Bipartite Dichotomous Ordinal Graphs

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Introduction

Dichotomous Ordinal Graphs

- Given:
  - Graph $G = (V, E)$
  - A partition of edges into short and long, $E = E_s \cup E_\ell$

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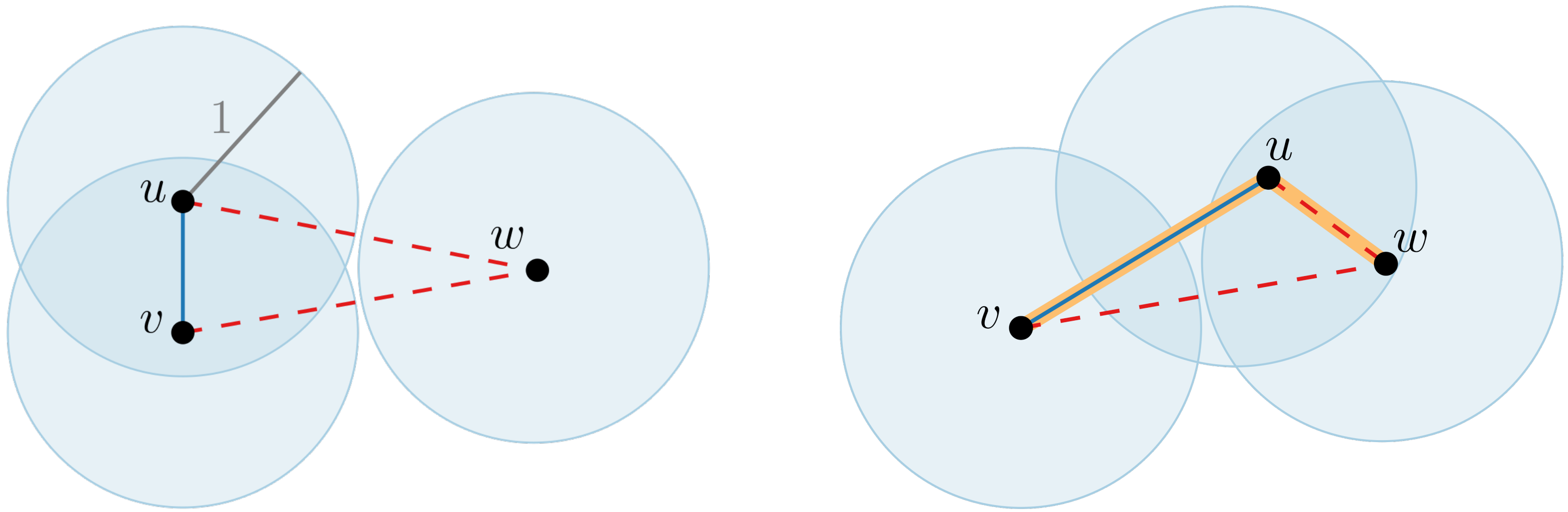

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  - $G \equiv K_{n,m}$ ($\exists\mathbb{R}$-complete) [Peters (2017)]

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  - $G \equiv K_{n,m}$ ($\exists\mathbb{R}$-complete) [Peters (2017)]
- Angelini et al. (2019)

Positive Instances

- double-wheel
- 2-degenerate
- subcubic
- 4-colorable and the short edges induce a caterpillar

Negative Instances

- double wheel plus one edge

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✔ $G \subseteq K_{3,m}$ ✔ $G \subseteq K_{4,6}$

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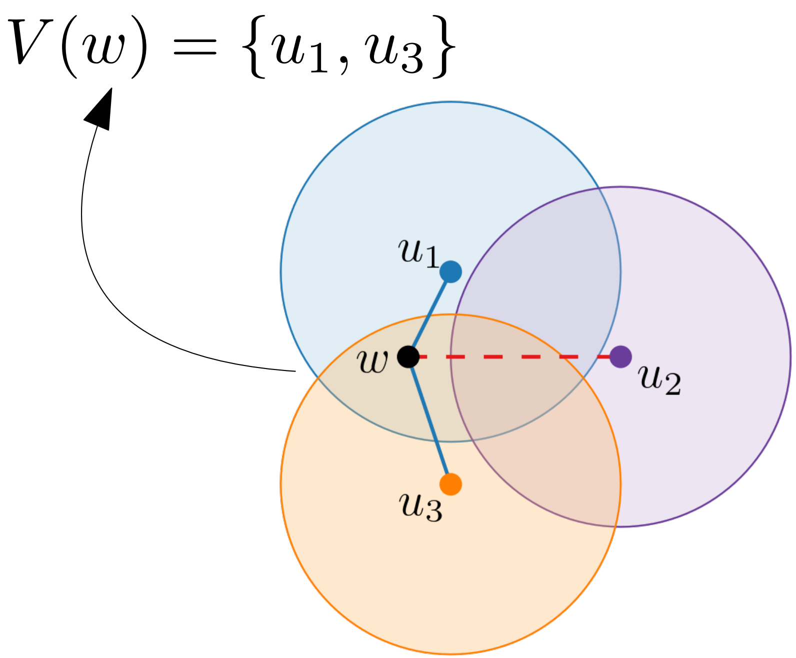
Complete Bipartite Graphs

- $G = (U \cup W, E)$
- $U = \{u_1, u_2, \ldots, u_n\}$
  - C_i the unit circle centered at u_i
- If edge (u_i, w) is short, then w should be placed in C_i.
- $V(w) \subseteq U$ is the "short neighborhood" of $w \in W$



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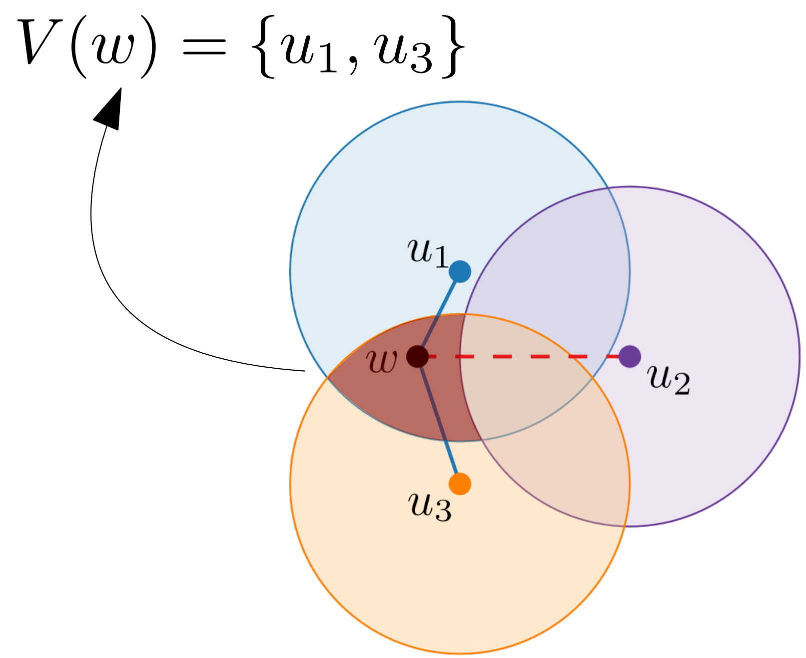
Goal: To find an arrangement $\mathcal{C}$ of $C_1, C_2, \ldots, C_n$ such that every $V(w)$ corresponds to a cell r



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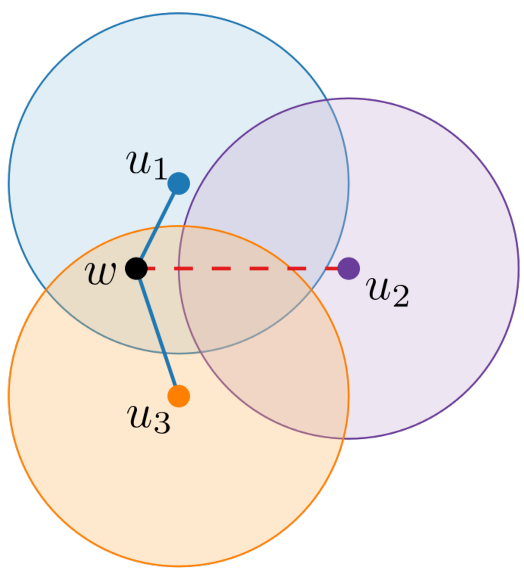
- $K_{3,m}$:
  - all eight subsets of $U = \{u_1, u_2, u_3\}$ can be realized



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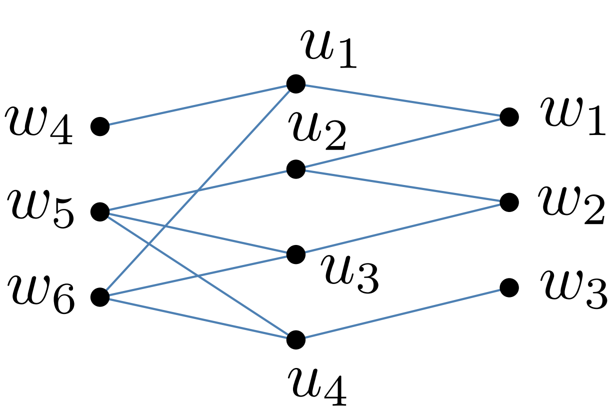

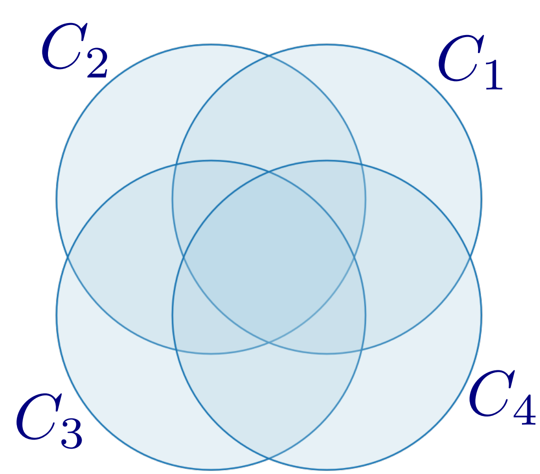

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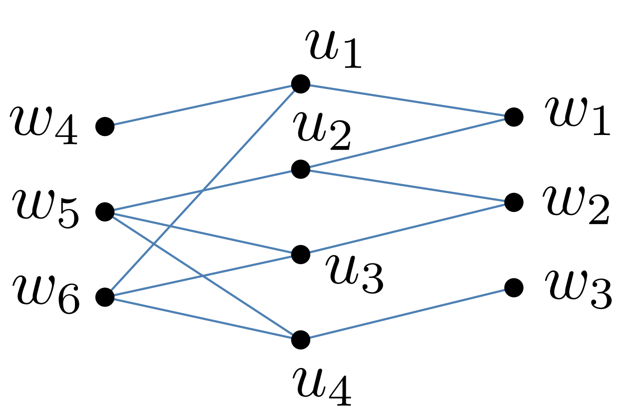

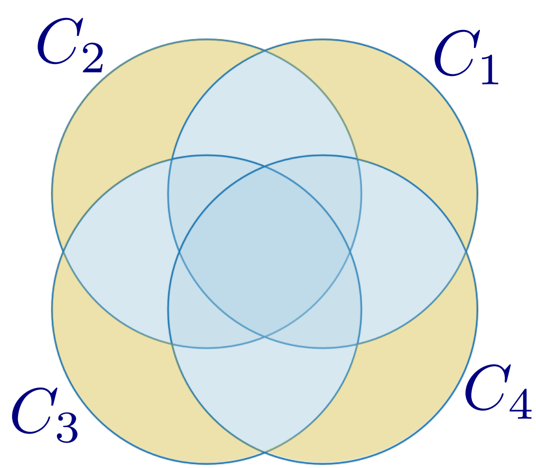

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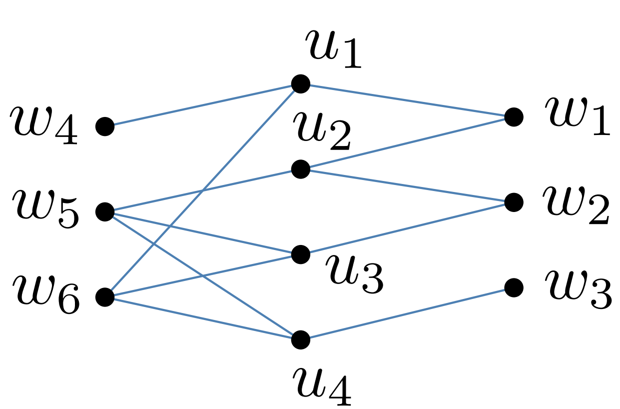

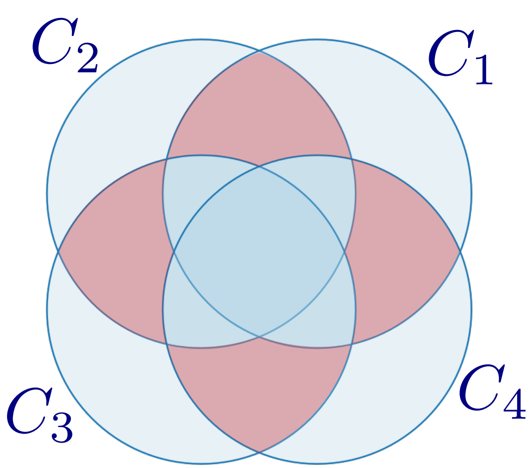

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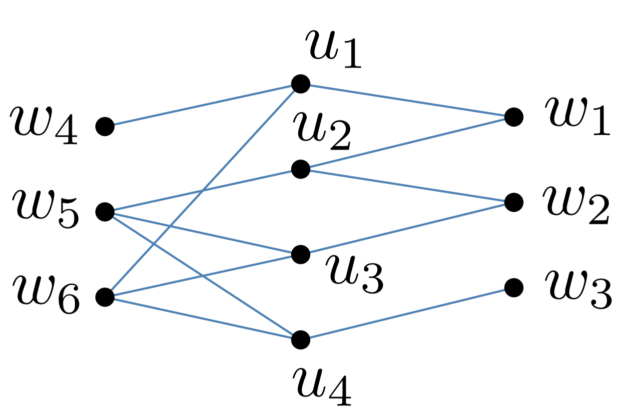

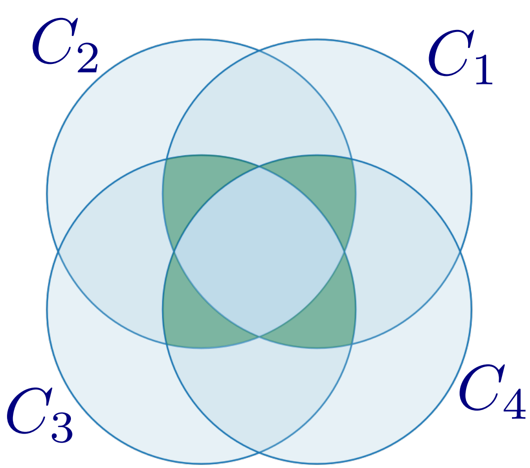

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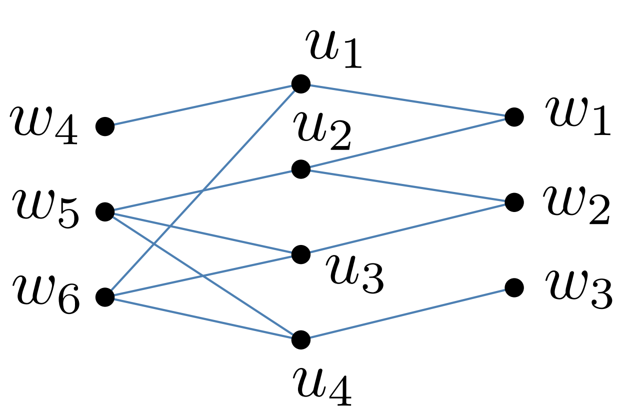

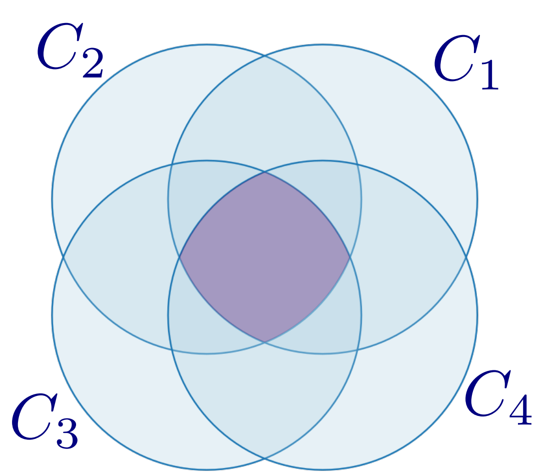

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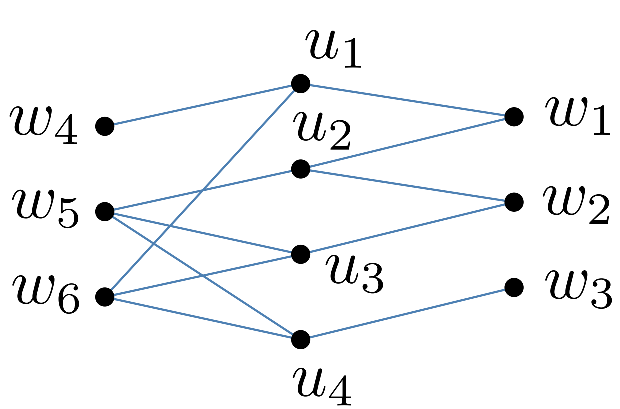

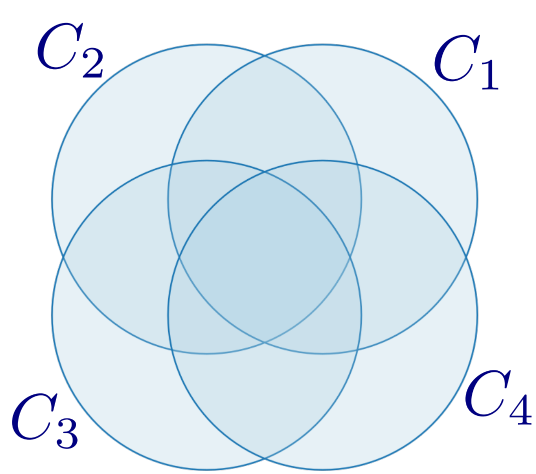

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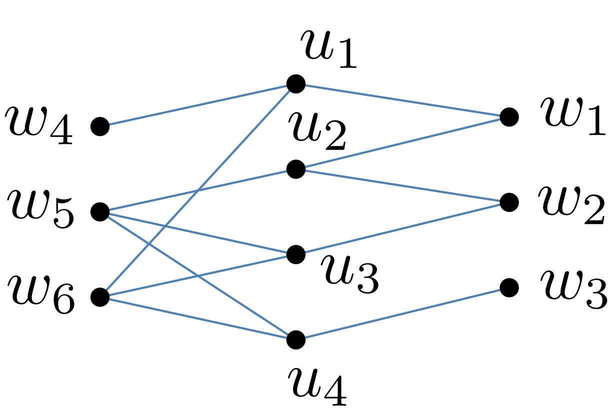

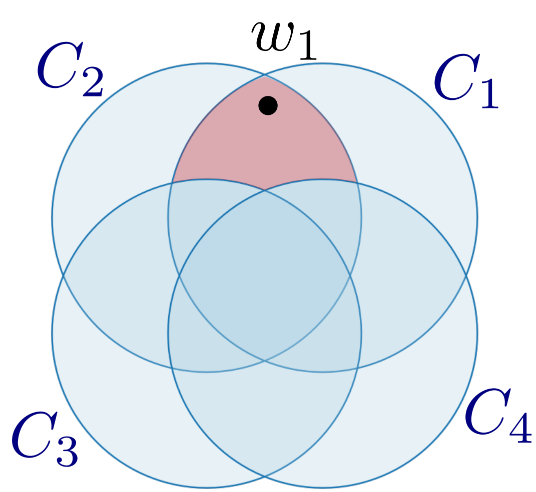

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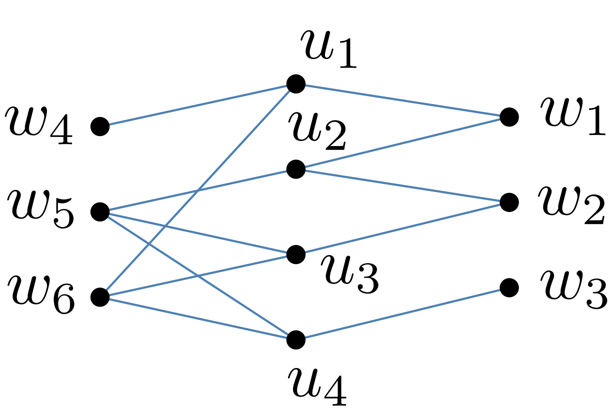

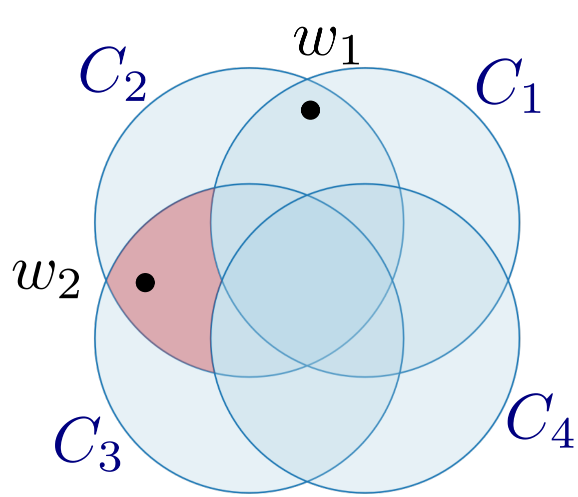

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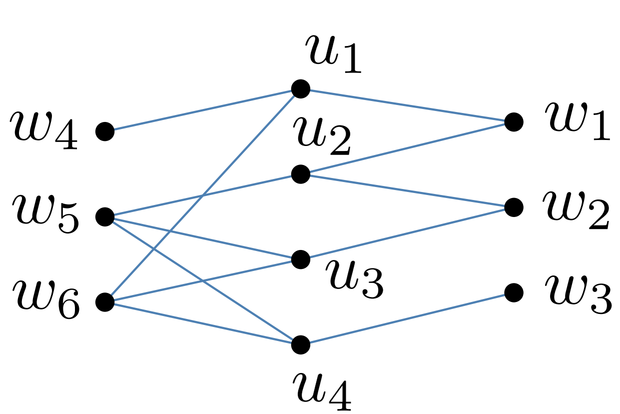

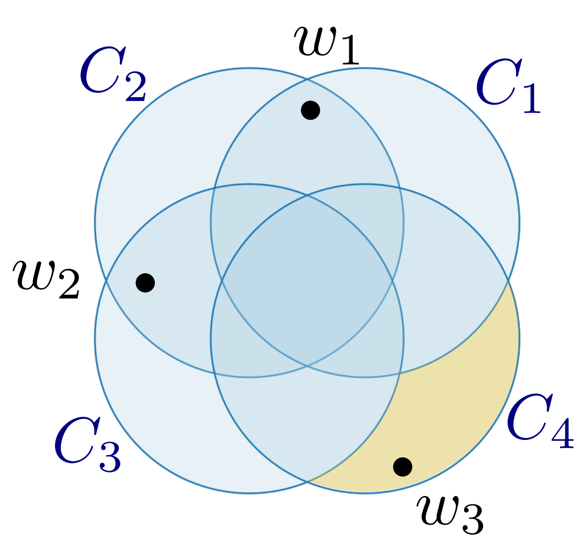

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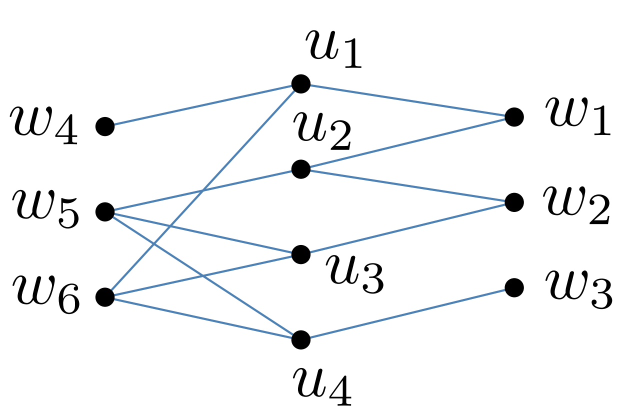

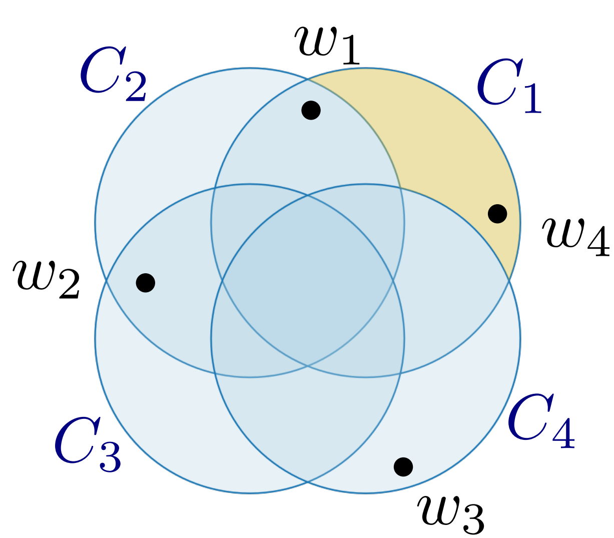

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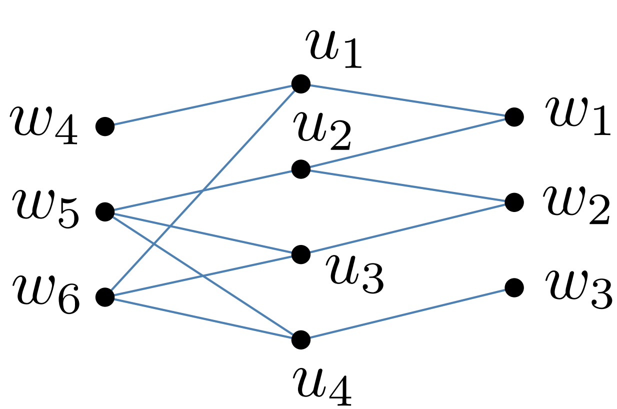

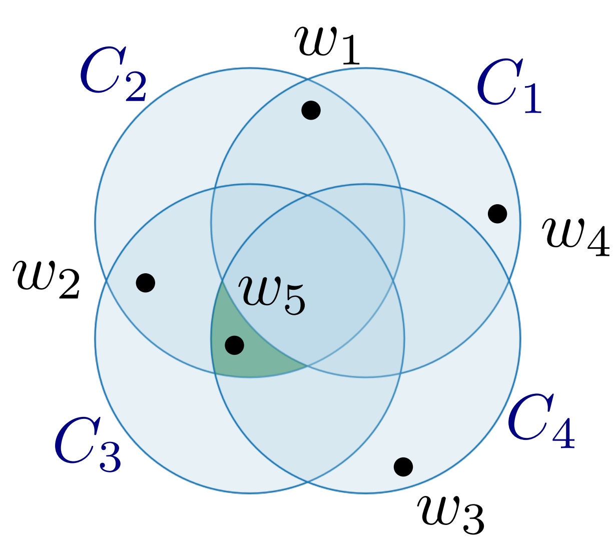

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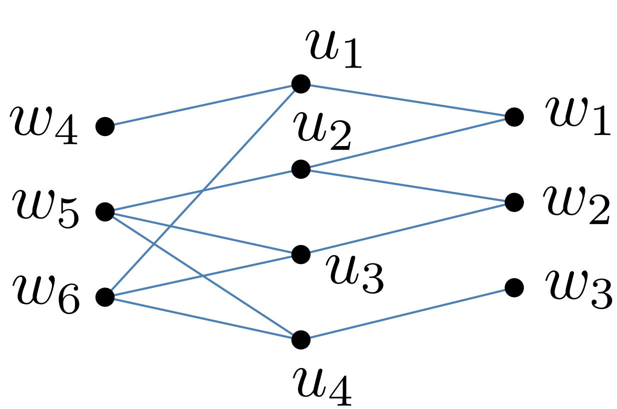

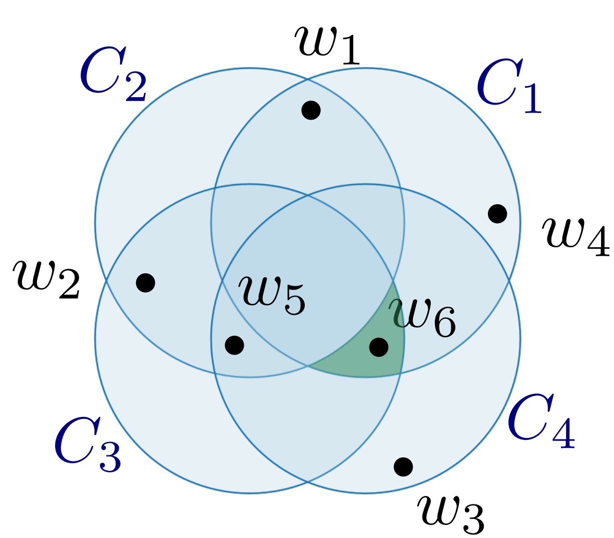

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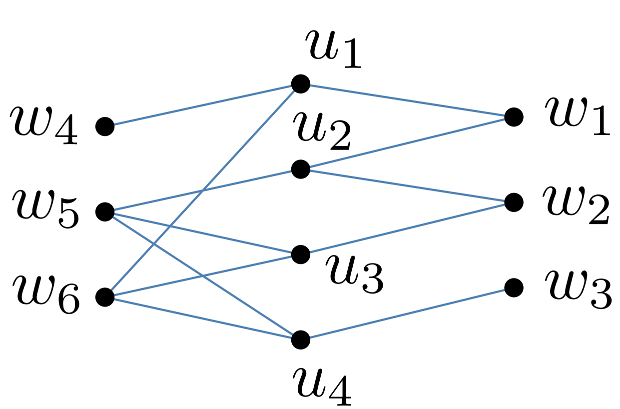

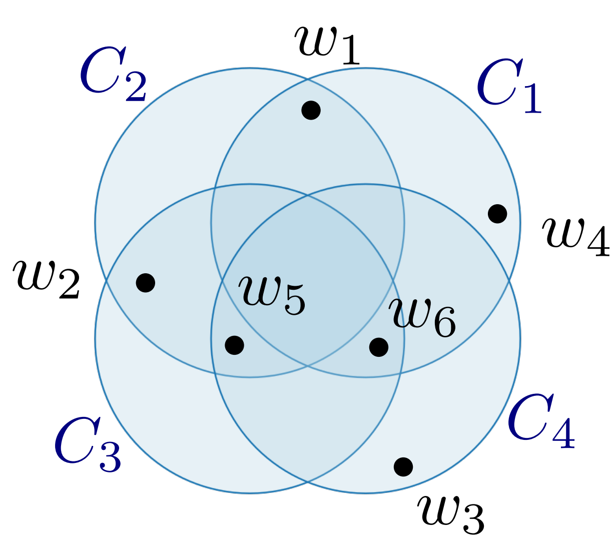

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- $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$
  - Case 2: At least three vertices from W are shortly connected to a pair of U.

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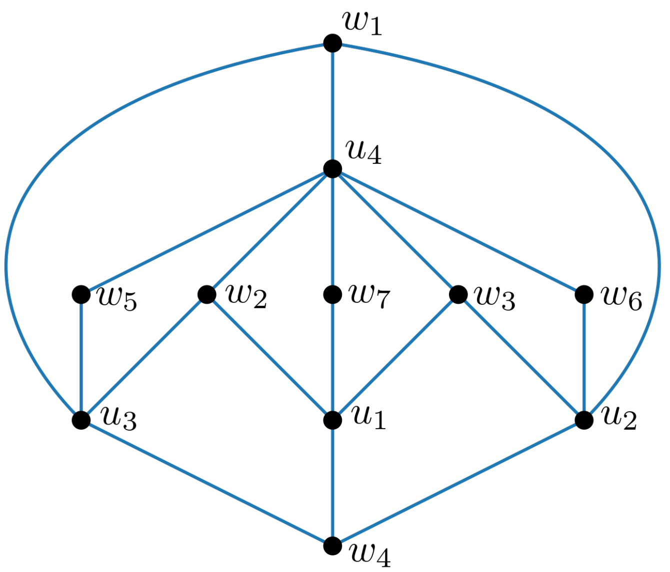
- $U = \{u_1, u_2, u_3, u_4\}, W = \{w_1, \ldots, w_7\}$
- Counterexample:
  - All four triplets
  - The three pairs that contain u_4



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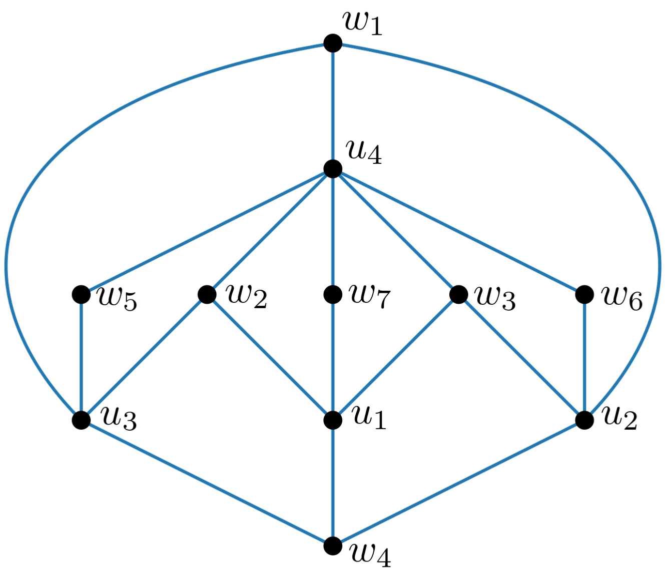

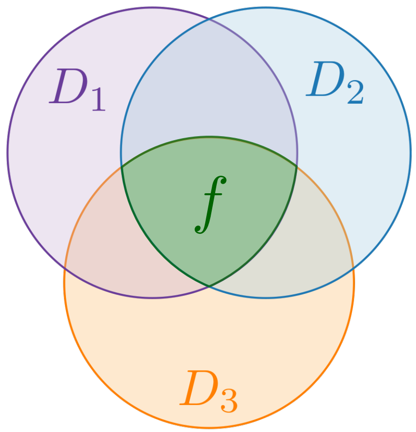

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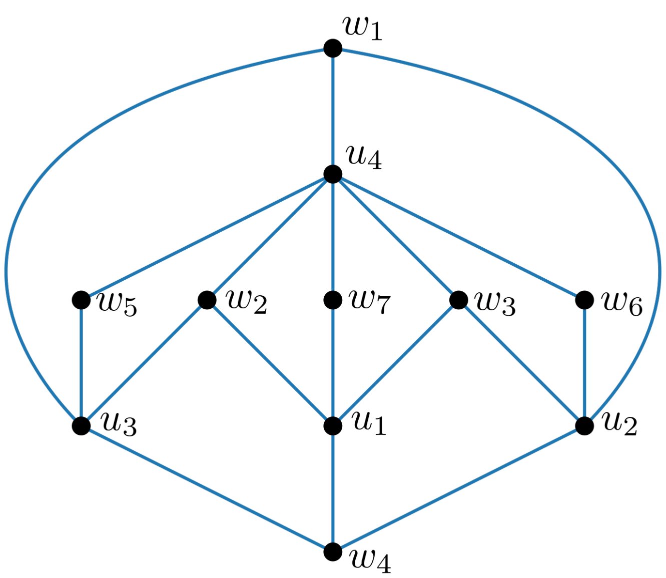

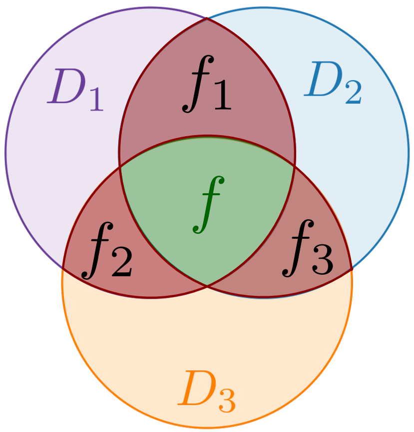

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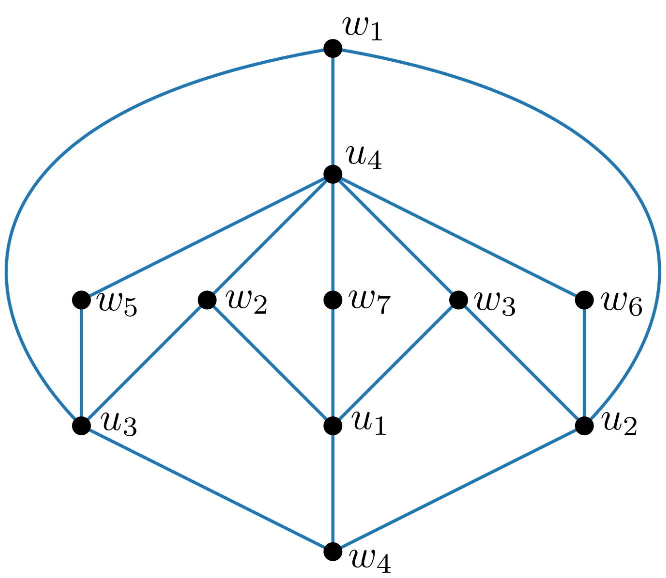
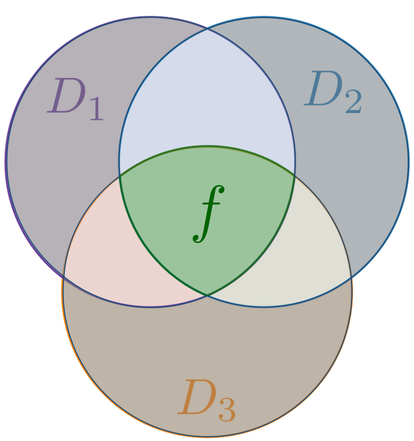
- $U = \{u_1, u_2, u_3, u_4\}, W = \{w_1, \ldots, w_7\}$
- Counterexample:
  - All four triplets
  - The three pairs that contain u_4





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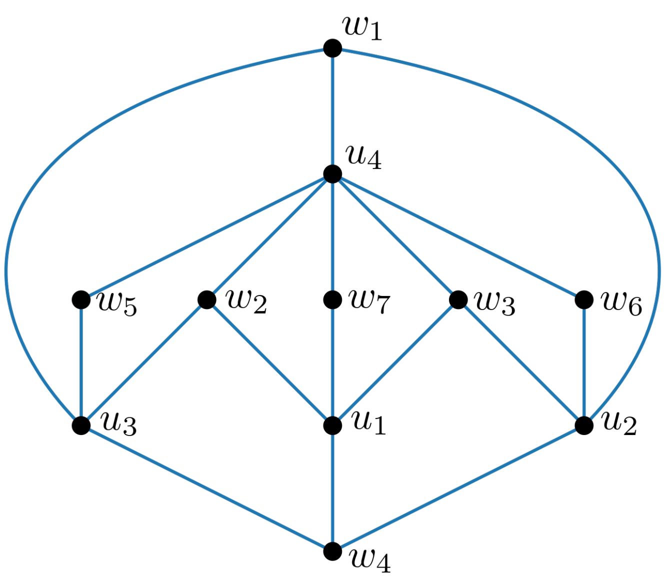

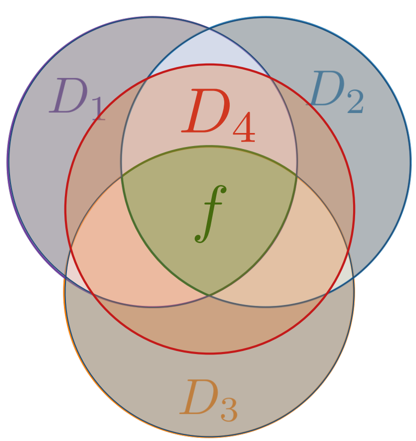

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Short Subgraphs of the Grid

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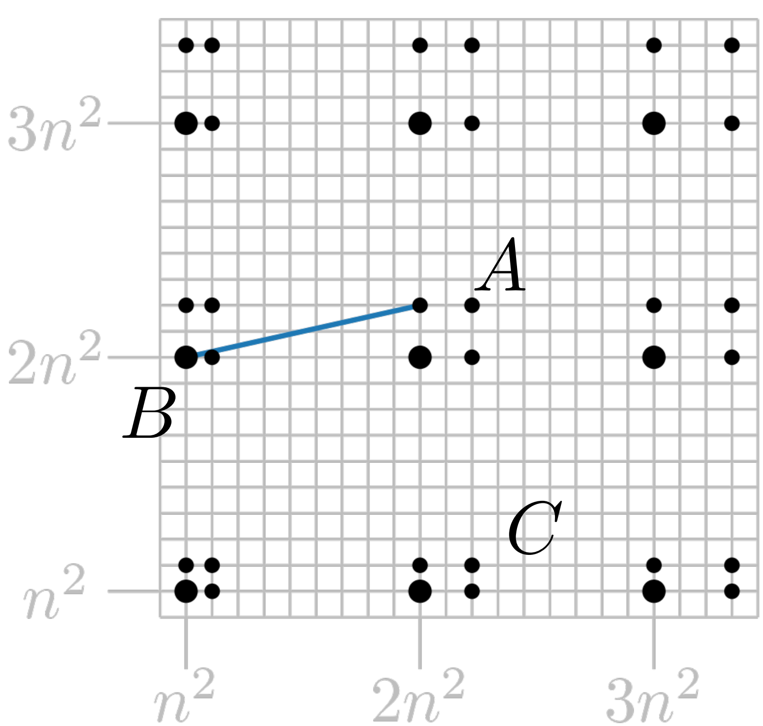

- Four possible choices for each grid point (i, j)
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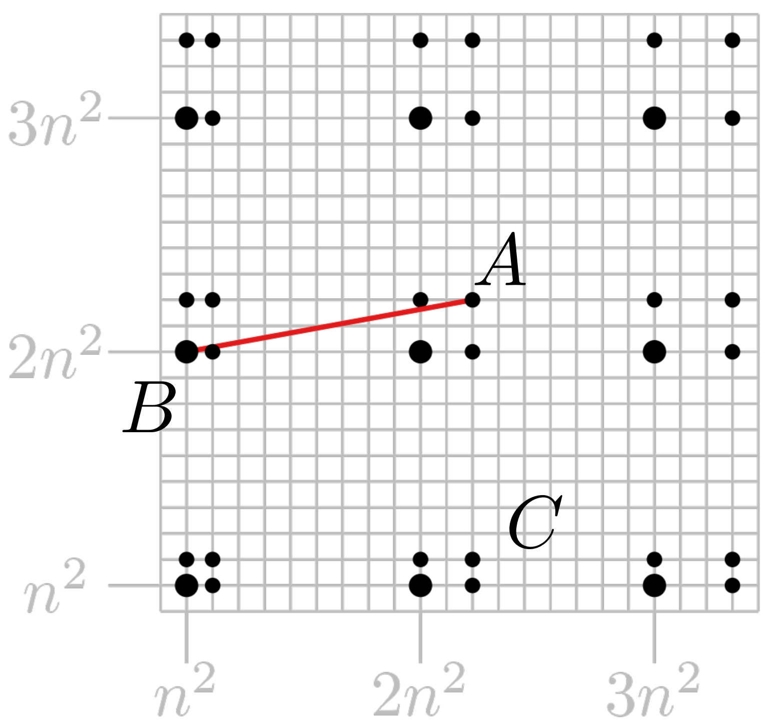

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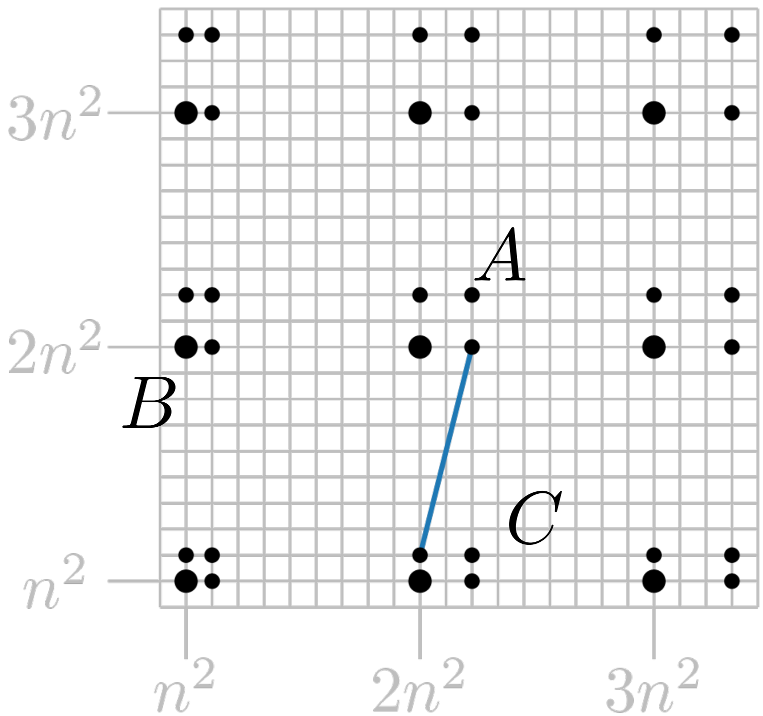

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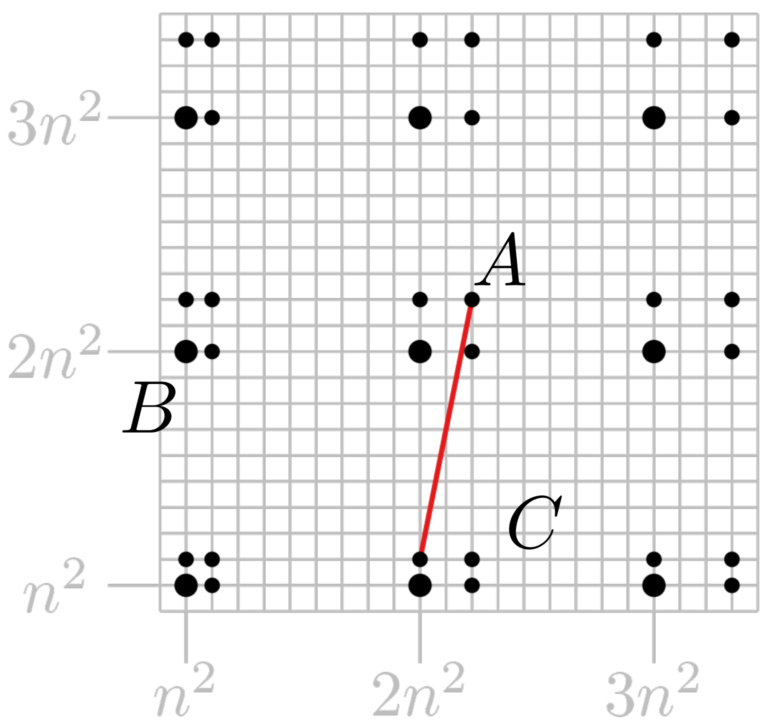

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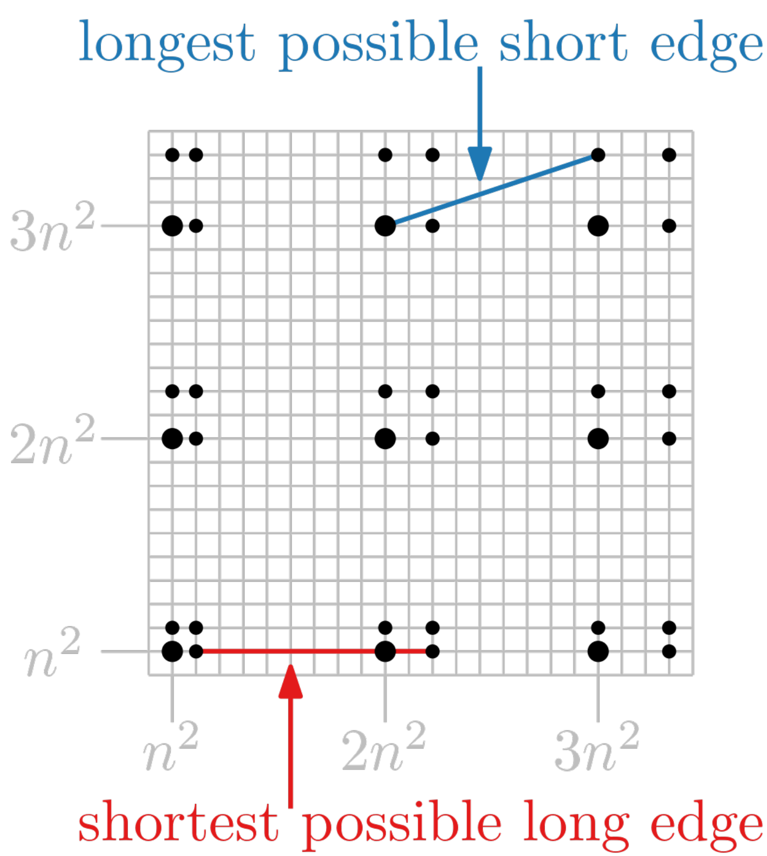

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- Long edges have length $\geq n^2 + 1$
- Short edges have length $\leq n^2 + \frac{1}{2}$

Conclusion

Open Problems

Do bipartite dichotomous ordinal graphs always admit a geometric realization when:
(i) the underlying graph is planar?
(ii) the underlying graph is 3-degenerate?
(iii) the graph induced by the short edges is a 2-tree?

Questions (i) and (ii) are open even for non-bipartite dichotomous ordinal graphs.

Complete Bipartite Graphs

Theorem

Every dichotomous ordinal $K_{3,m}$, for $m \in \mathbb{N}$, and every dichotomous ordinal $K_{4,m}$, for $m \leq 6$, admits a geometric representation.

Fact [Steiner (1826)] : n unit circles form at most $n(n-1)+2$ cells

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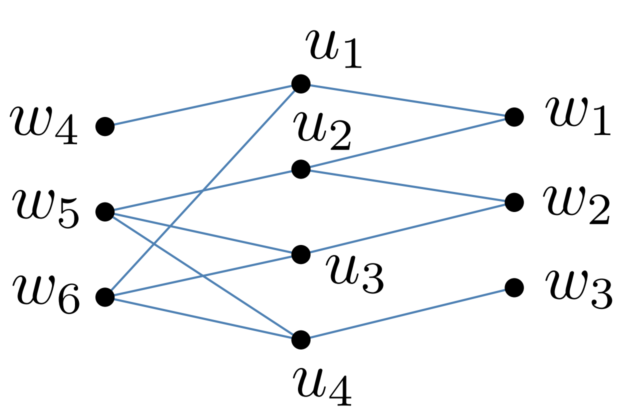

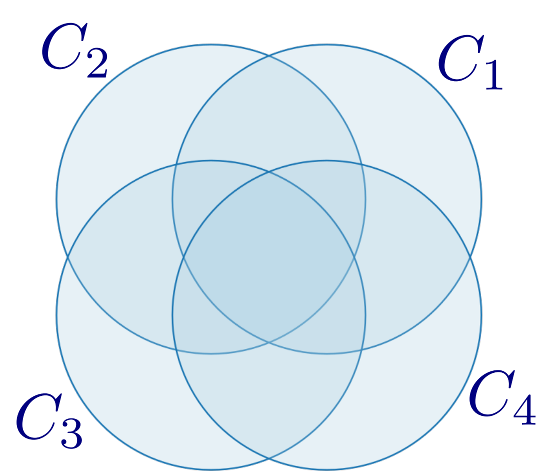

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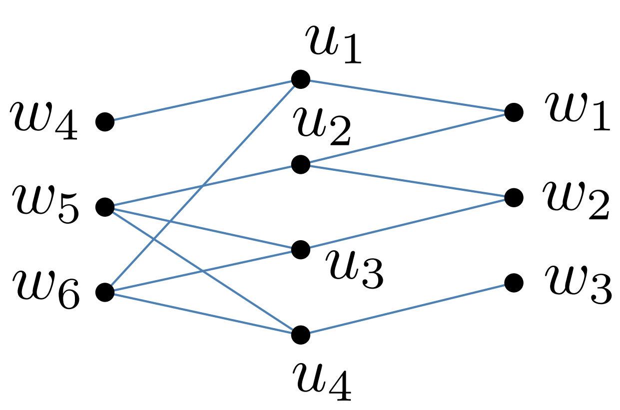

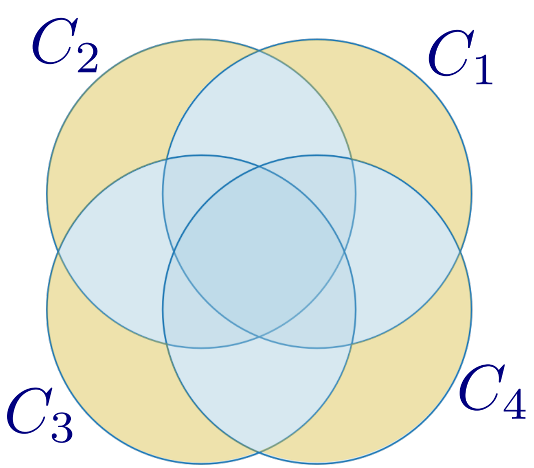

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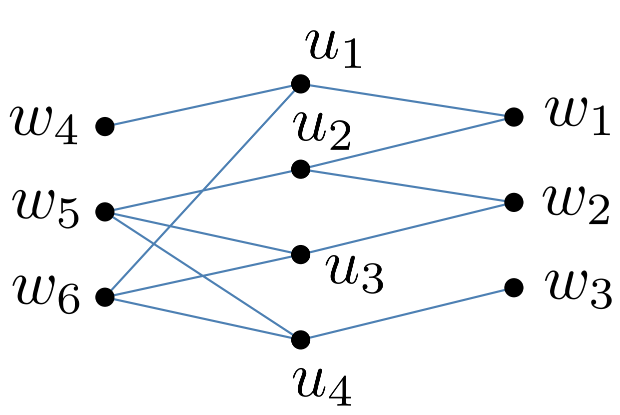

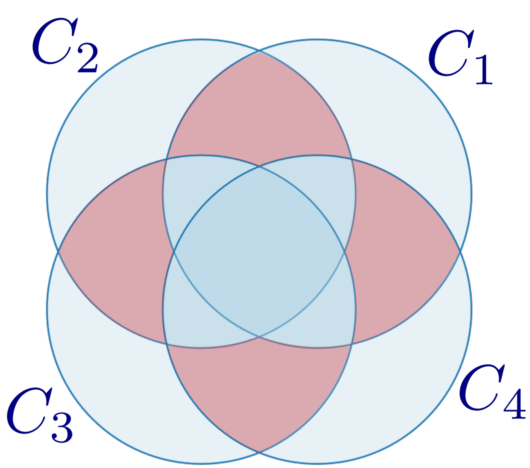

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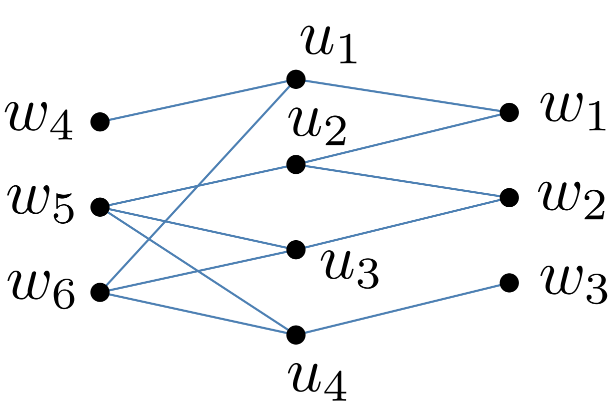

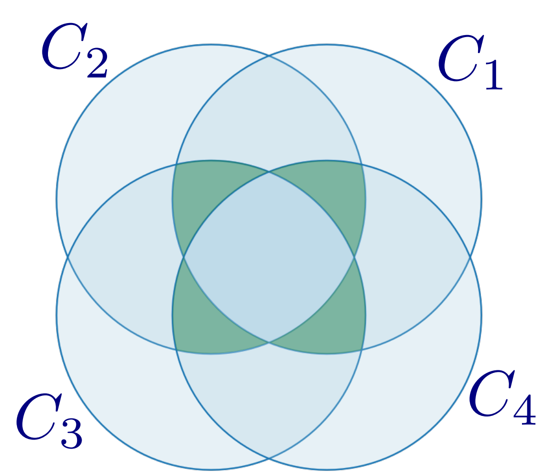

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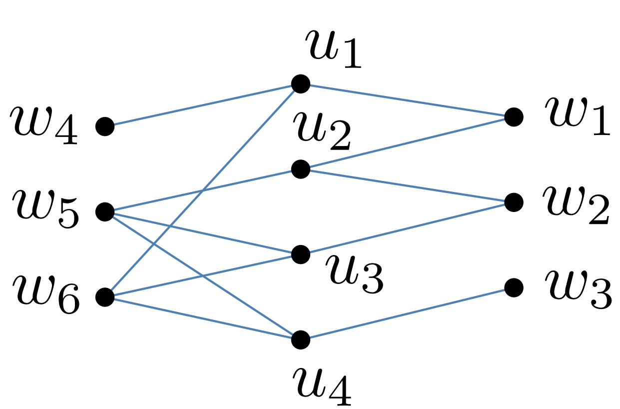

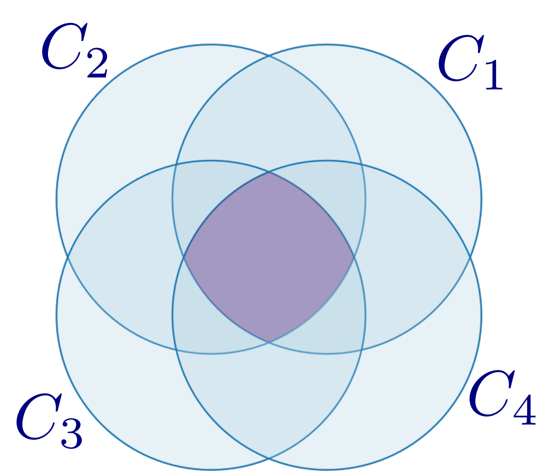

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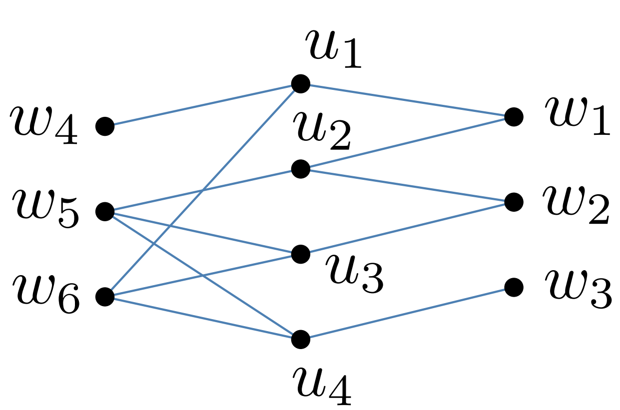

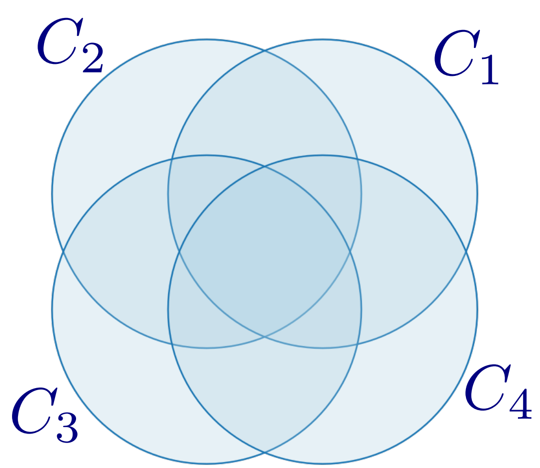

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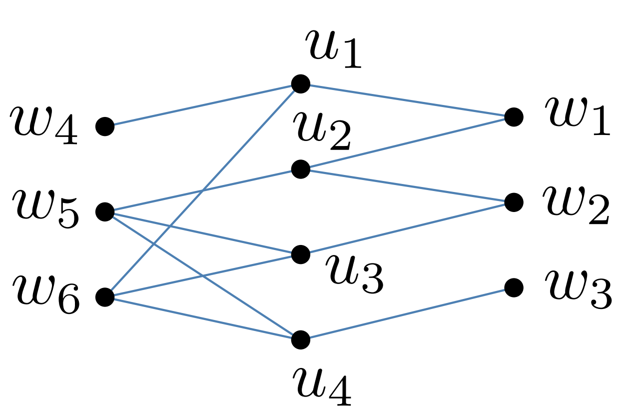

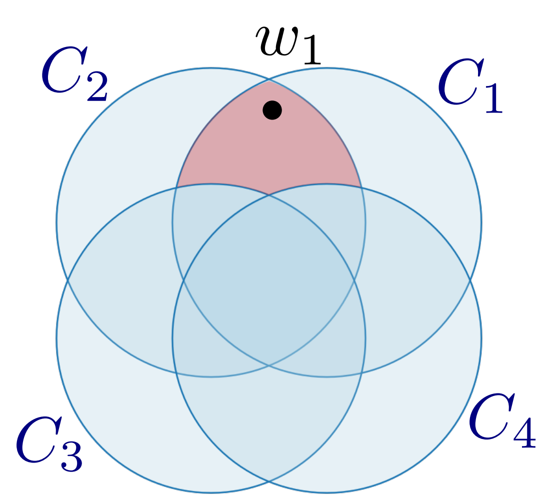

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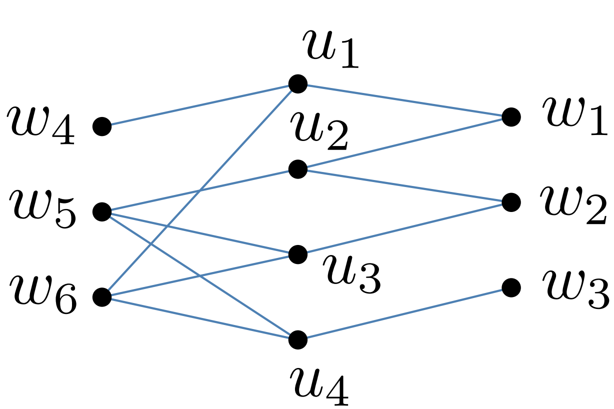

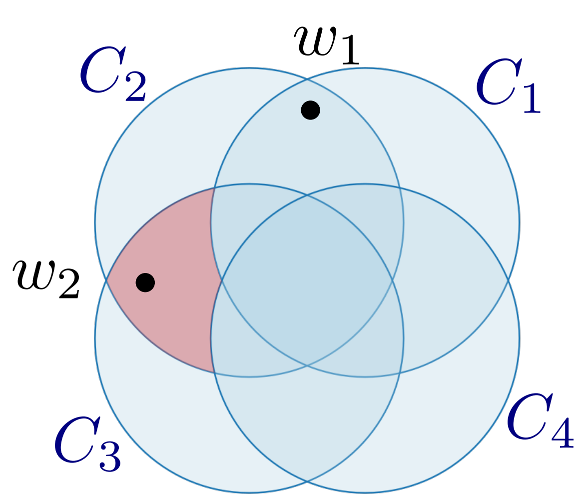

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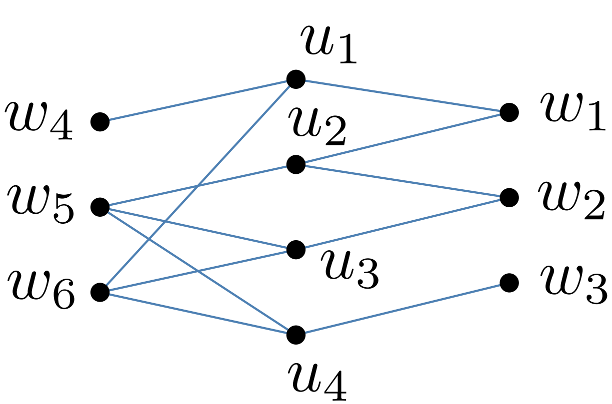

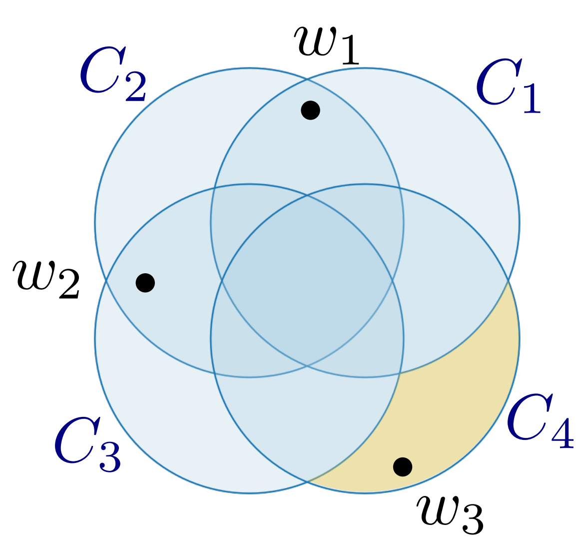

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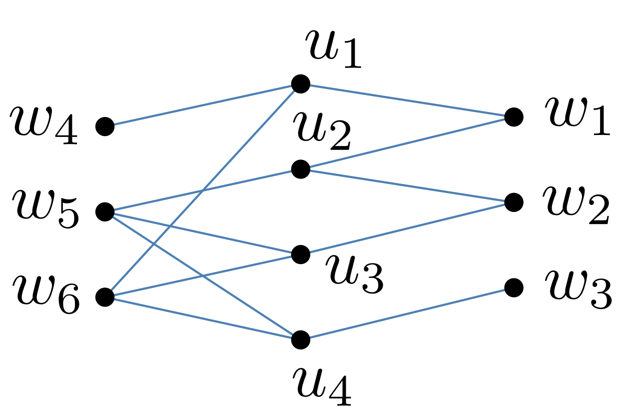

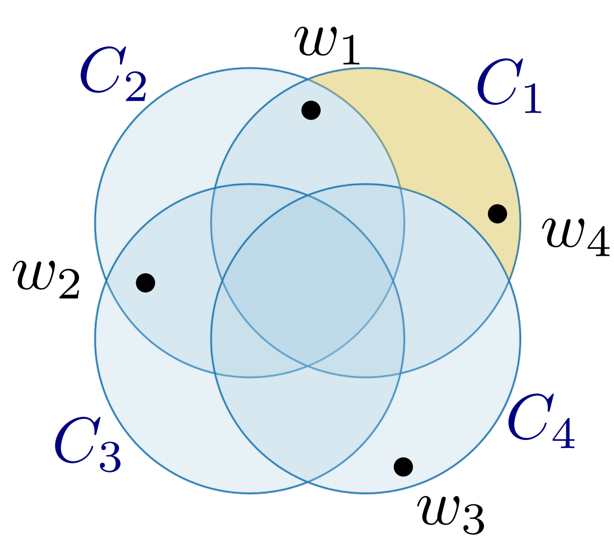

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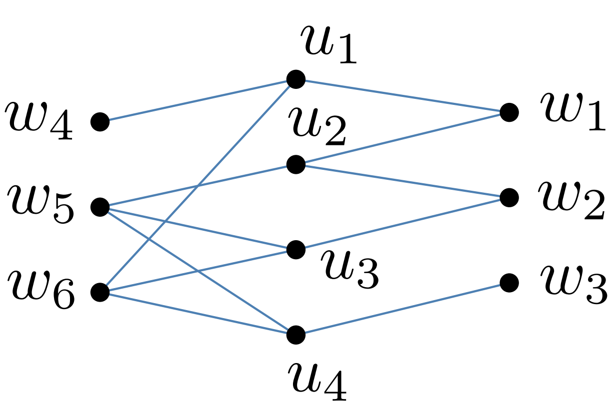

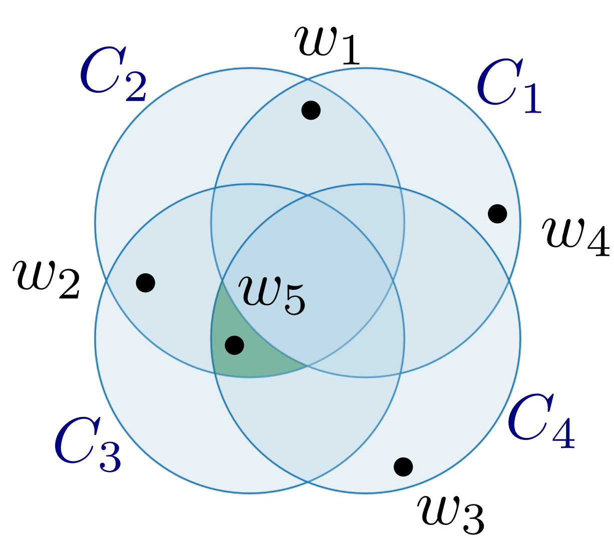

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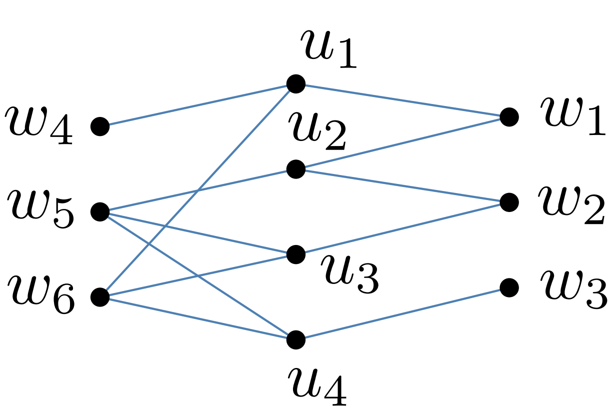

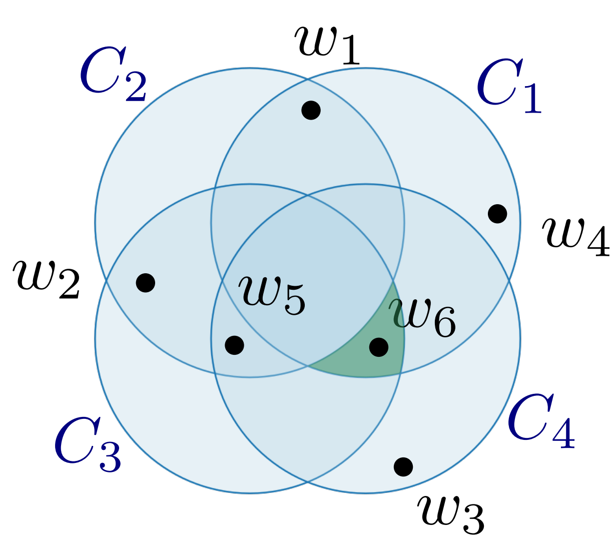

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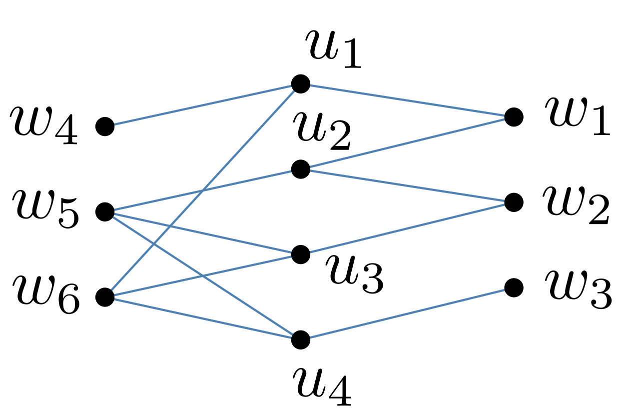

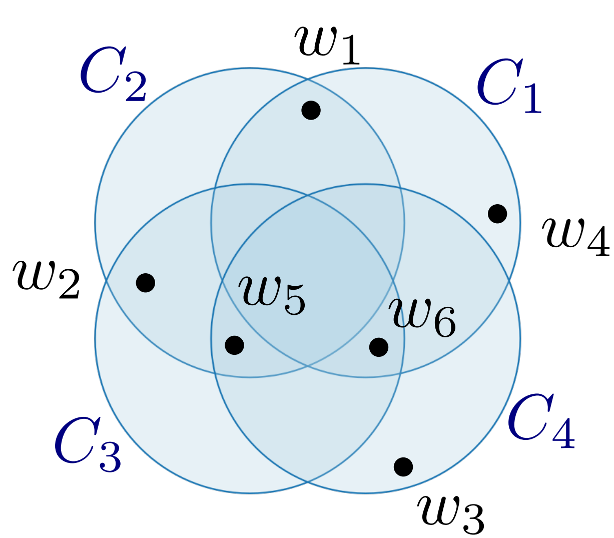

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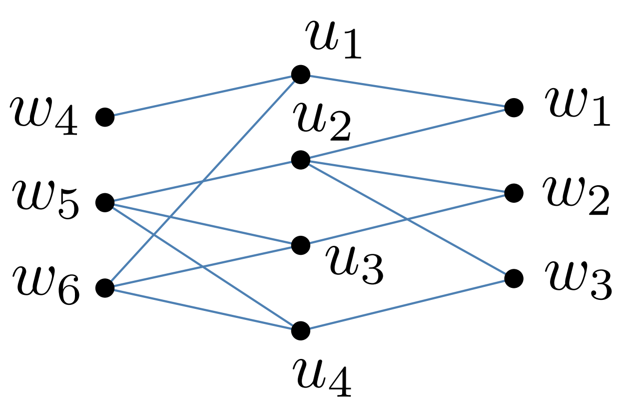

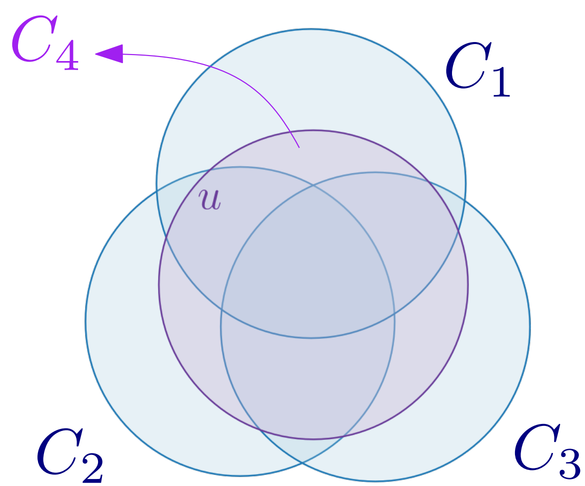

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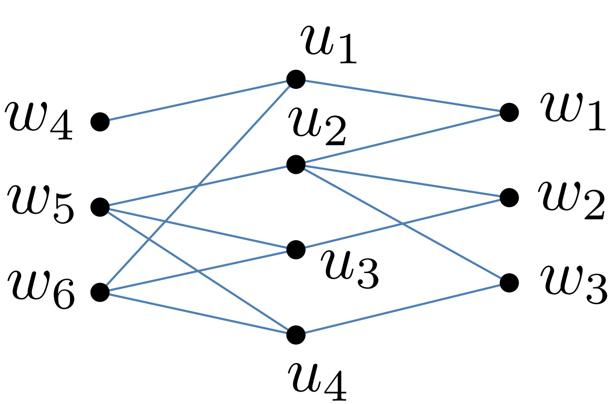

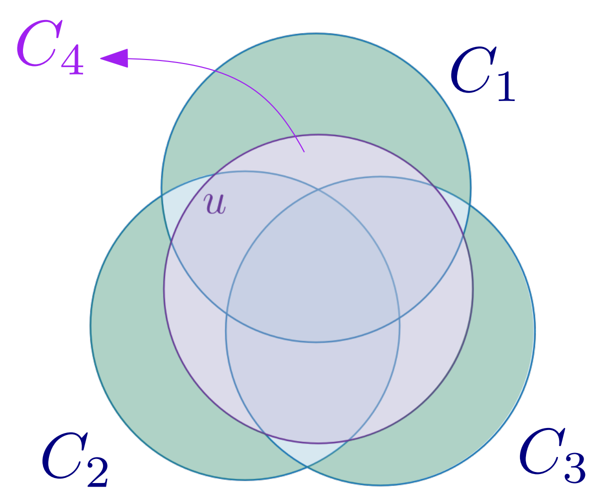

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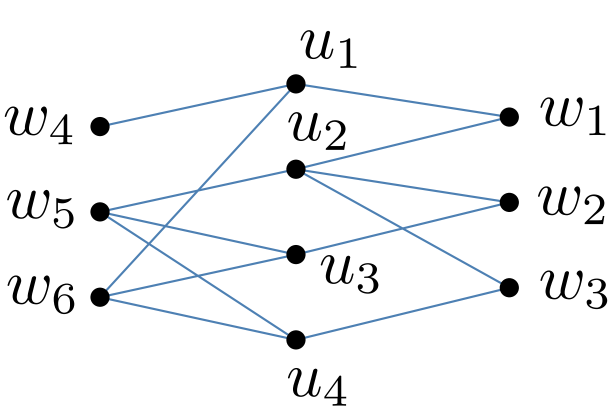

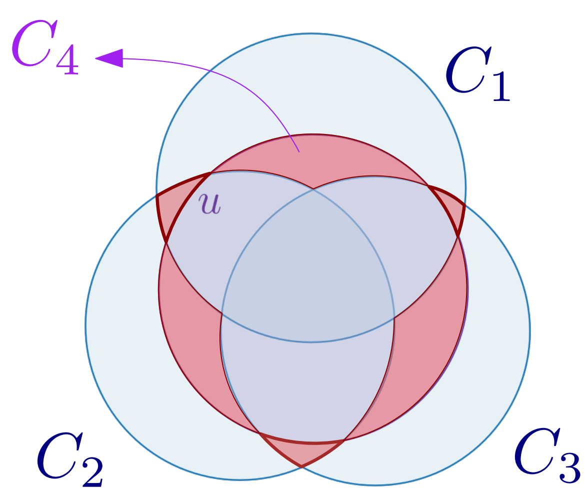

Missing Cells:

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Complete Bipartite Graphs

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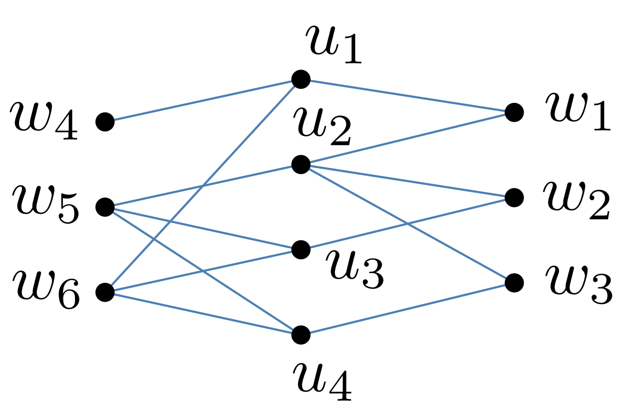

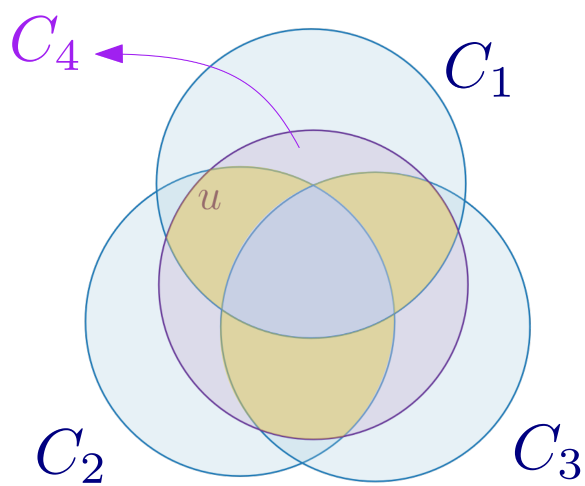

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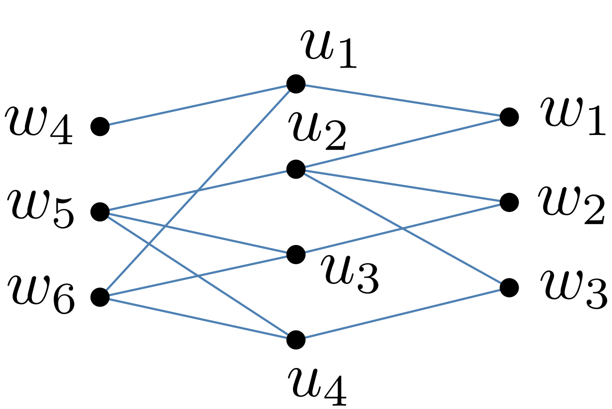

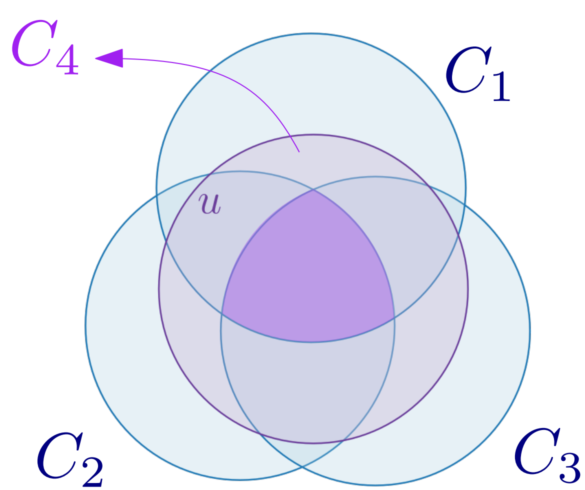

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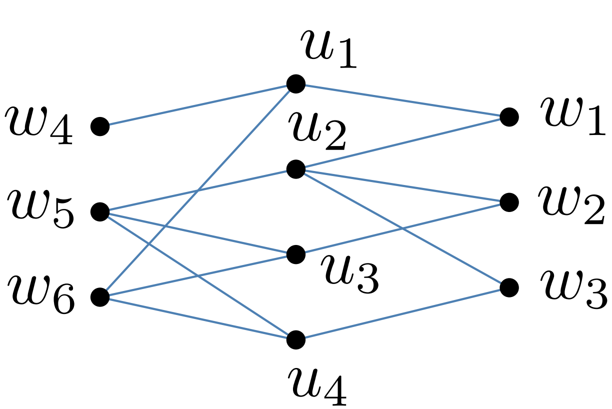
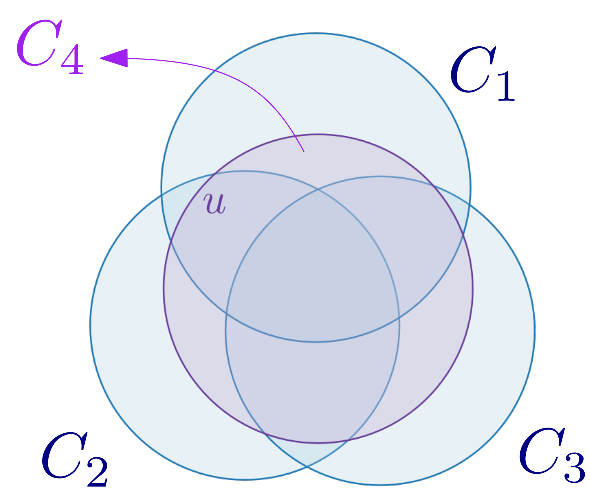
  $\Rightarrow$ We have at most three singletons and triplets





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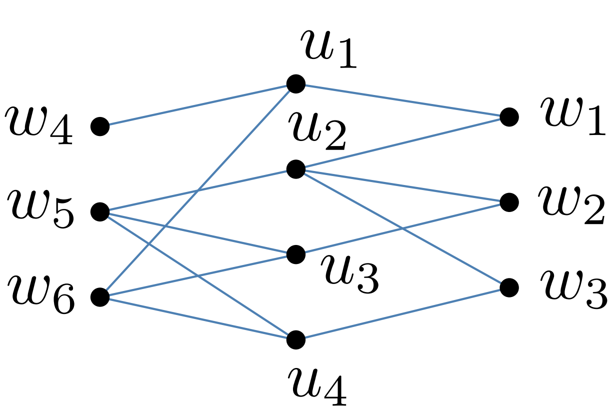

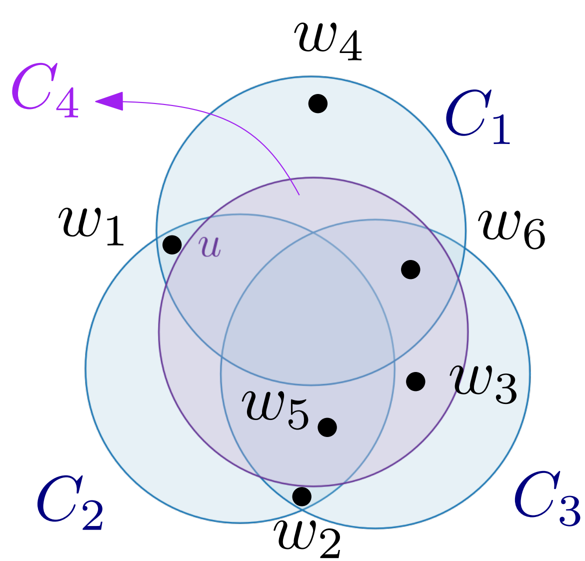

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There is a dichotomous ordinal $K_{5,5}$ that does not admit a geometric representation.

Complete Bipartite Graphs

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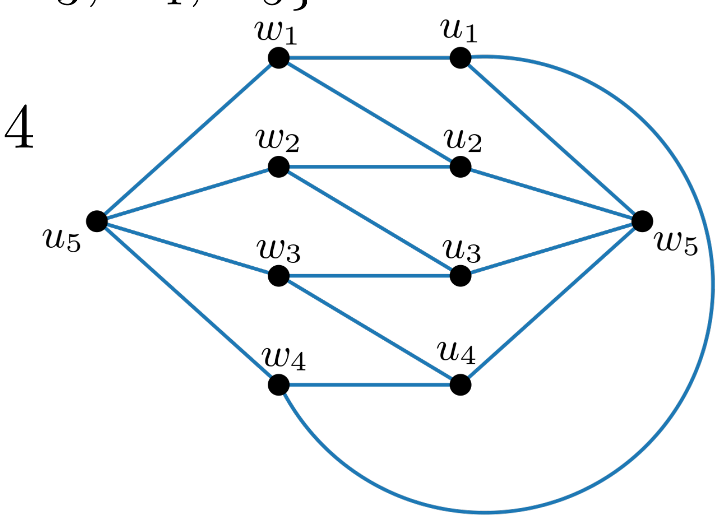
- $U = \{u_1, u_2, u_3, u_4, u_5\}, W = \{w_1, w_2, w_3, w_4, w_5\}$
- Counterexample:
  - $V(w_i) = \{u_i, u_{i \oplus 1}, u_5\}$, for $1 \leq i \leq 4$
  - $V(w_5) = U \setminus \{u_5\}$



- Each $V(w_i)$ corresponds to a cell in arrangement $\mathcal{C}$
- Analyze $\mathcal{C}$ geometrically to show that it cannot be realized

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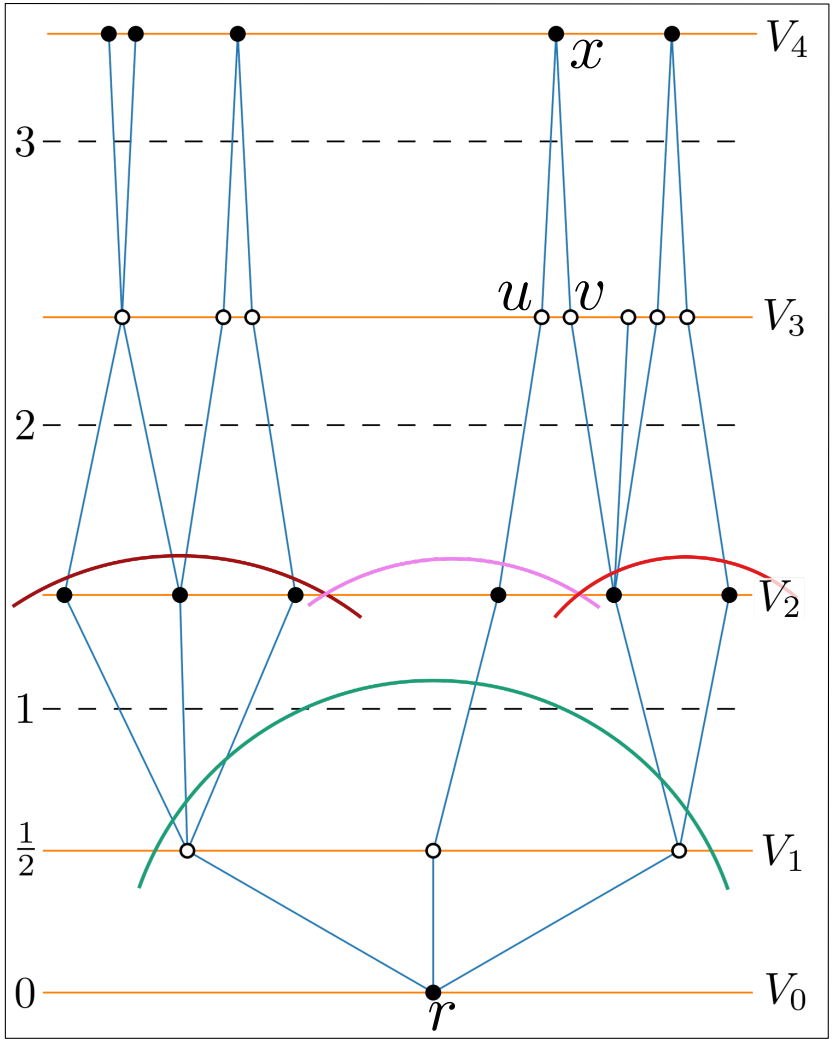

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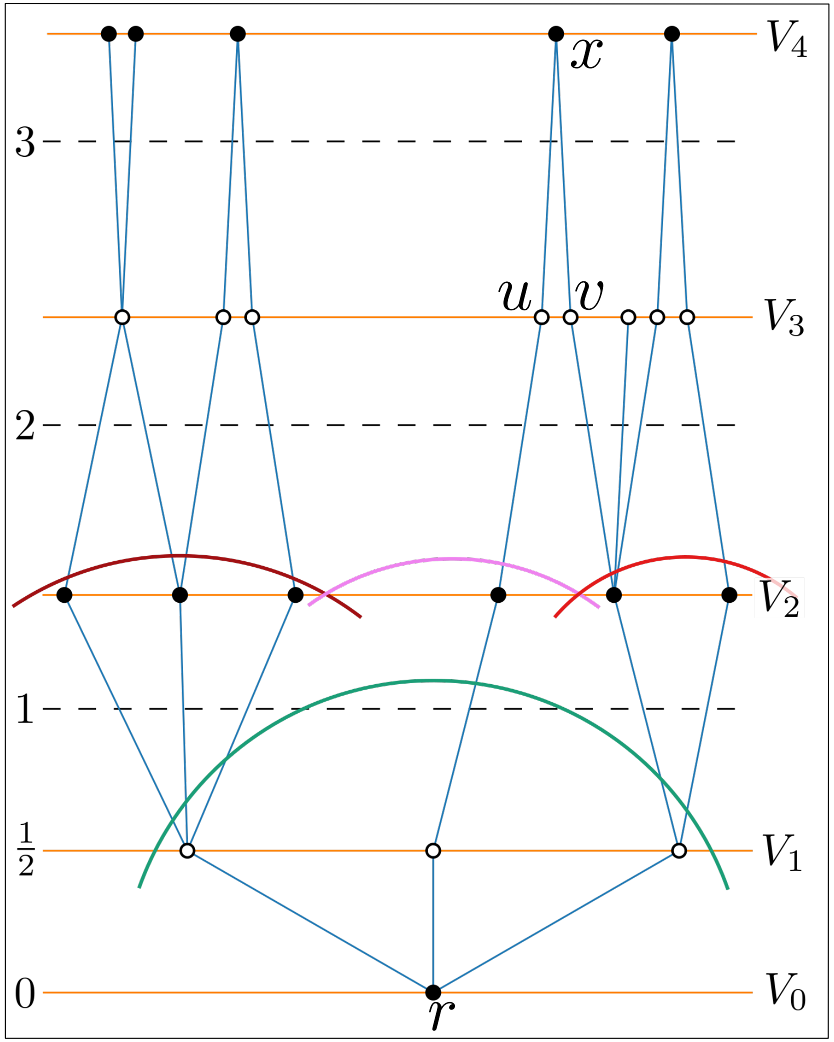

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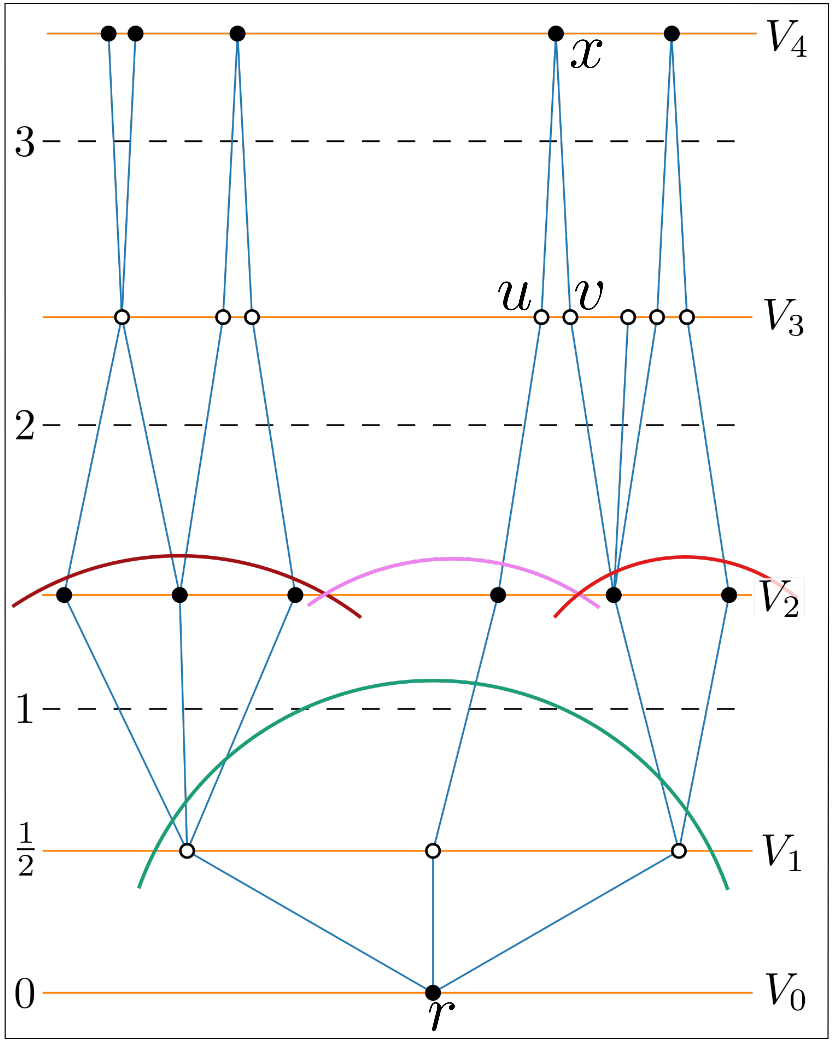

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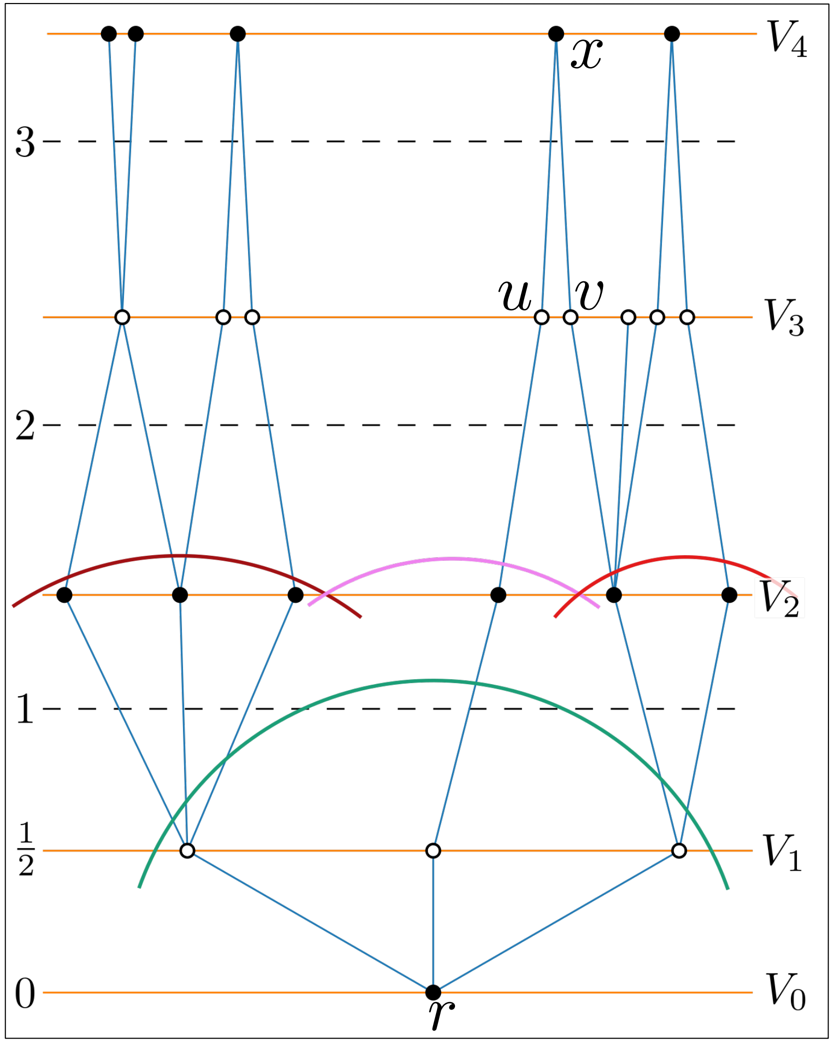

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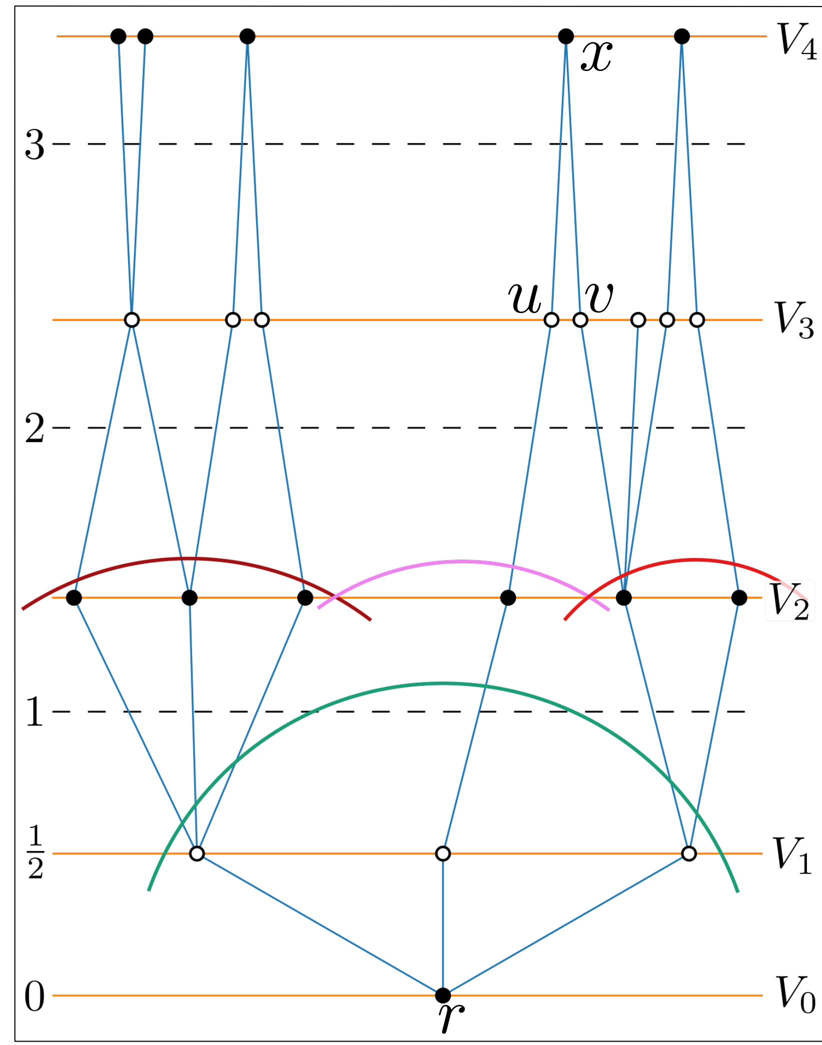

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