

Bipartite Dichotomous Ordinal Graphs

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EuroCG 2024

Dichotomous Ordinal Graphs

- ▶ Given:
 - Graph $G = (V, E)$
 - A partition of edges into short and long, $E = E_s \cup E_\ell$

Introduction

Dichotomous Ordinal Graphs

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Question:

Does G admit a geometric representation?

Introduction

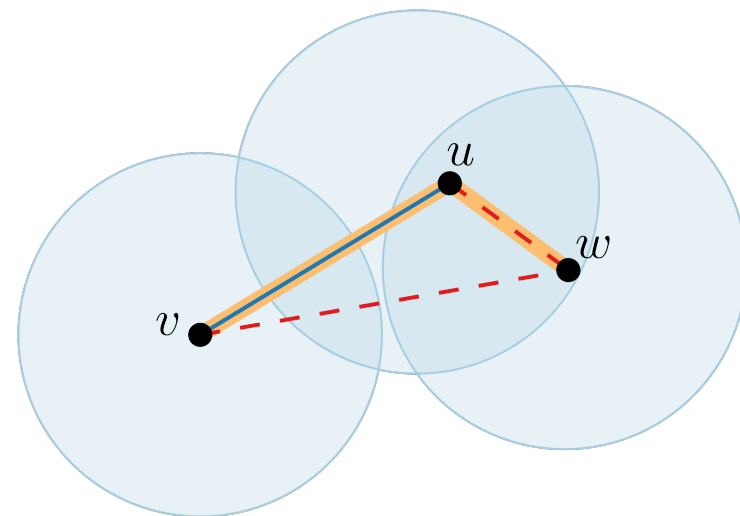
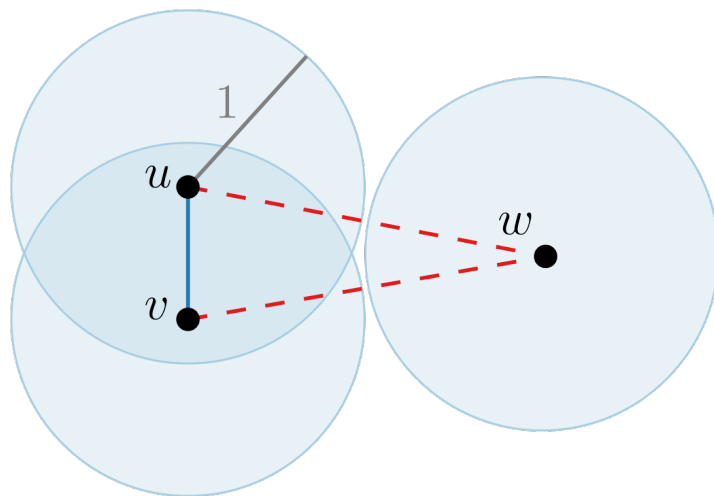
Dichotomous Ordinal Graphs

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- Graph $G = (V, E)$
- A partition of edges into short and long, $E = E_s \cup E_\ell$

Question:

Does G admit a geometric representation?



Known Results

3 - 1

- ▶ NP-hard to decide whether a dichotomous ordinal graph $G = (V, E_\ell \cup E_s)$ admits a geometric representation, even if:
 - $G \equiv K_n$ and $G_s = (V, E_s)$ is a planar graph [Alam et al. (2015)]
 - $G \equiv K_{n,m}$ ($\exists\mathbb{R}$ -complete) [Peters (2017)]

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 - $G \equiv K_n$ and $G_s = (V, E_s)$ is a planar graph [Alam et al. (2015)]
 - $G \equiv K_{n,m}$ ($\exists\mathbb{R}$ -complete) [Peters (2017)]
- ▶ Angelini et al. (2019)

Positive Instances

- ▶ double-wheel
- ▶ 2-degenerate
- ▶ subcubic
- ▶ 4-colorable and the short edges induce a caterpillar

Negative Instances

- ▶ double wheel plus one edge

Our Results

4 - 1

Our Results

4 - 2

A characterization of complete bipartite graphs

$$\checkmark G \subseteq K_{3,m} \quad \checkmark G \subseteq K_{4,6}$$

$$\times G = K_{4,7} \quad \times G = K_{5,5}$$

Our Results

4 - 3

A characterization of complete bipartite graphs

✓ $G \subseteq K_{3,m}$ ✓ $G \subseteq K_{4,6}$

✗ $G = K_{4,7}$ ✗ $G = K_{5,5}$

✓ $G_s = (V, E_s)$ is a subgraph of the rectangular grid

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✓ $G = (V, E_s \cup E_\ell)$ is bipartite and $G_s = (V, E_s)$ is outerplanar

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4 - 5

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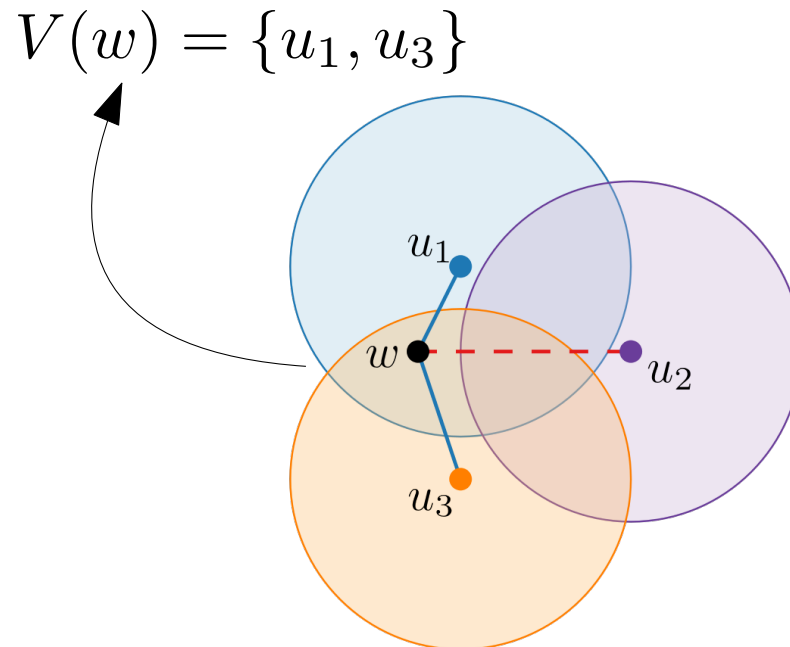
Complete Bipartite Graphs

5 - 1

Complete Bipartite Graphs

5 - 2

- ▶ $G = (U \cup W, E)$
- ▶ $U = \{u_1, u_2, \dots, u_n\}$
 - C_i the unit circle centered at u_i
- ▶ If edge (u_i, w) is short, then w should be placed in C_i .
- ▶ $V(w) \subseteq U$ is the "short neighborhood" of $w \in W$

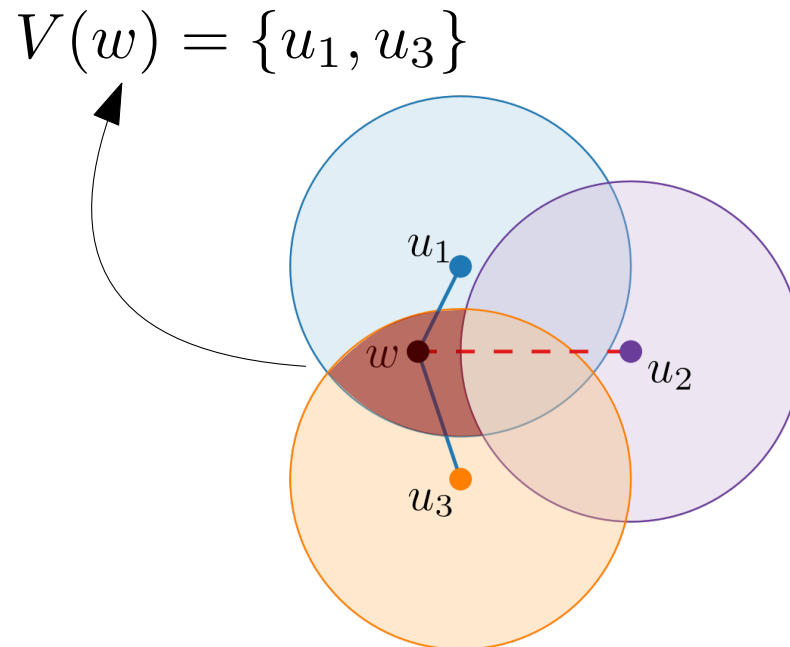


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Goal: To find an arrangement \mathcal{C} of C_1, C_2, \dots, C_n such that every $V(w)$ corresponds to a cell r



Complete Bipartite Graphs

5 - 4

Theorem

Every dichotomous ordinal $K_{3,m}$, for $m \in \mathbb{N}$, and every dichotomous ordinal $K_{4,m}$, for $m \leq 6$, admits a geometric representation.

Complete Bipartite Graphs

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Fact [Steiner (1826)] : n unit circles form at most $n(n - 1) + 2$ cells

Complete Bipartite Graphs

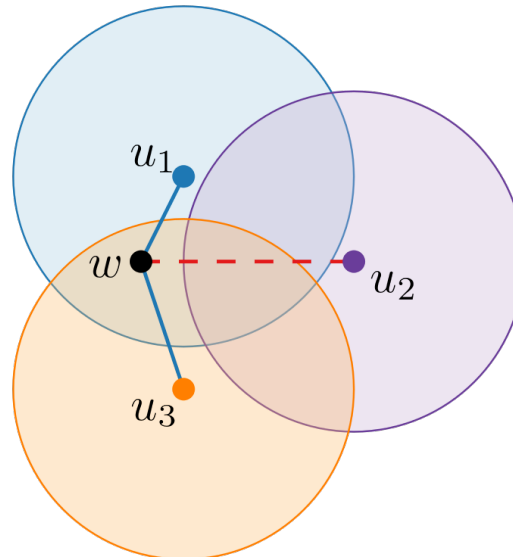
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- ▶ $K_{3,m}$:
 - all eight subsets of $U = \{u_1, u_2, u_3\}$ can be realized



Complete Bipartite Graphs

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Complete Bipartite Graphs

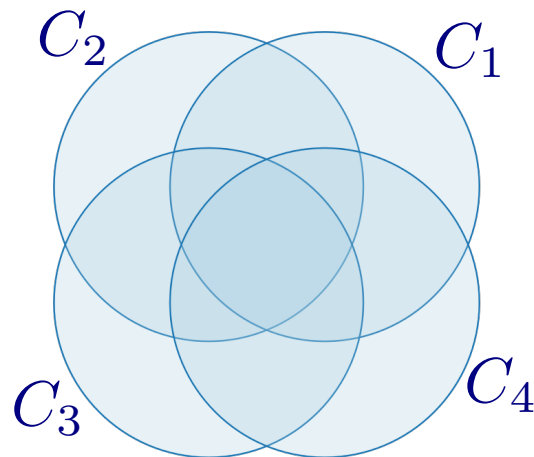
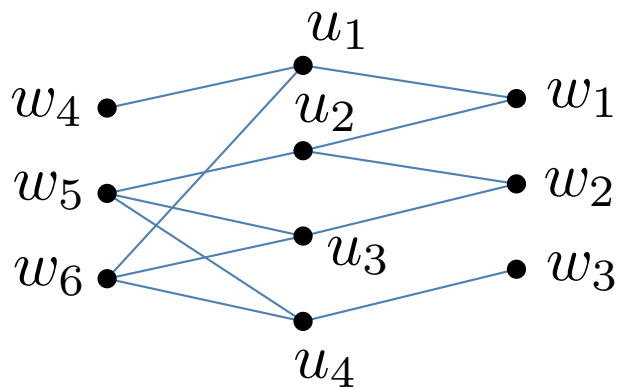
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- ▶ $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$
 - Case 1: At most two vertices from W are shortly connected to a pair of U .
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Complete Bipartite Graphs

5 - 9

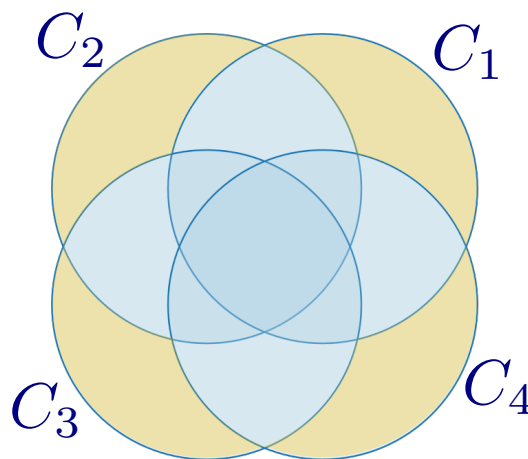
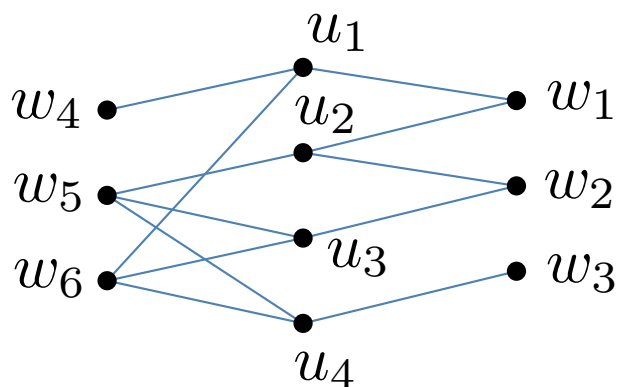
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Complete Bipartite Graphs

5 - 10

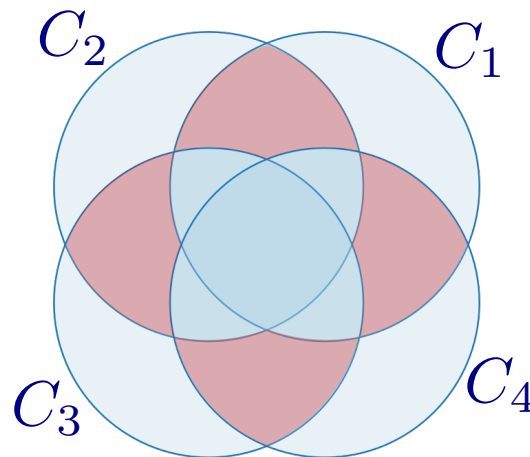
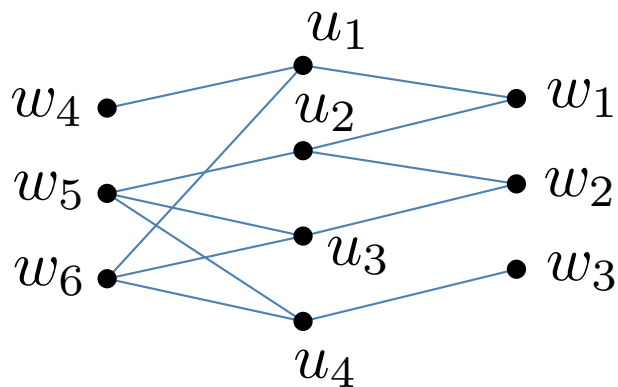
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Missing Cells:

- $C_1 \cap C_3$
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Complete Bipartite Graphs

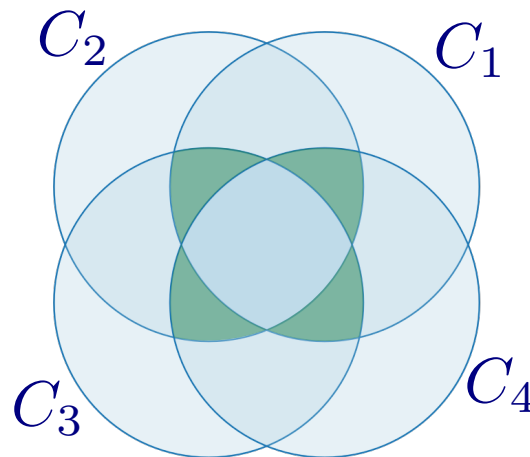
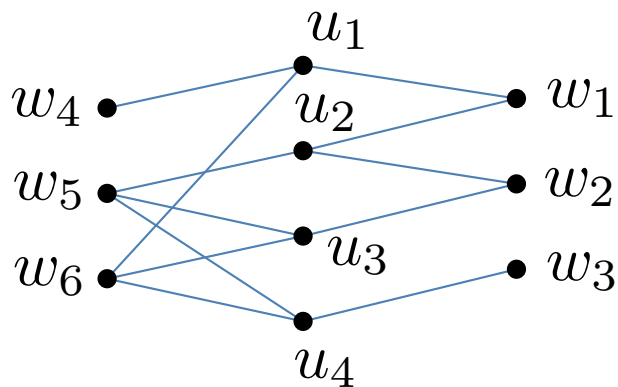
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Complete Bipartite Graphs

5 - 12

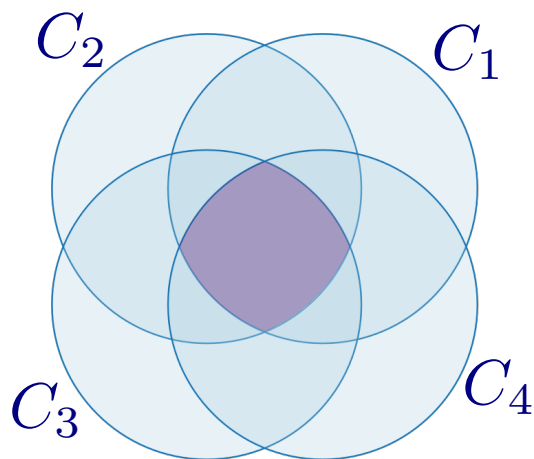
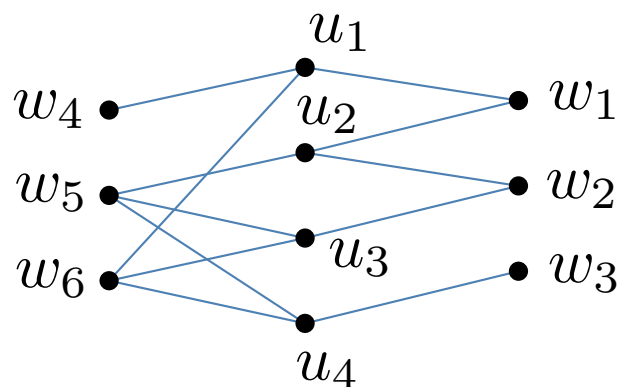
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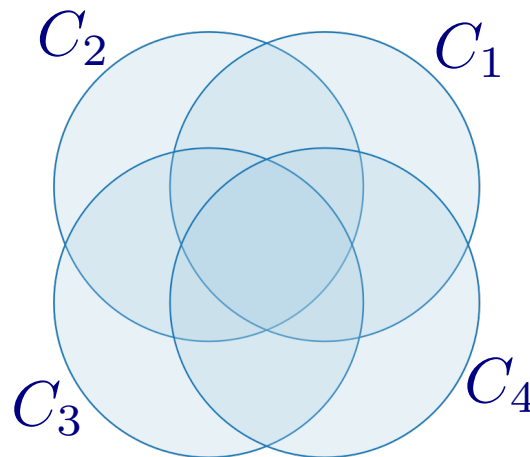
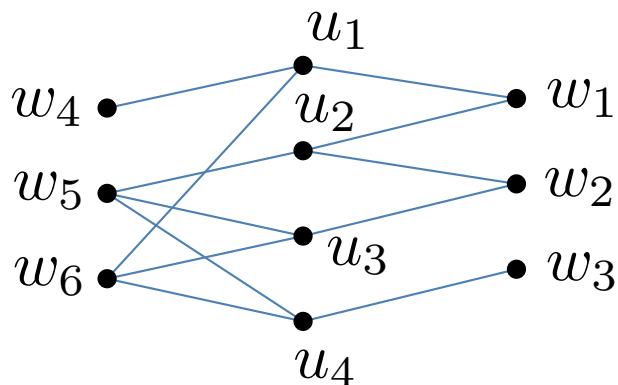
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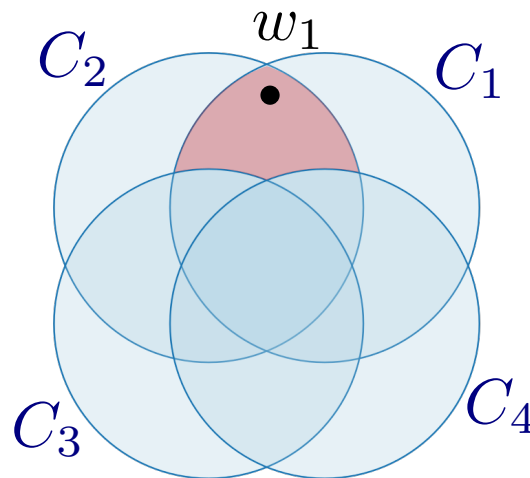
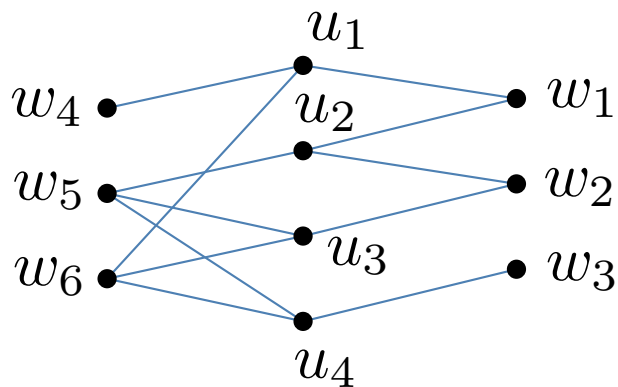
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Complete Bipartite Graphs

5 - 15

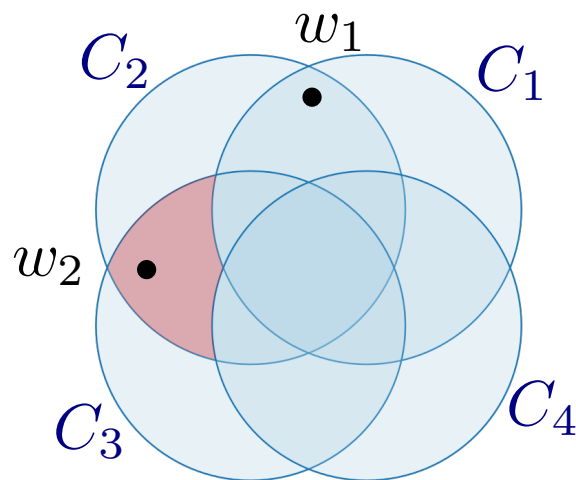
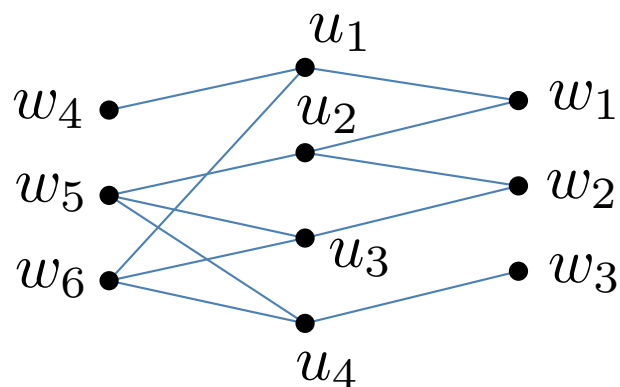
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Complete Bipartite Graphs

5 - 16

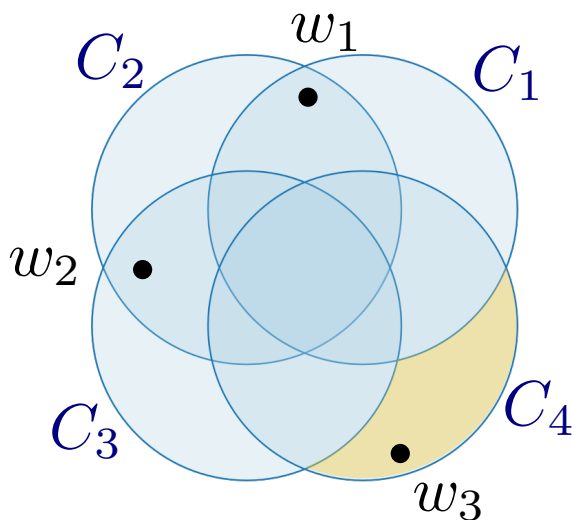
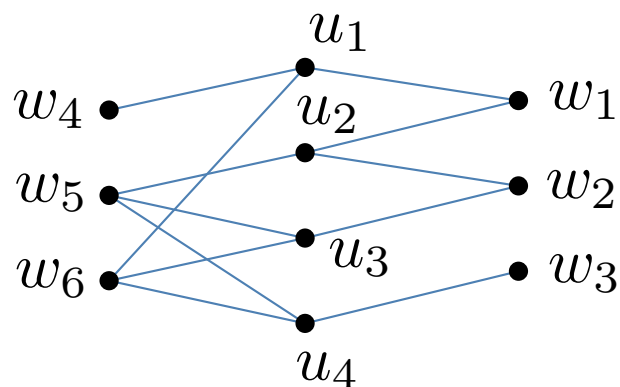
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Complete Bipartite Graphs

5 - 17

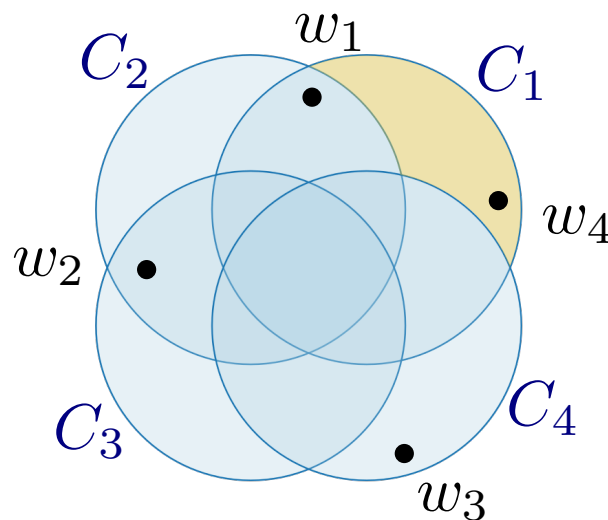
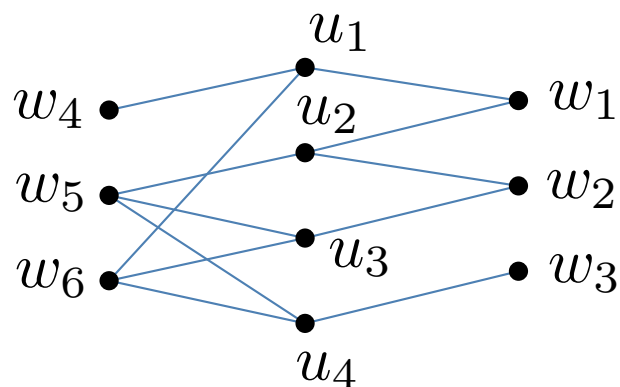
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Complete Bipartite Graphs

5 - 18

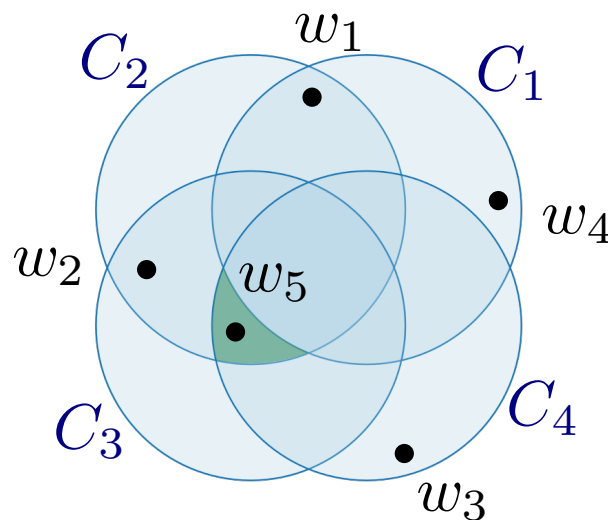
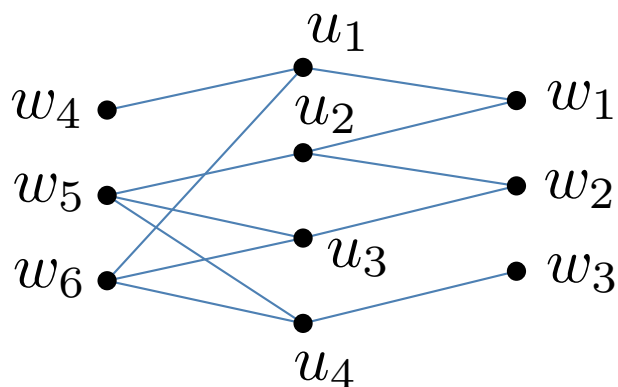
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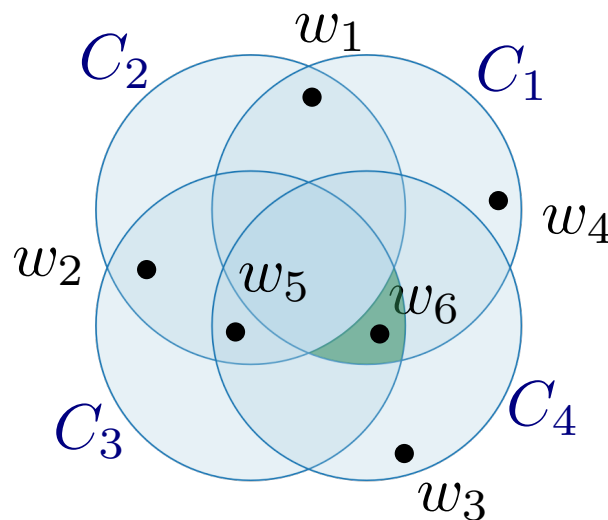
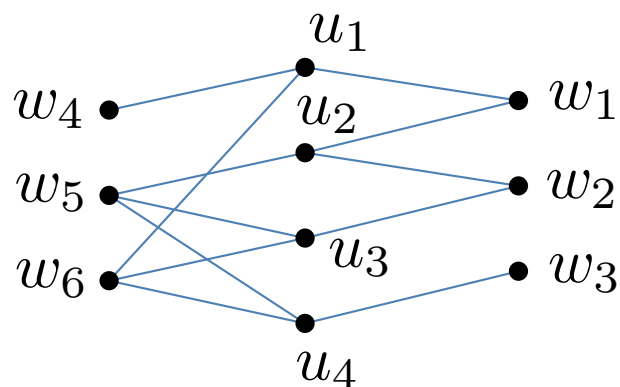
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Complete Bipartite Graphs

5 - 20

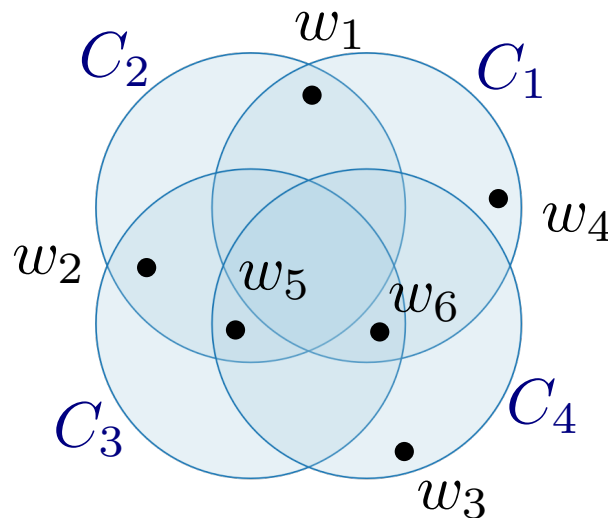
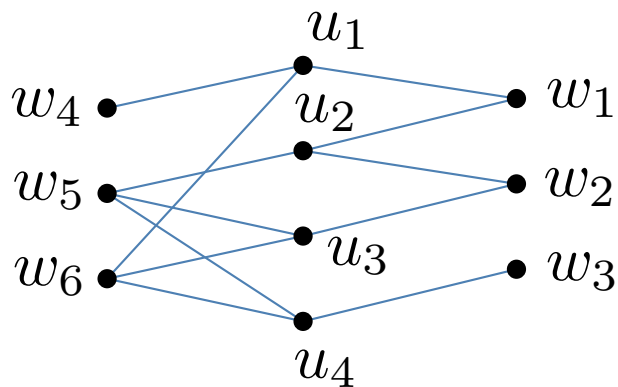
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- ▶ $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$
 - Case 2: At least three vertices from W are shortly connected to a pair of U .

Complete Bipartite Graphs

6 - 1

Theorem

There is a dichotomous ordinal $K_{4,7}$ that does not admit a geometric representation.

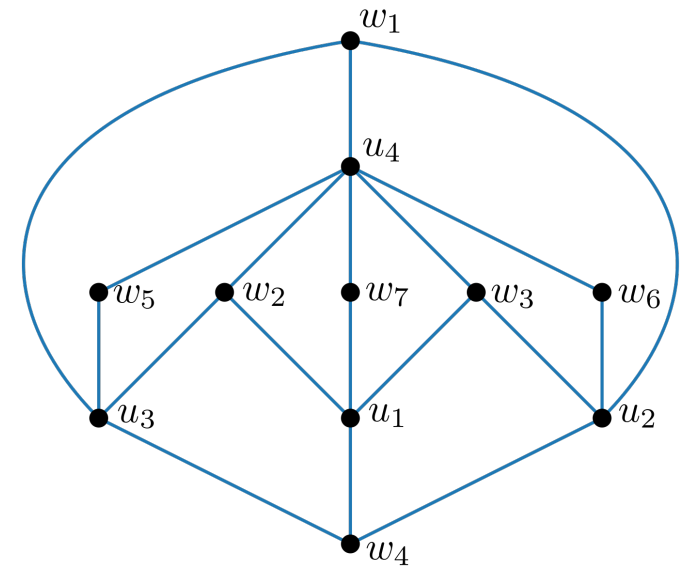
Complete Bipartite Graphs

6 - 2

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- ▶ $U = \{u_1, u_2, u_3, u_4\}, W = \{w_1, \dots, w_7\}$
- ▶ Counterexample:
 - All four triplets
 - The three pairs that contain u_4



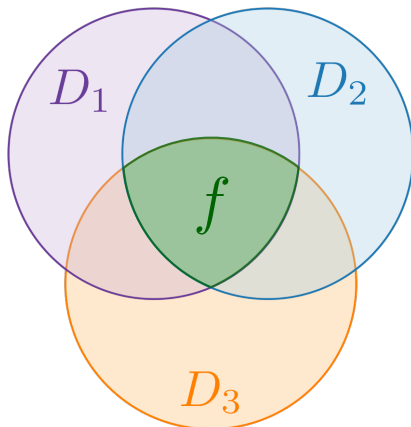
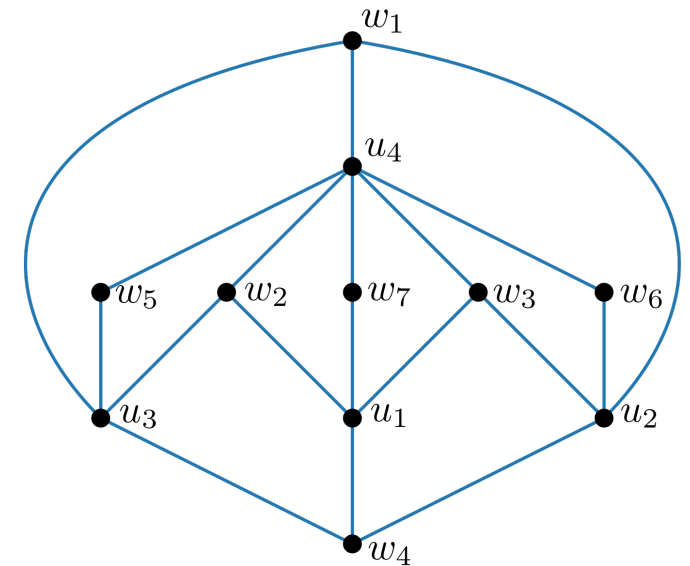
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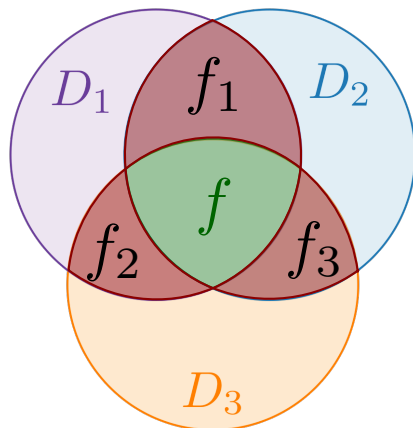
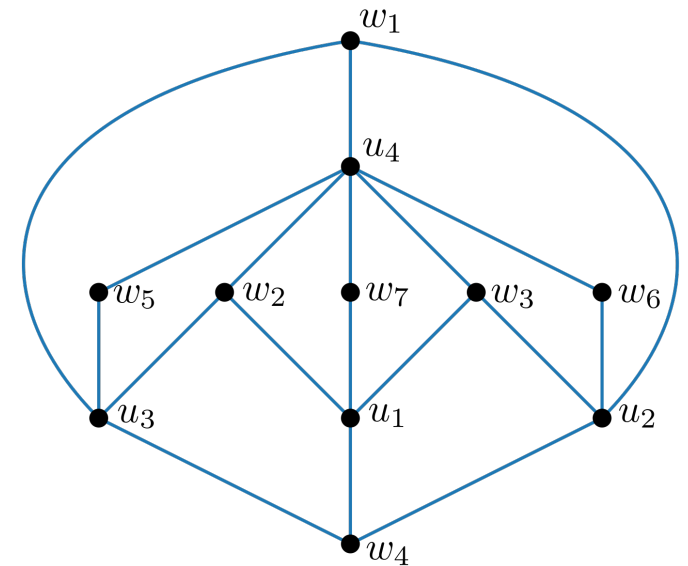
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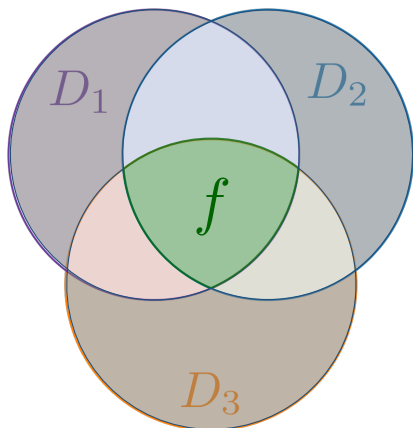
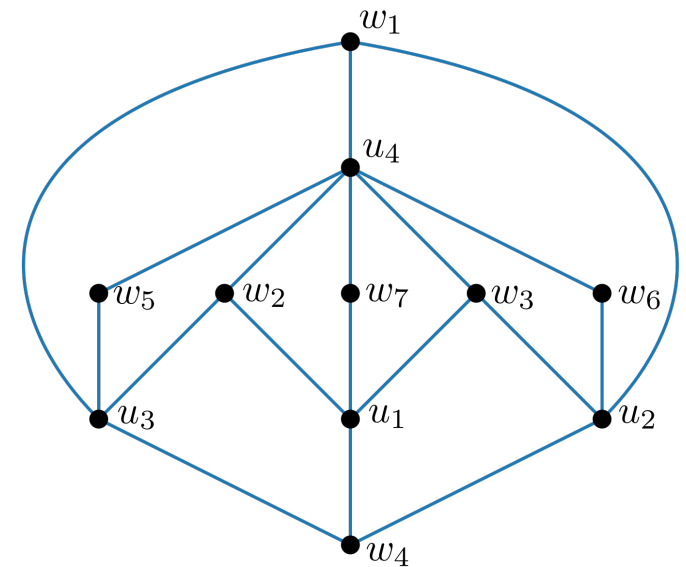
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- ▶ We require to cross all singletons, but not fully cover f .

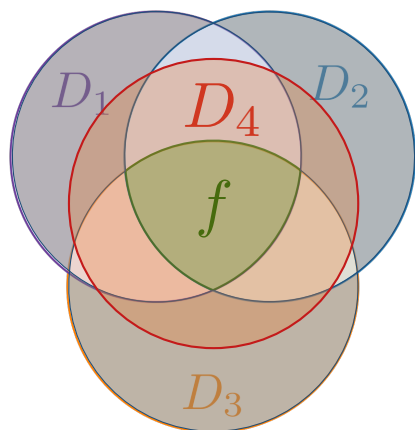
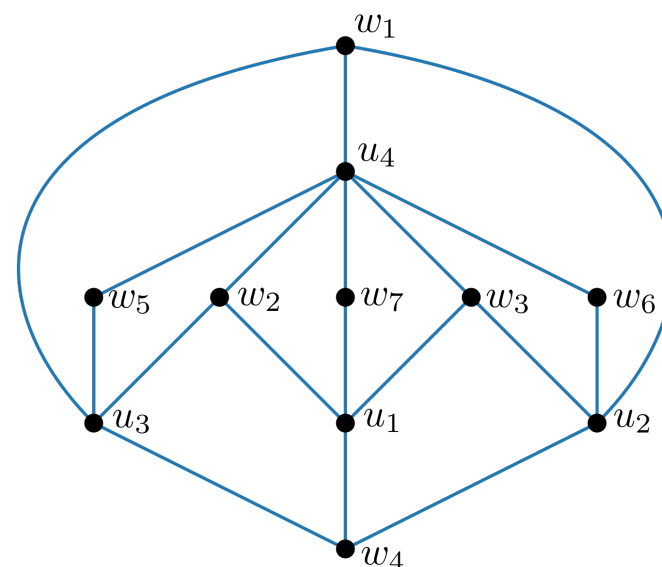
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Theorem

There is a dichotomous ordinal $K_{4,7}$ that does not admit a geometric representation.

- ▶ $U = \{u_1, u_2, u_3, u_4\}$, $W = \{w_1, \dots, w_7\}$
- ▶ Counterexample:
 - All four triplets
 - The three pairs that contain u_4



- ▶ We require to cross all singletons, but not fully cover f .
⇒ A contradiction.

Short Subgraphs of the Grid

7 - 1

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

Short Subgraphs of the Grid

7 - 2

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges

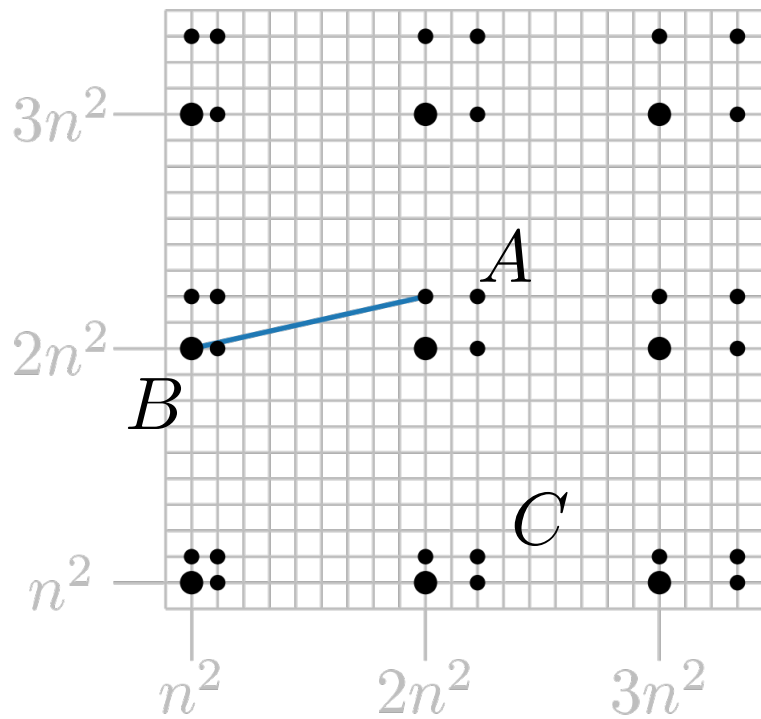
Short Subgraphs of the Grid

7 - 3

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges



- ▶ Four possible choices for each grid point (i, j)
 - x-coordinates in^2 and $in^2 + i$
 - y-coordinates jn^2 and $jn^2 + j$

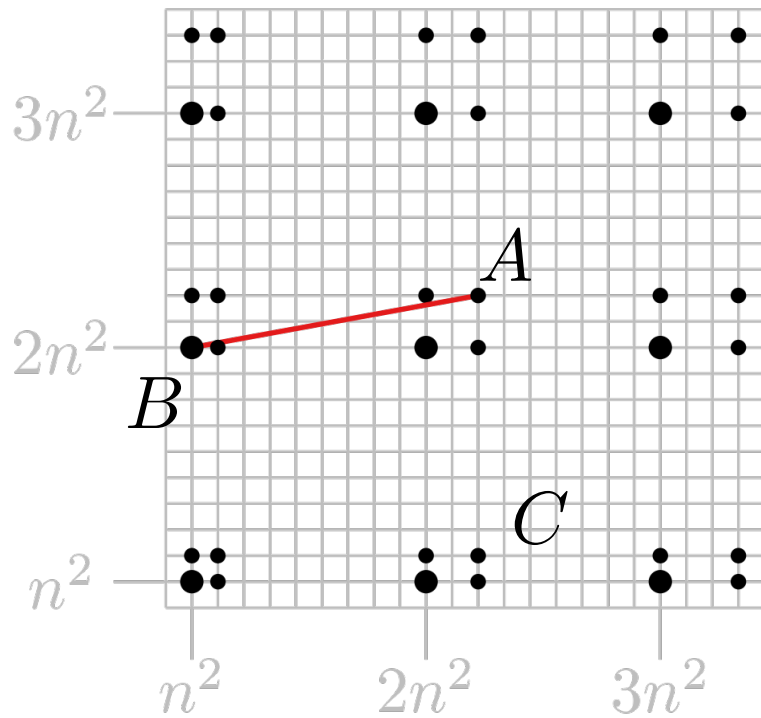
Short Subgraphs of the Grid

7 - 4

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges



- ▶ Four possible choices for each grid point (i, j)
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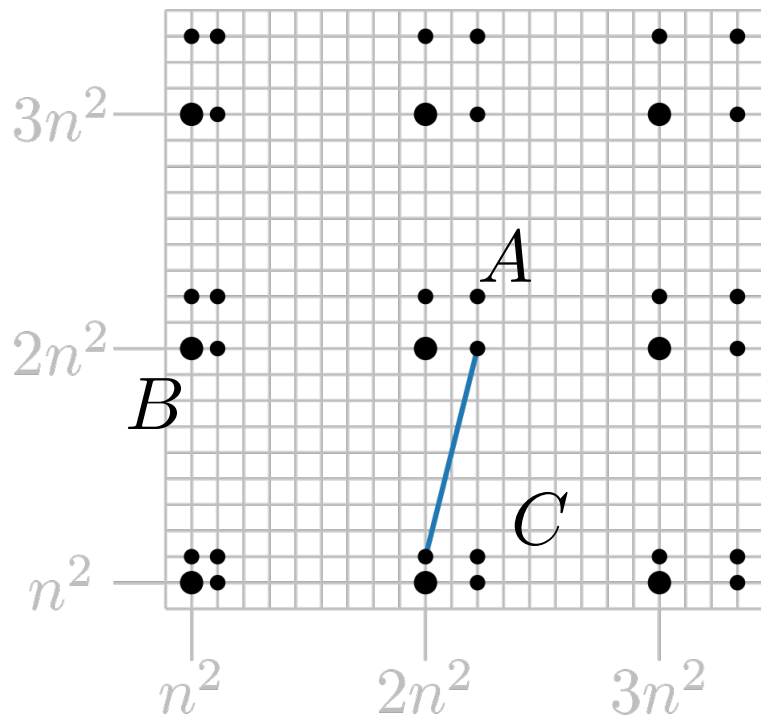
Short Subgraphs of the Grid

7 - 5

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges



- ▶ Four possible choices for each grid point (i, j)
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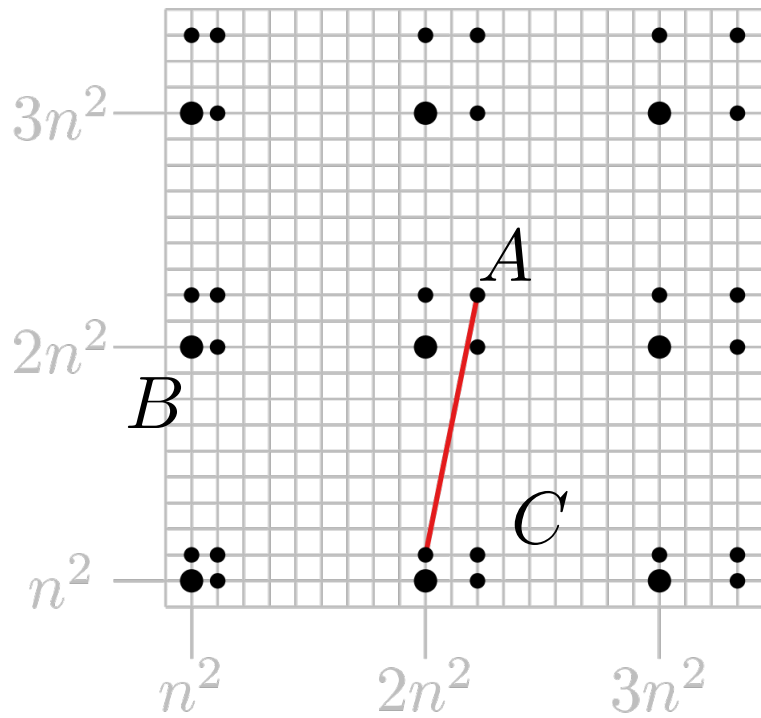
Short Subgraphs of the Grid

7 - 6

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges



- ▶ Four possible choices for each grid point (i, j)
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Short Subgraphs of the Grid

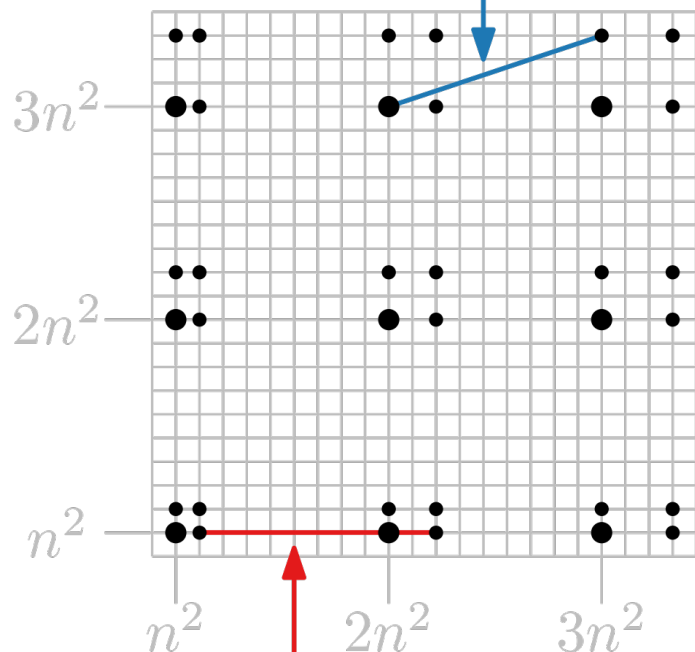
7 - 7

Theorem

A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric representation if the set of short edges induces a subgraph of the grid.

- ▶ Extend G_s by remaining *long* edges

longest possible short edge



shortest possible long edge

- ▶ Four possible choices for each grid point (i, j)
 - x-coordinates in^2 and $in^2 + i$
 - y-coordinates jn^2 and $jn^2 + j$
- ▶ **Long** edges have length $\geq n^2 + 1$
- ▶ **Short** edges have length $\leq n^2 + \frac{1}{2}$

Conclusion

Open Problems

Do bipartite dichotomous ordinal graphs always admit a geometric realization when:

- (i) the underlying graph is planar?
- (ii) the underlying graph is 3-degenerate?
- (iii) the graph induced by the short edges is a 2-tree?

Questions (i) and (ii) are open even for non-bipartite dichotomous ordinal graphs.

Complete Bipartite Graphs

9 - 1

Theorem

Every dichotomous ordinal $K_{3,m}$, for $m \in \mathbb{N}$, and every dichotomous ordinal $K_{4,m}$, for $m \leq 6$, admits a geometric representation.

Fact [Steiner (1826)] : n unit circles form at most $n(n - 1) + 2$ cells

► $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$

Complete Bipartite Graphs

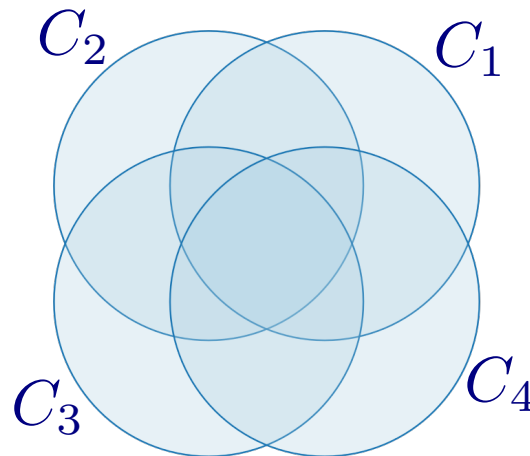
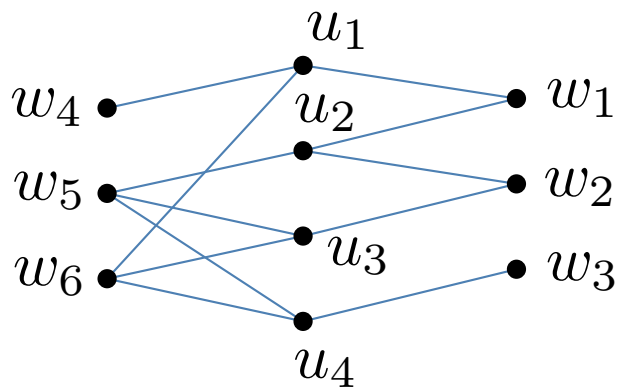
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 - Case 1: At most two vertices from W are shortly connected to a pair of U .
 - All subsets, except for the two pairs that correspond to opposite circles, are realized



Complete Bipartite Graphs

9 - 3

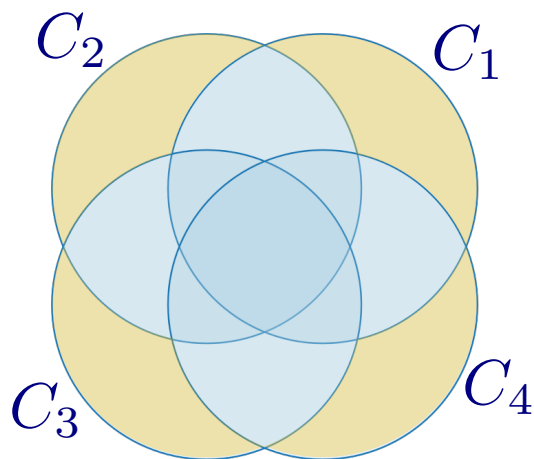
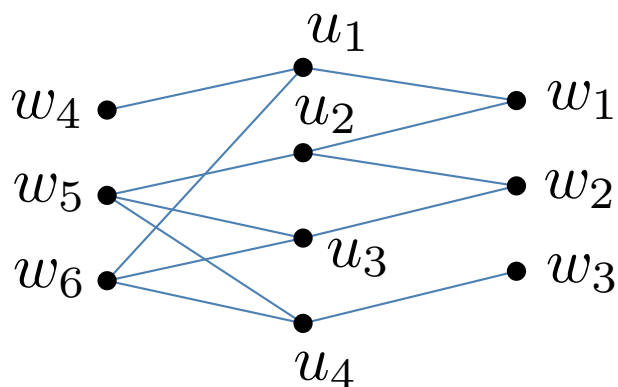
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Complete Bipartite Graphs

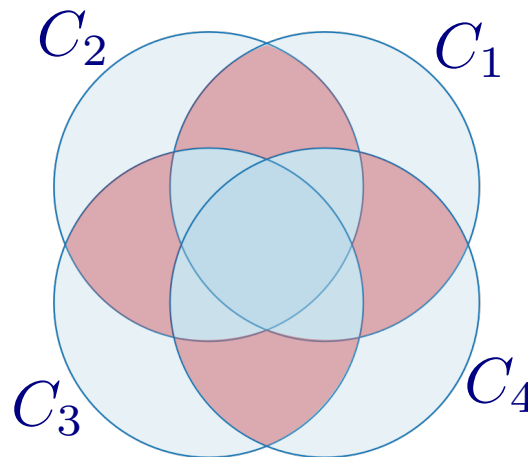
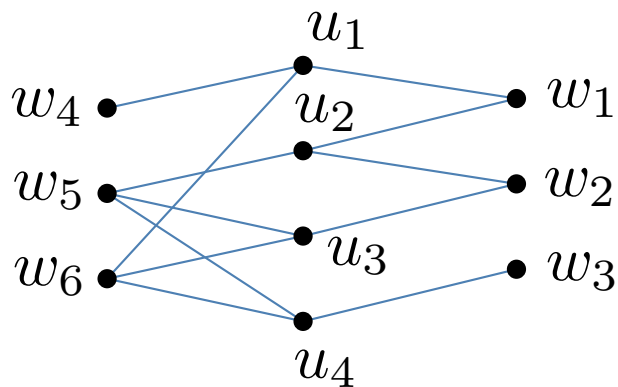
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Missing Cells:

- $C_1 \cap C_3$
- $C_2 \cap C_4$

Complete Bipartite Graphs

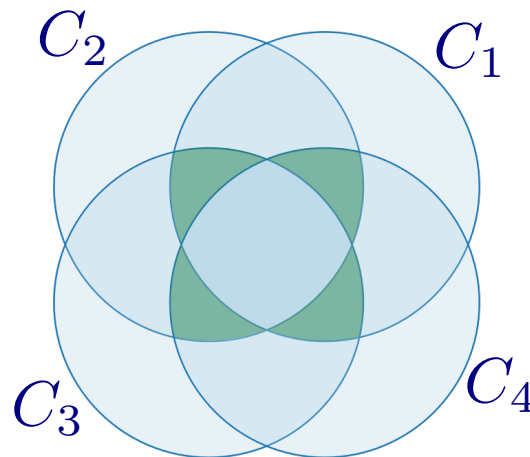
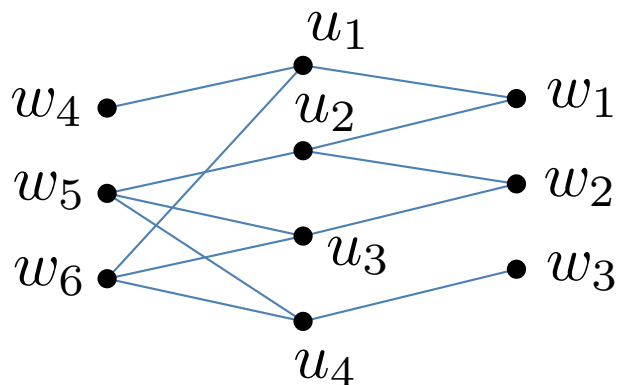
9 - 5

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Complete Bipartite Graphs

9 - 6

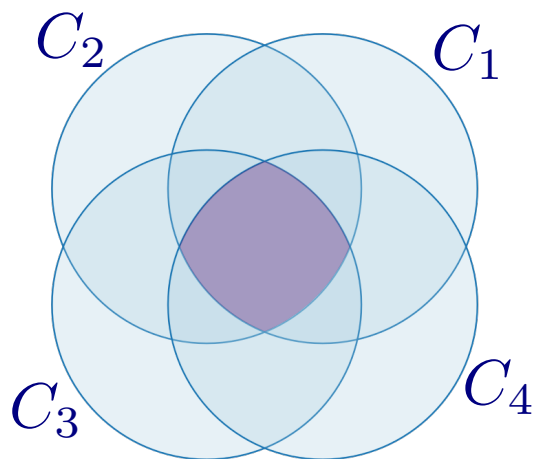
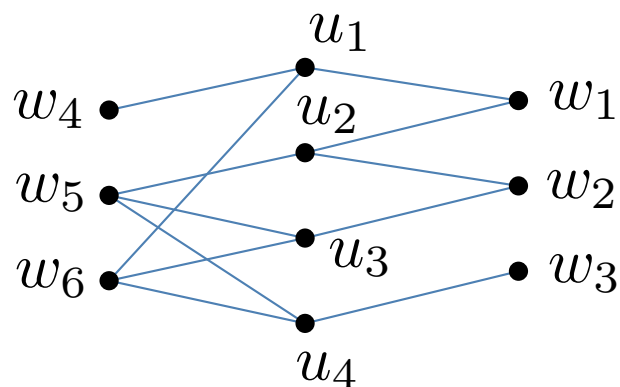
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Complete Bipartite Graphs

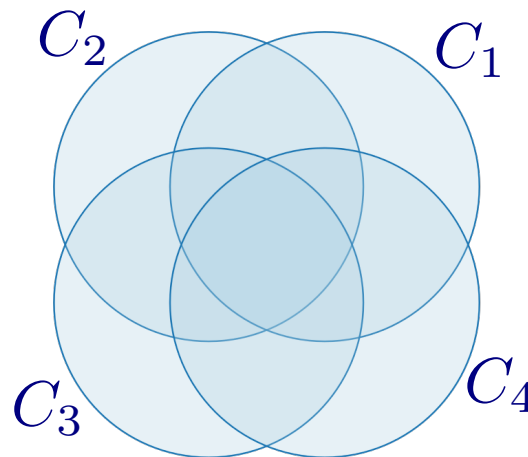
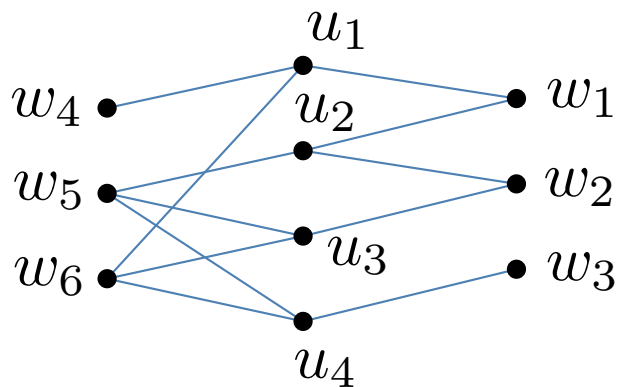
9 - 7

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Complete Bipartite Graphs

9 - 8

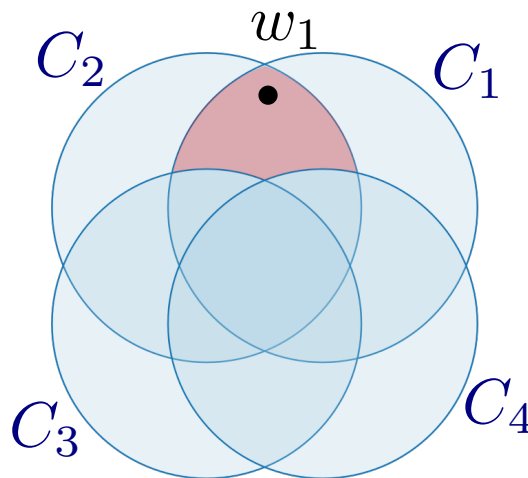
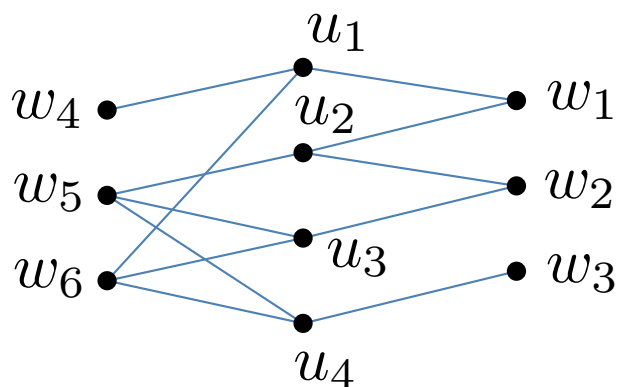
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Complete Bipartite Graphs

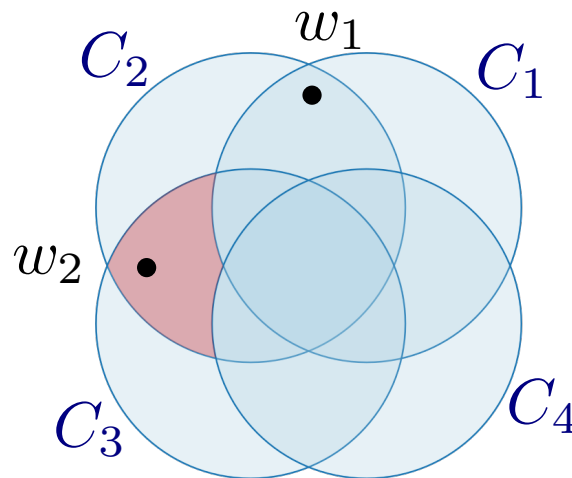
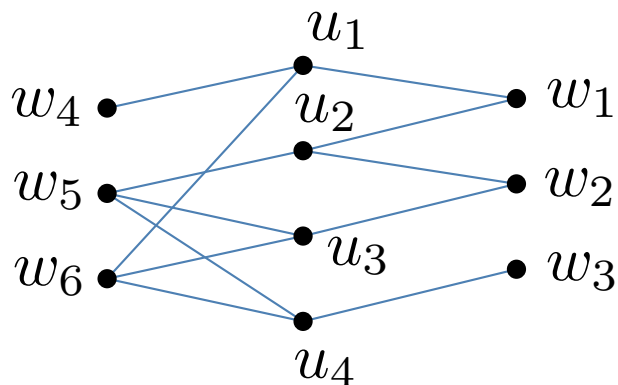
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Complete Bipartite Graphs

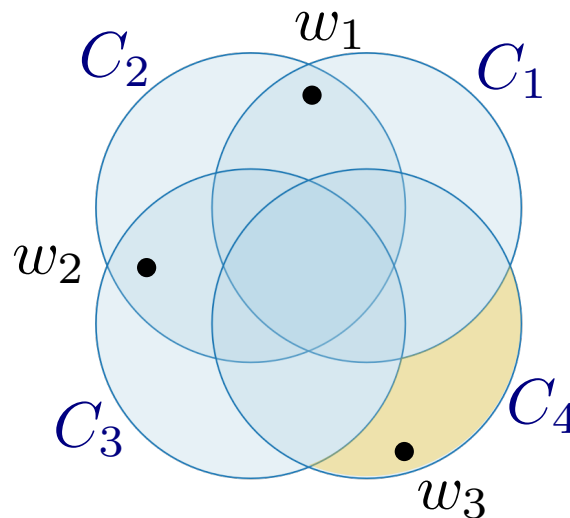
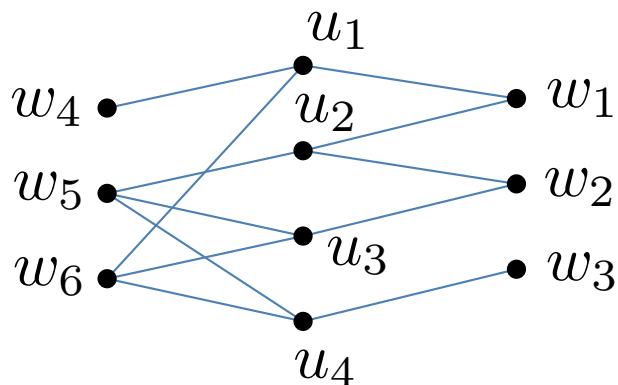
9 - 10

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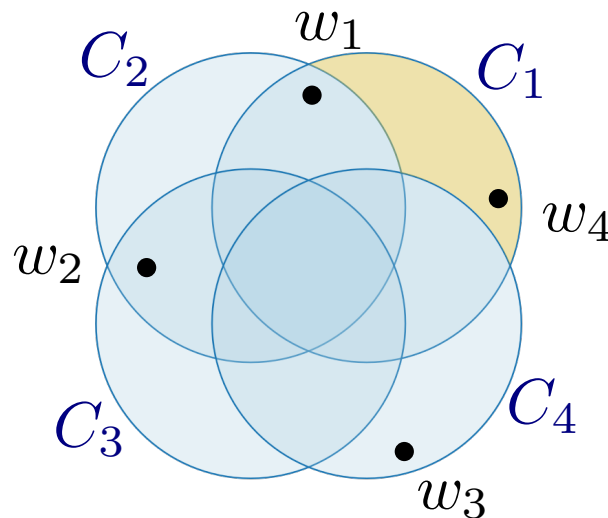
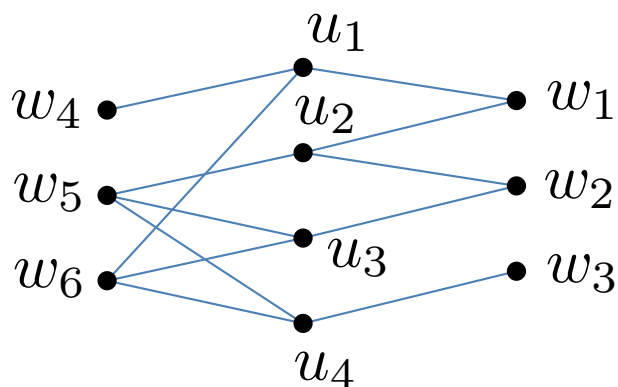
Complete Bipartite Graphs

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- $C_2 \cap C_4$

Complete Bipartite Graphs

9 - 12

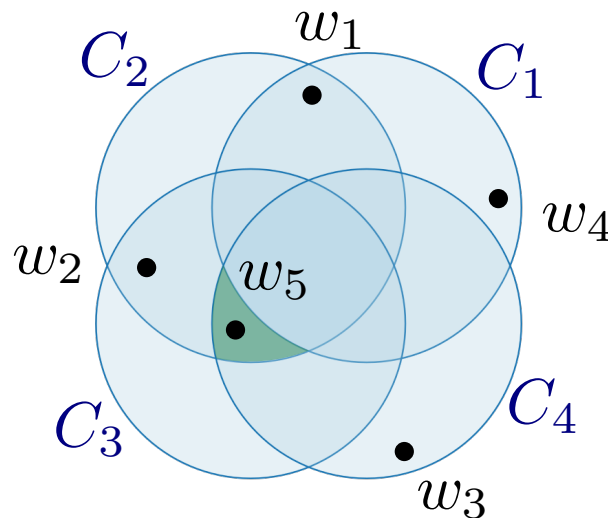
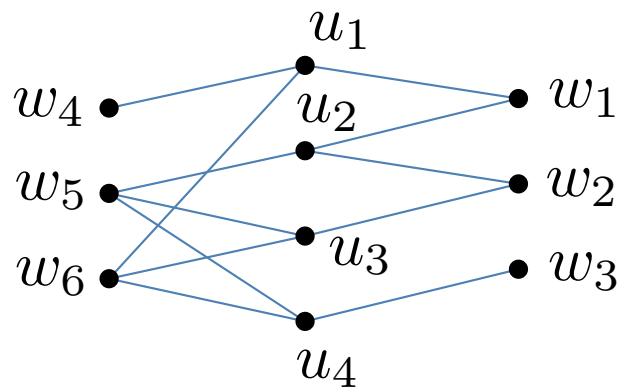
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Complete Bipartite Graphs

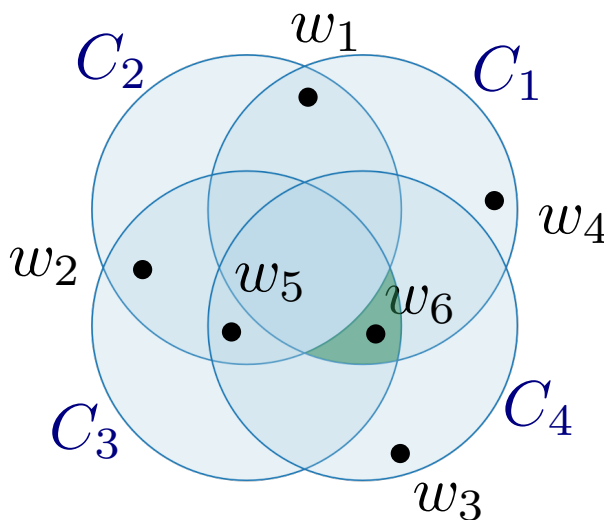
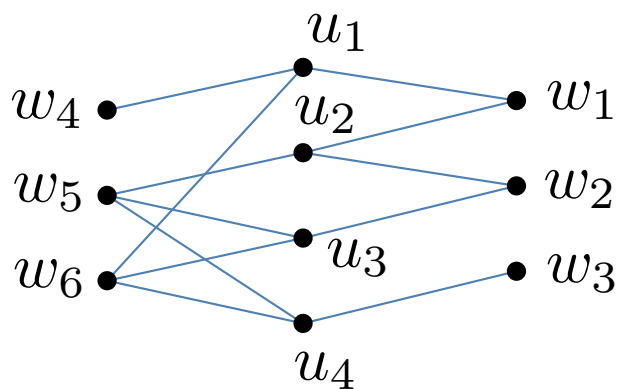
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Missing Cells:

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Complete Bipartite Graphs

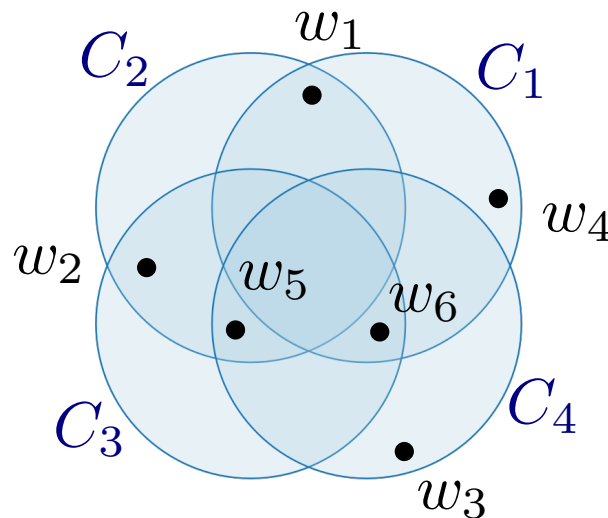
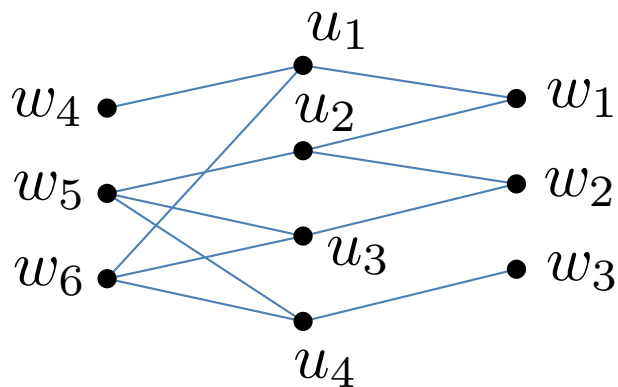
9 - 14

Theorem

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Missing Cells:

- $C_1 \cap C_3$
- $C_2 \cap C_4$

Complete Bipartite Graphs

9 - 15

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- ▶ $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$
 - Case 2: At least three vertices from W are shortly connected to a pair of U .

Complete Bipartite Graphs

9 - 16

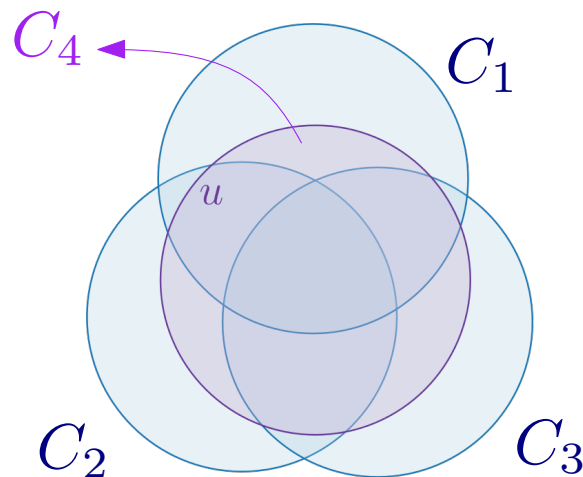
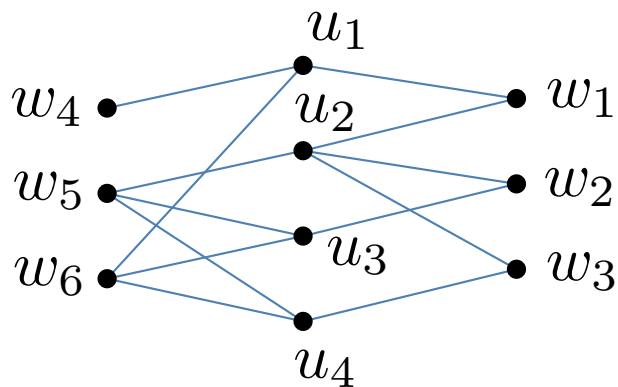
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Complete Bipartite Graphs

9 - 17

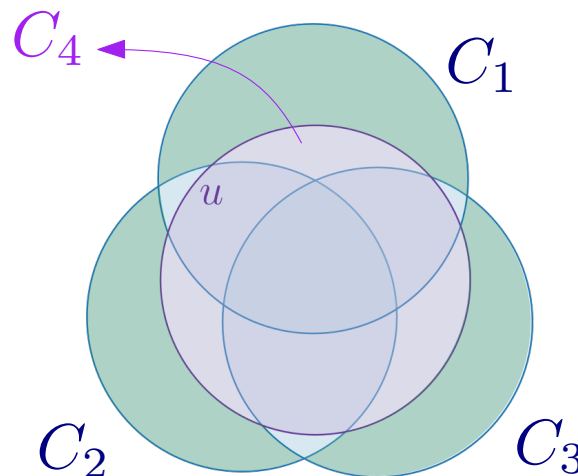
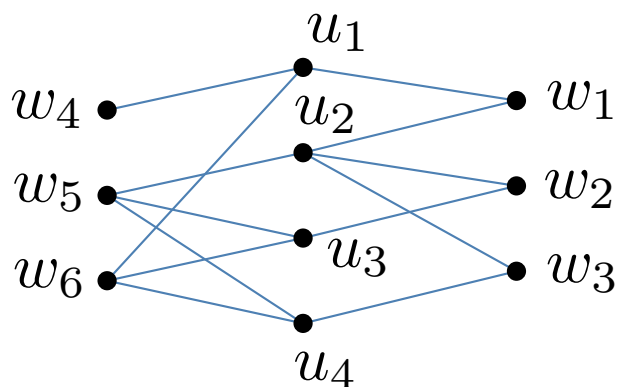
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Missing Cells:

• C_4

Complete Bipartite Graphs

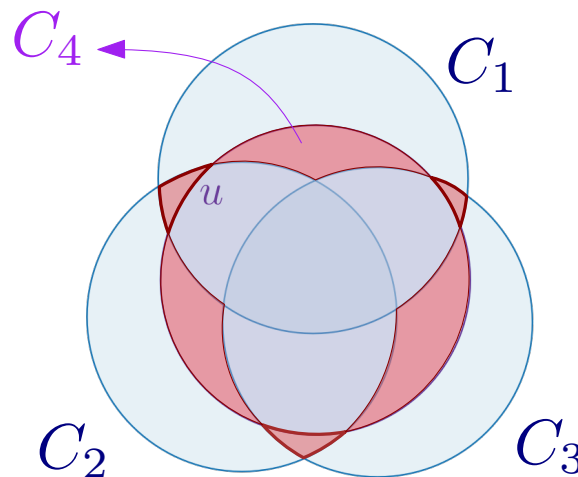
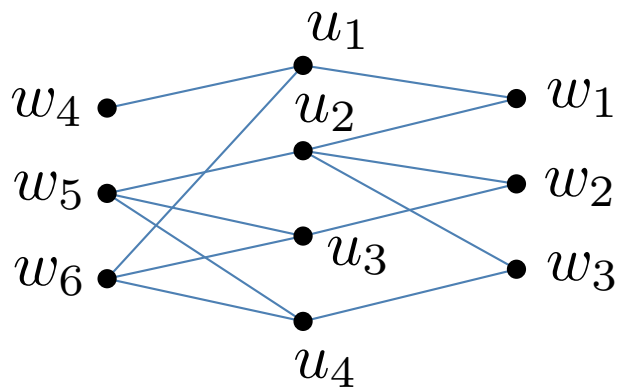
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Missing Cells:

- C_4

Complete Bipartite Graphs

9 - 19

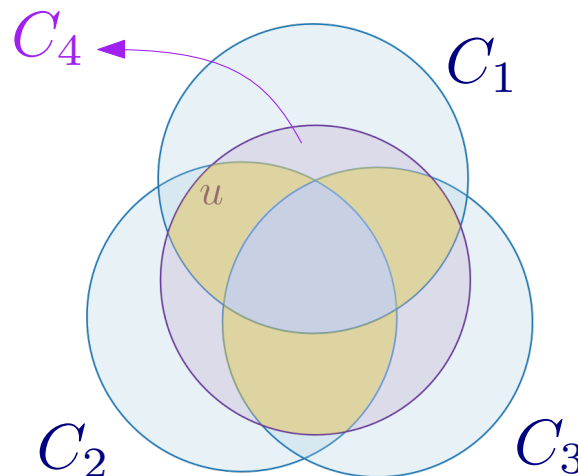
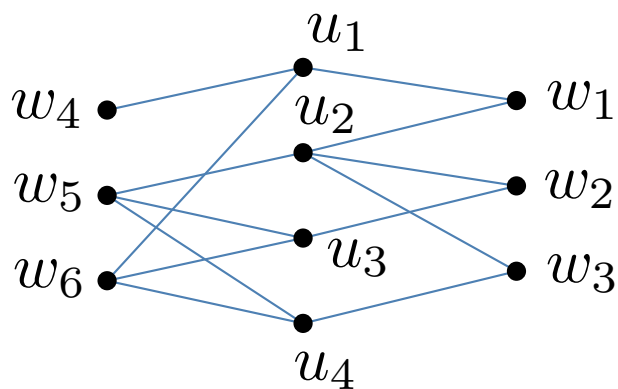
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Missing Cells:

- C_4
- $C_1 \cap C_2 \cap C_3$

Complete Bipartite Graphs

9 - 20

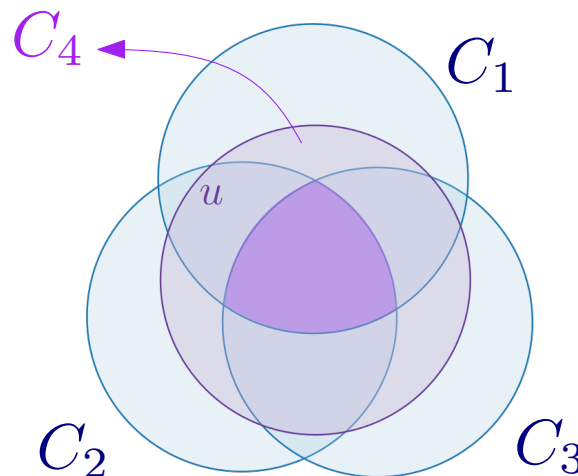
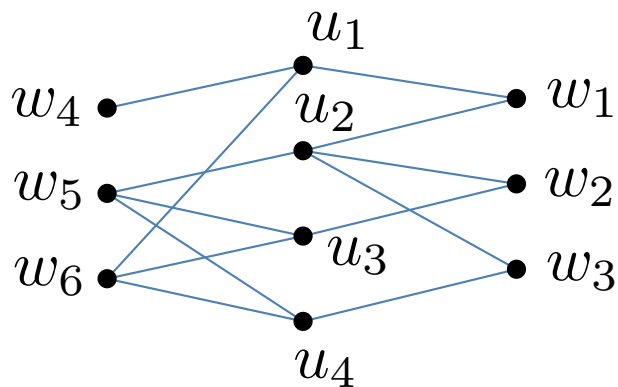
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Missing Cells:

- C_4
- $C_1 \cap C_2 \cap C_3$

Complete Bipartite Graphs

9 - 21

Theorem

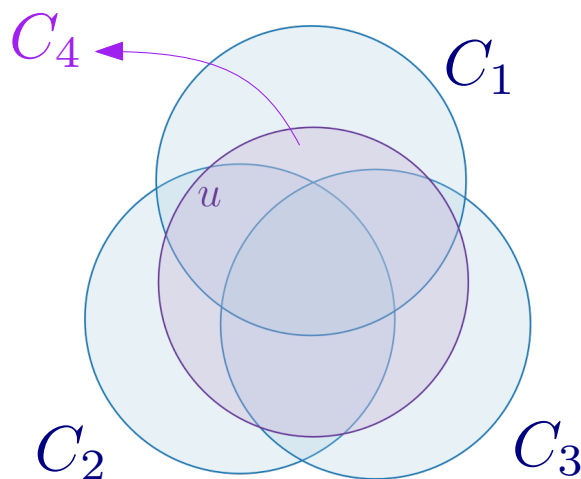
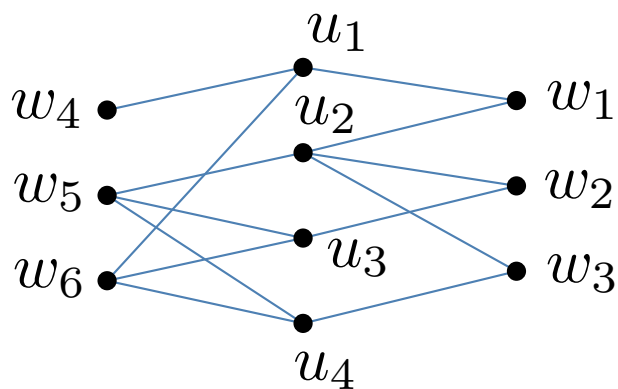
Every dichotomous ordinal $K_{3,m}$, for $m \in \mathbb{N}$, and every dichotomous ordinal $K_{4,m}$, for $m \leq 6$, admits a geometric representation.

Fact [Steiner (1826)] : n unit circles form at most $n(n - 1) + 2$ cells

► $K_{4,6}$: $U = \{u_1, u_2, u_3, u_4\}$

– Case 2: At least three vertices from W are shortly connected to a pair of U .

⇒ We have at most three singletons and triplets



Missing Cells:

- C_4
- $C_1 \cap C_2 \cap C_3$

Complete Bipartite Graphs

9 - 22

Theorem

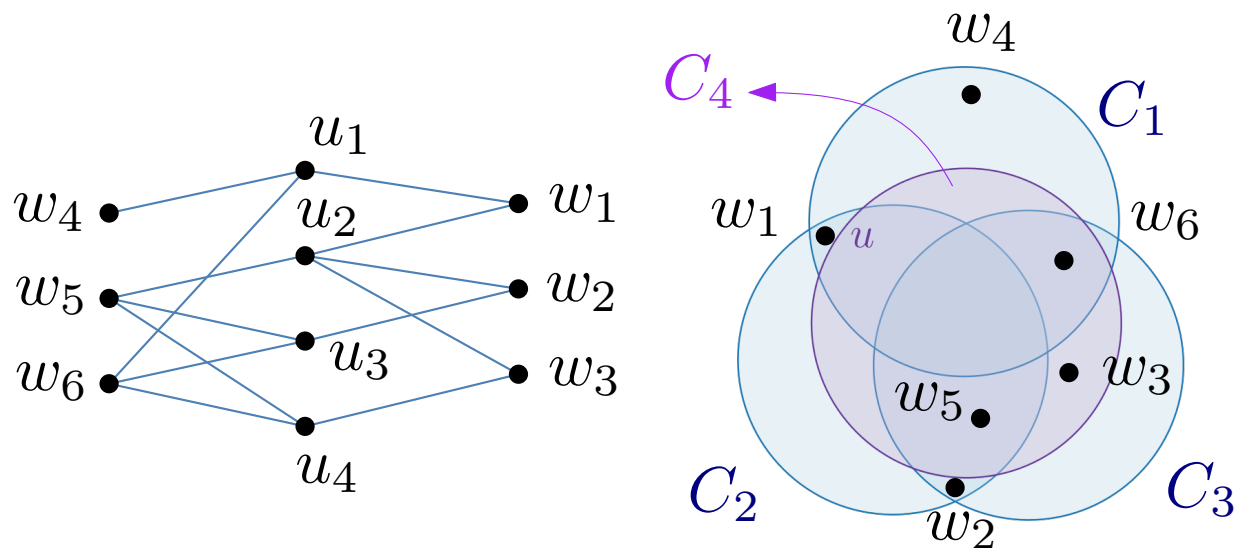
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Complete Bipartite Graphs

10 - 1

Theorem

There is a dichotomous ordinal $K_{5,5}$ that does not admit a geometric representation.

Complete Bipartite Graphs

10 - 2

Theorem

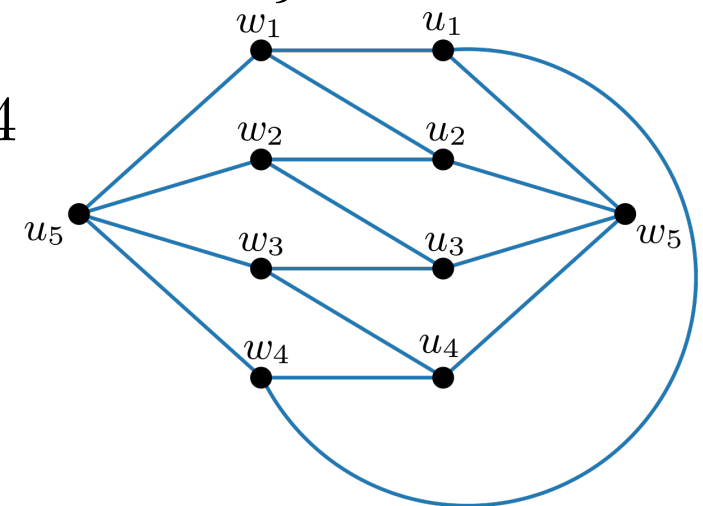
There is a dichotomous ordinal $K_{5,5}$ that does not admit a geometric representation.

▶ $U = \{u_1, u_2, u_3, u_4, u_5\}$, $W = \{w_1, w_2, w_3, w_4, w_5\}$

▶ Counterexample:

– $V(w_i) = \{u_i, u_{i \oplus 1}, u_5\}$, for $1 \leq i \leq 4$

– $V(w_5) = U \setminus \{u_5\}$



▶ Each $V(w_i)$ corresponds to a cell in arrangement \mathcal{C}

▶ Analyze \mathcal{C} geometrically to show that it cannot be realized

Short Outerplanar Graphs

11 - 1

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

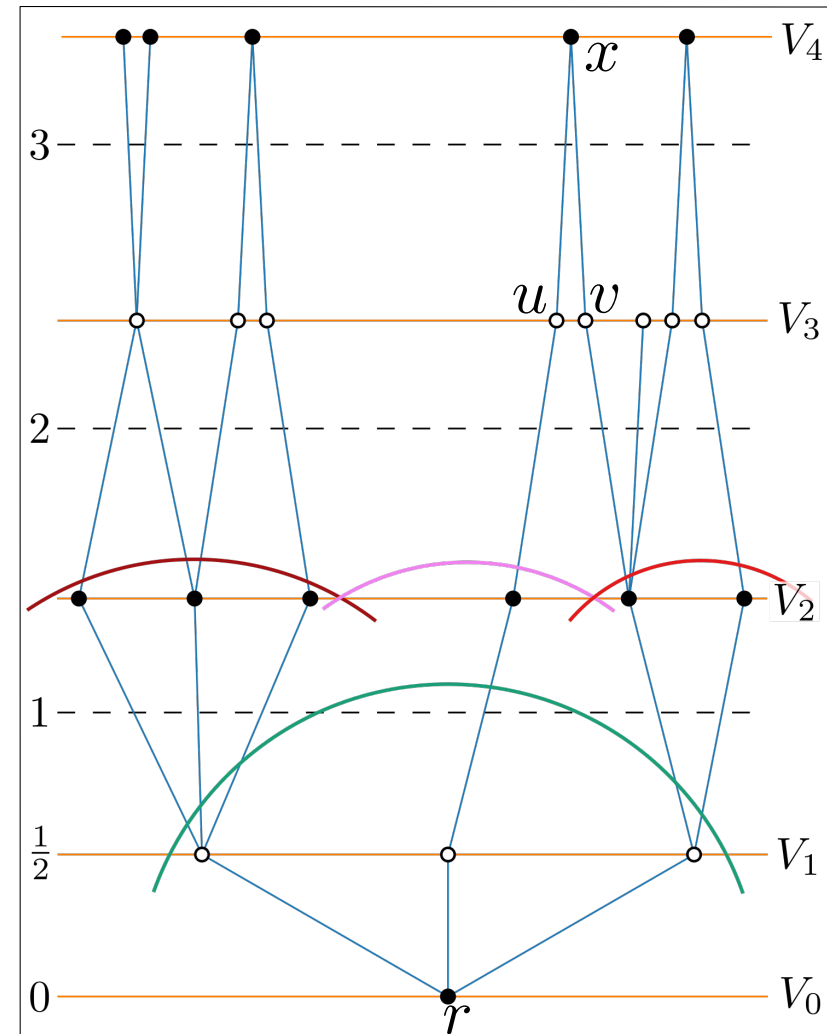
Short Outerplanar Graphs

11 - 2

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

Layered Drawing



Short Outerplanar Graphs

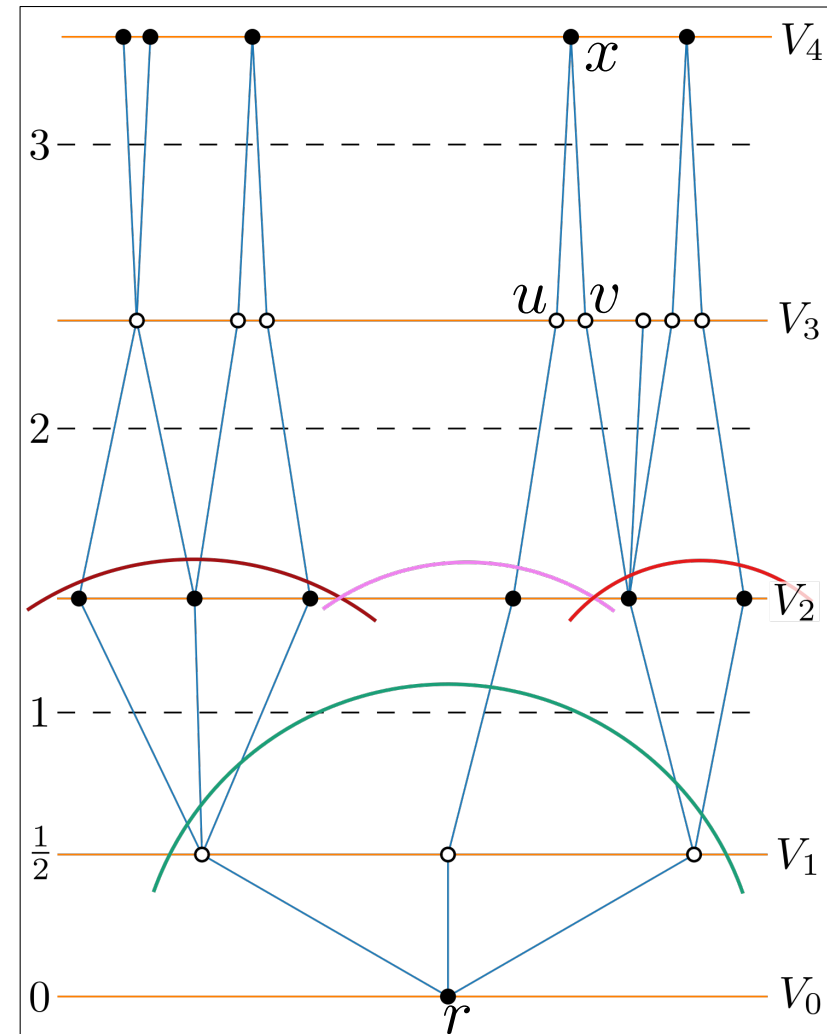
11 - 3

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

Layered Drawing

- ▶ root G_s at arbitrary vertex r



Short Outerplanar Graphs

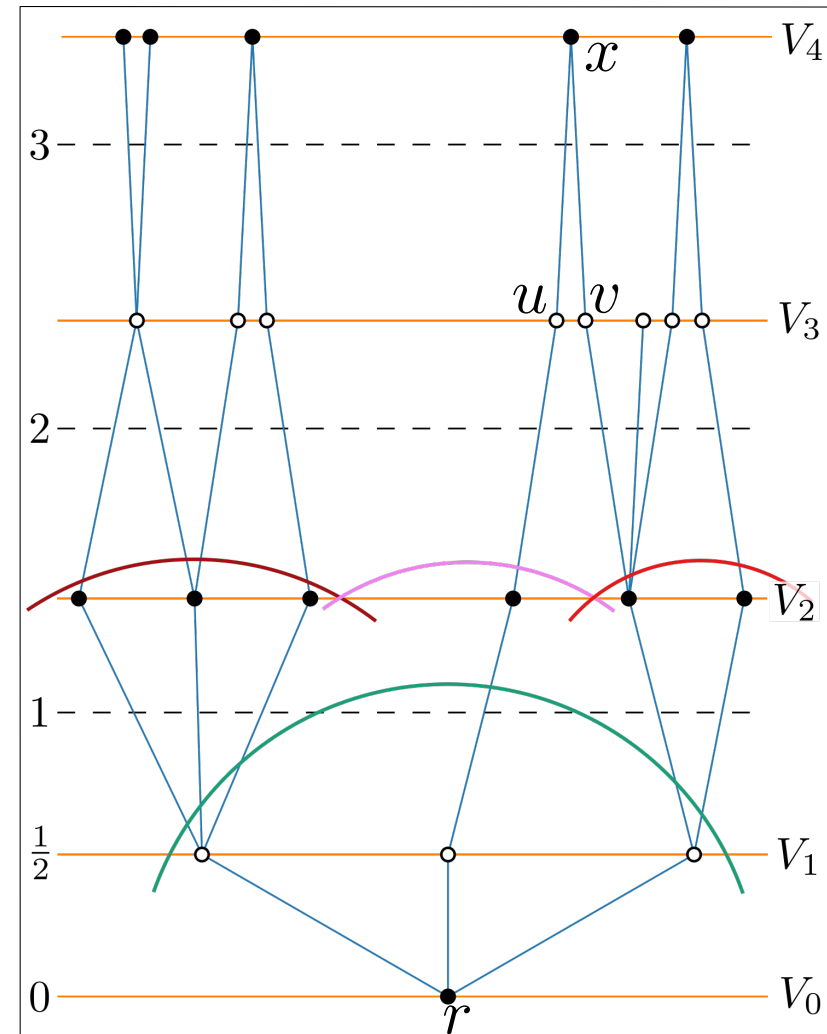
11 - 5

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

Layered Drawing

- ▶ root G_s at arbitrary vertex r
- ▶ $V_k, k = 0, \dots$ is the BFS layer of G_s
- ▶ each V_k is placed on horizontal line ℓ_k



Short Outerplanar Graphs

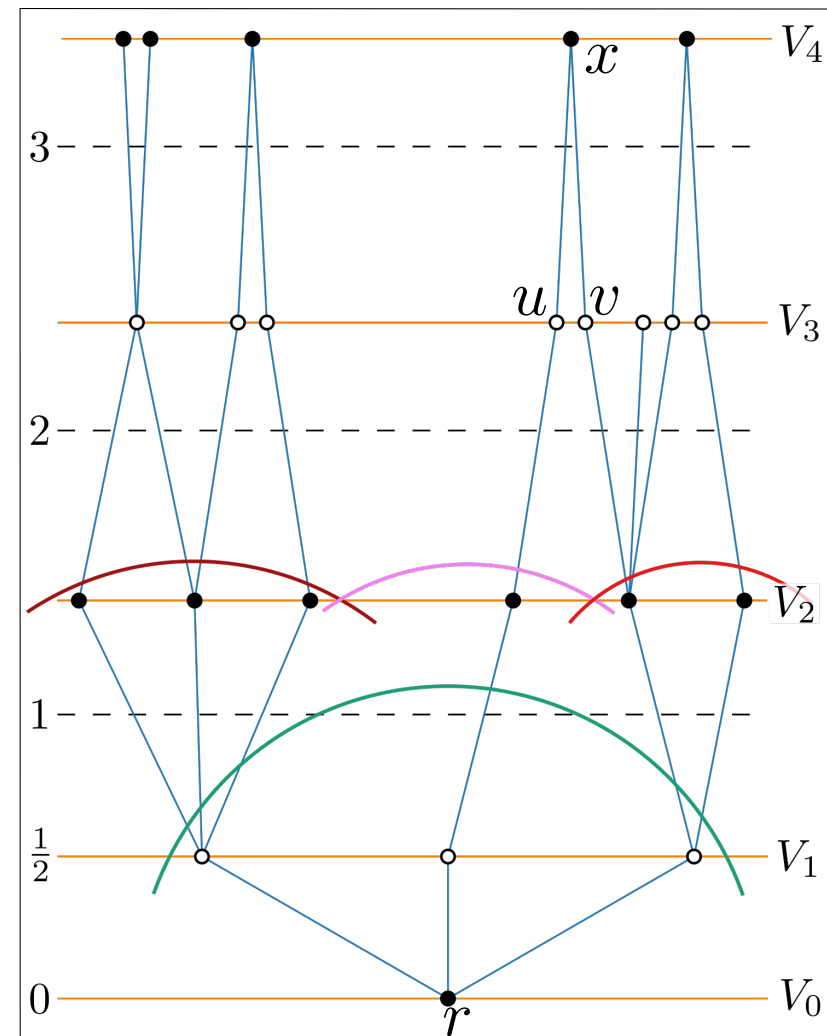
11 - 6

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

Layered Drawing

- ▶ root G_s at arbitrary vertex r
- ▶ $V_k, k = 0, \dots$ is the BFS layer of G_s
- ▶ each V_k is placed on horizontal line ℓ_k
- ▶ ℓ_k has y-coordinate
 - between $k - 1$ and k
 - at least the topmost intersection point of the unit circles of V_{k-1}



Short Outerplanar Graphs

11 - 7

Theorem

A bipartite dichotomous ordinal graph admits a geometric representation if the subgraph induced by the short edges is outerplanar.

Layered Drawing

- ▶ root G_s at arbitrary vertex r
- ▶ $V_k, k = 0, \dots$ is the BFS layer of G_s
- ▶ each V_k is placed on horizontal line ℓ_k
- ▶ ℓ_k has y-coordinate
 - between $k - 1$ and k
 - at least the topmost intersection point of the unit circles of V_{k-1}
- ▶ if x has two parents u and v we can guarantee that they are close

