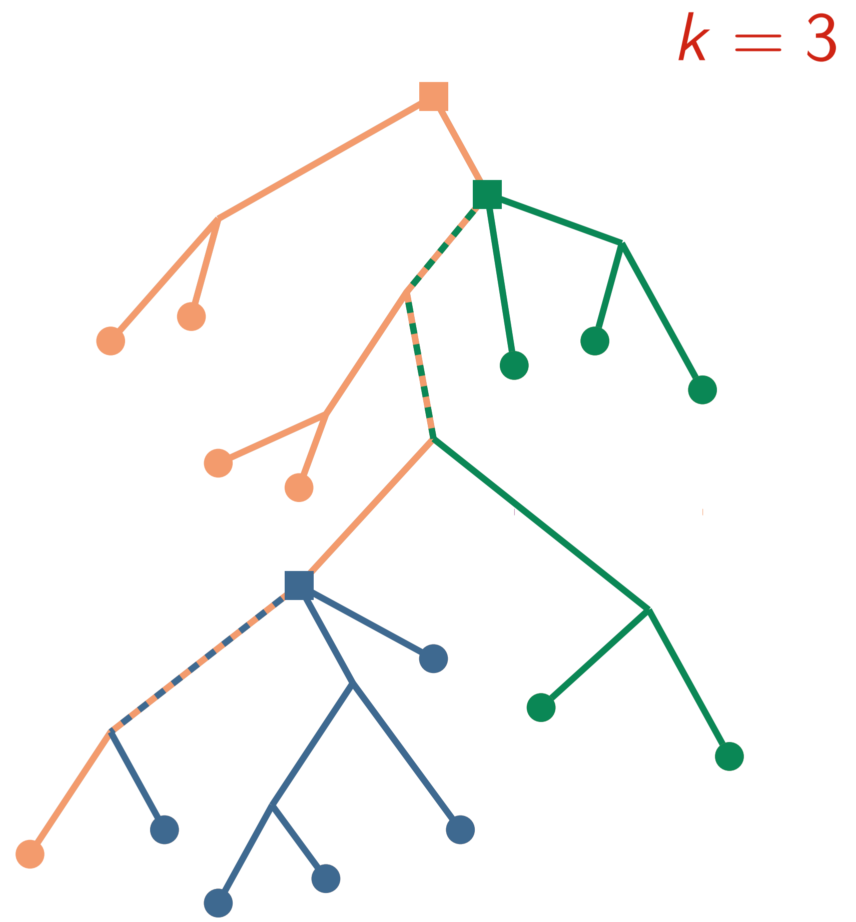
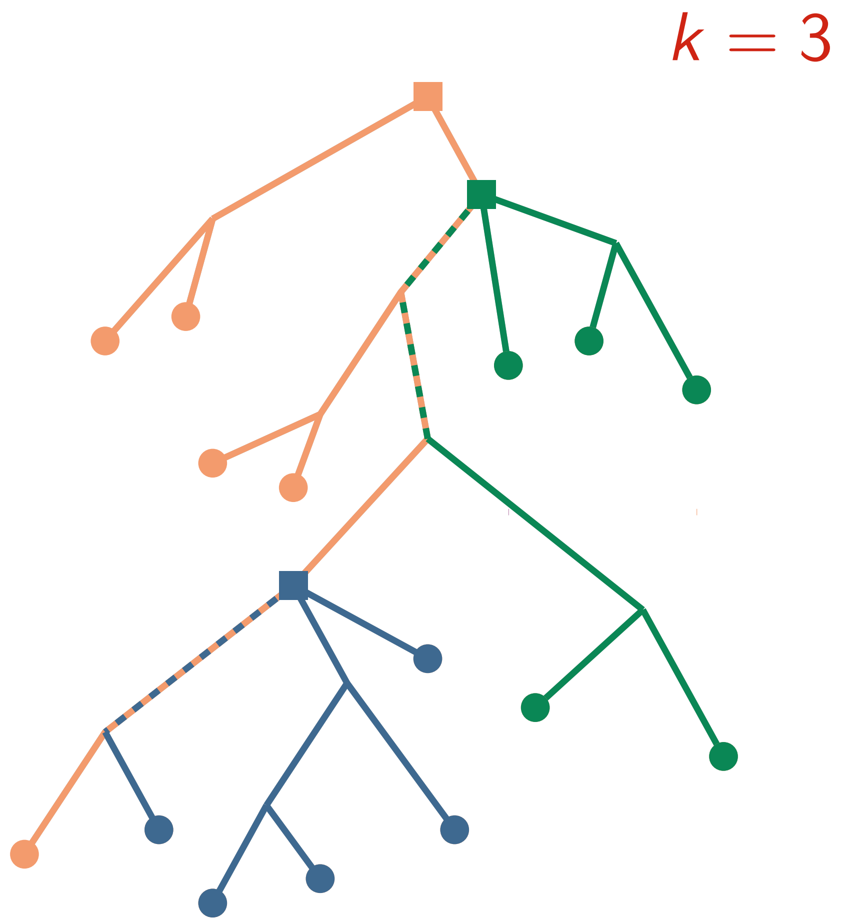


Where to place Steiner points?

Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal
2. Color the tree
3. Place Steiner points
4. Split tree into k subtrees



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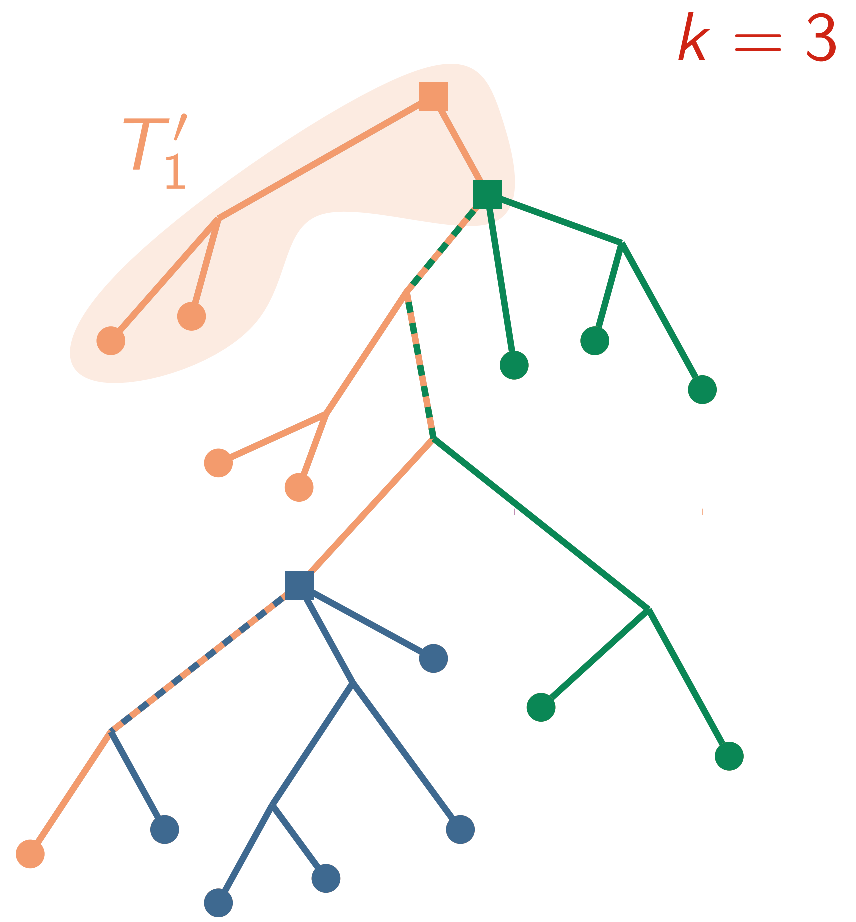
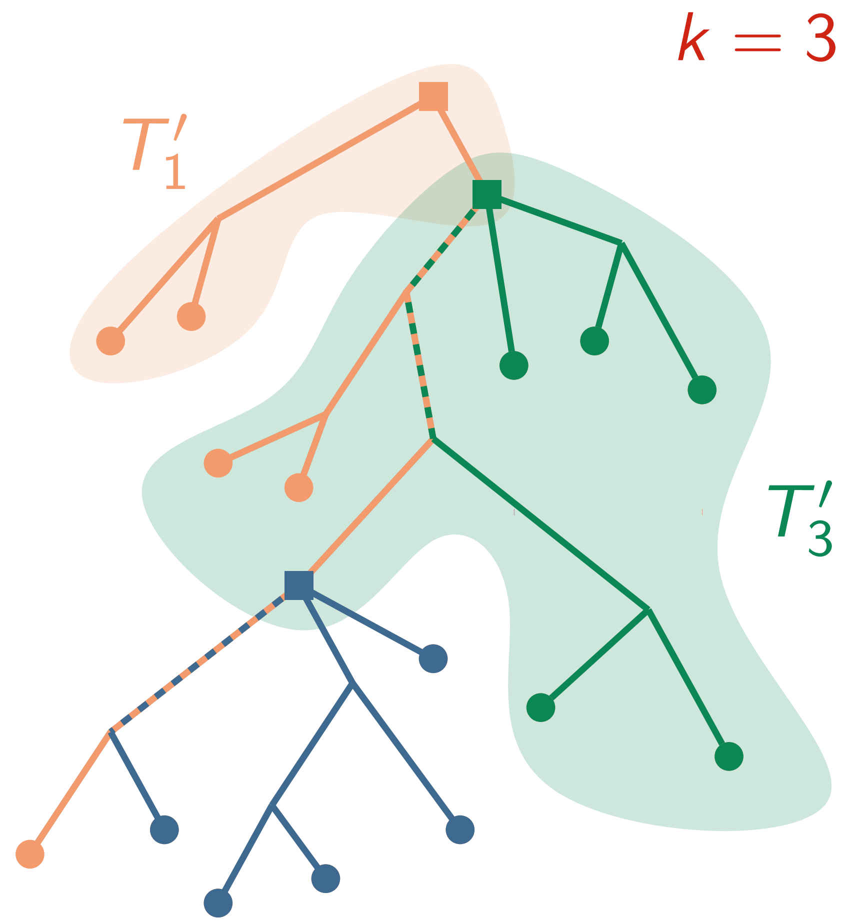
4. Split tree into k subtrees



Where to place Steiner points?

Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal
2. Color the tree
3. Place Steiner points
4. Split tree into k subtrees



Where to place Steiner points?

Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal
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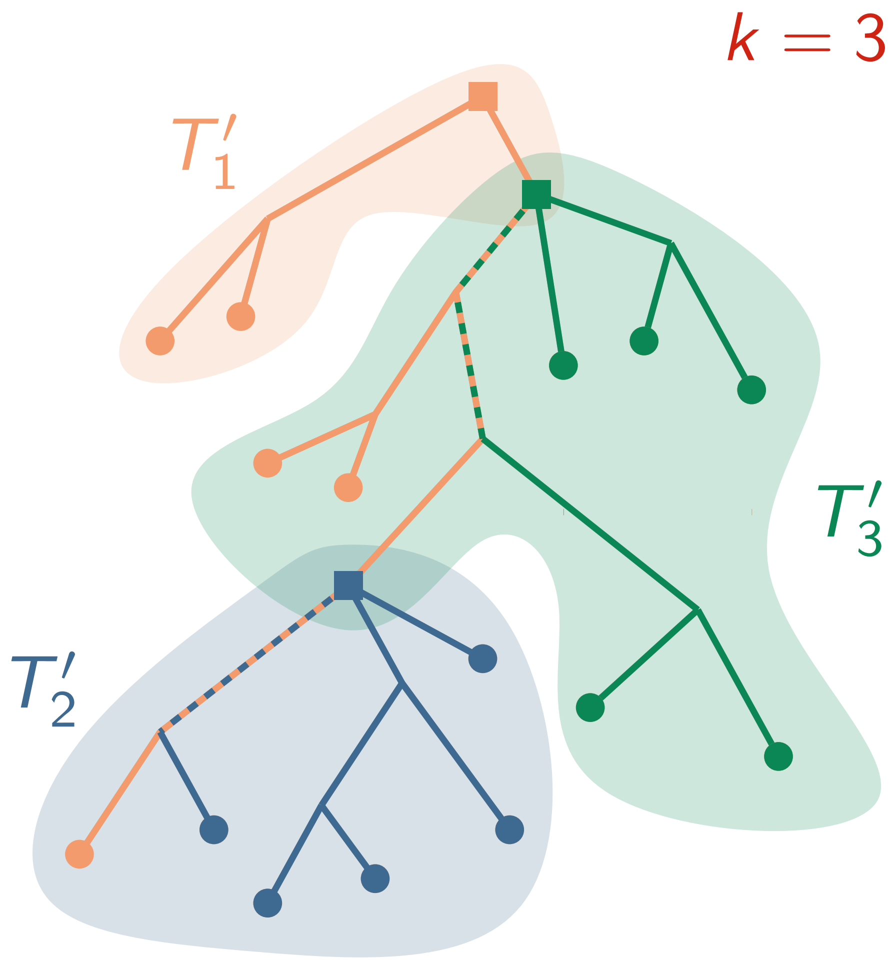

Where to place Steiner points?

Look at a simpler setting: a tree

1. Split vertices in k groups by inorder-traversal
2. Color the tree
3. Place Steiner points
4. Split tree into k subtrees
5. Build spanner on each subtree