Sibson's formula for higher order Voronoi diagrams

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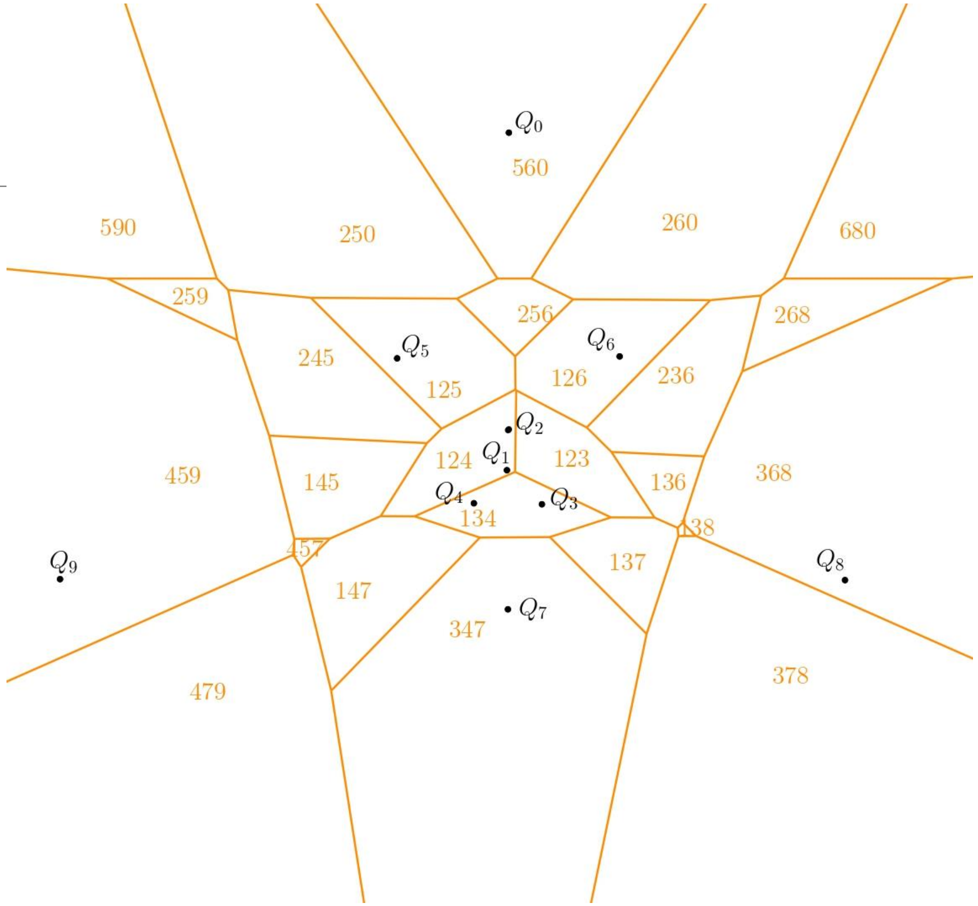
Voronoi Diagrams

Let S be a set of n points in general position in $\mathbb{R}^d$.

The Voronoi diagram of order k of S, $V_k(S)$ is a subdivision of the space into cells such that points in the same cell have the same k nearest points of S.



Voronoi diagram of order 3 of a set S of $n = 10$ points in the plane.
Corresponding indices of the three nearest neighbours are written in each region.

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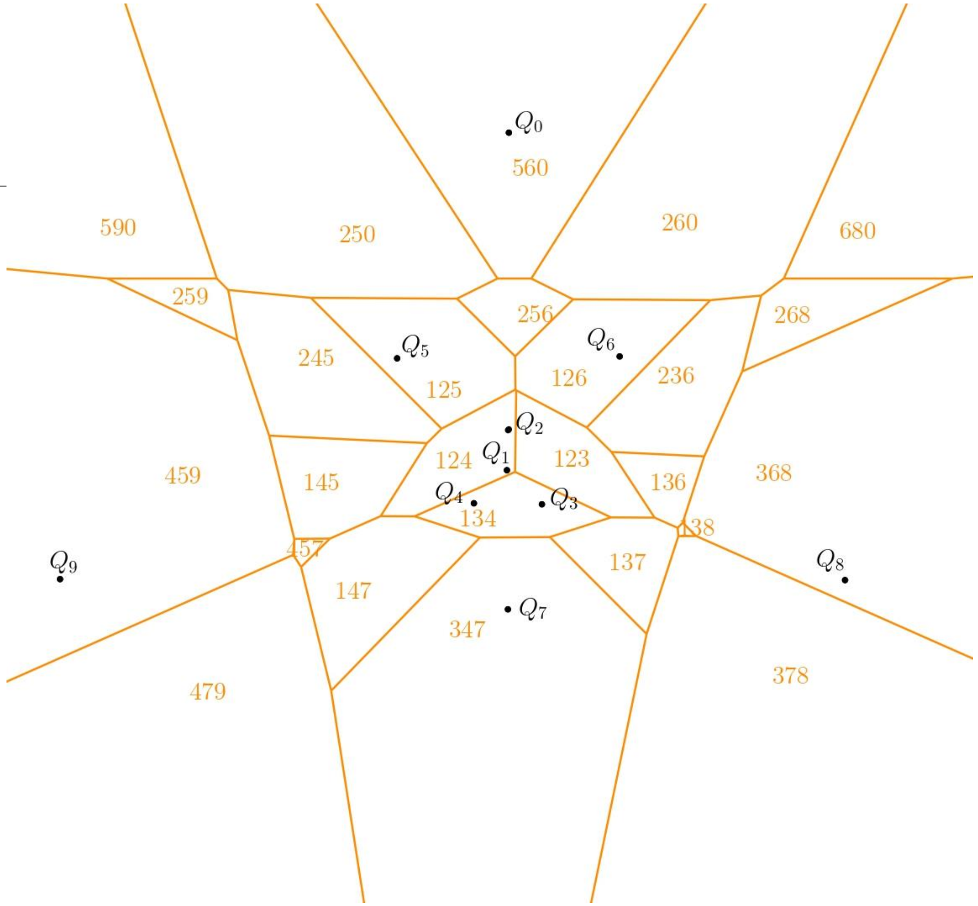
Thus, each cell $f(P_k)$ of $V_k(S)$ is defined by a subset P_k of k elements of S.



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Sibson's formula

LOCAL COORDINATES PROPERTY.

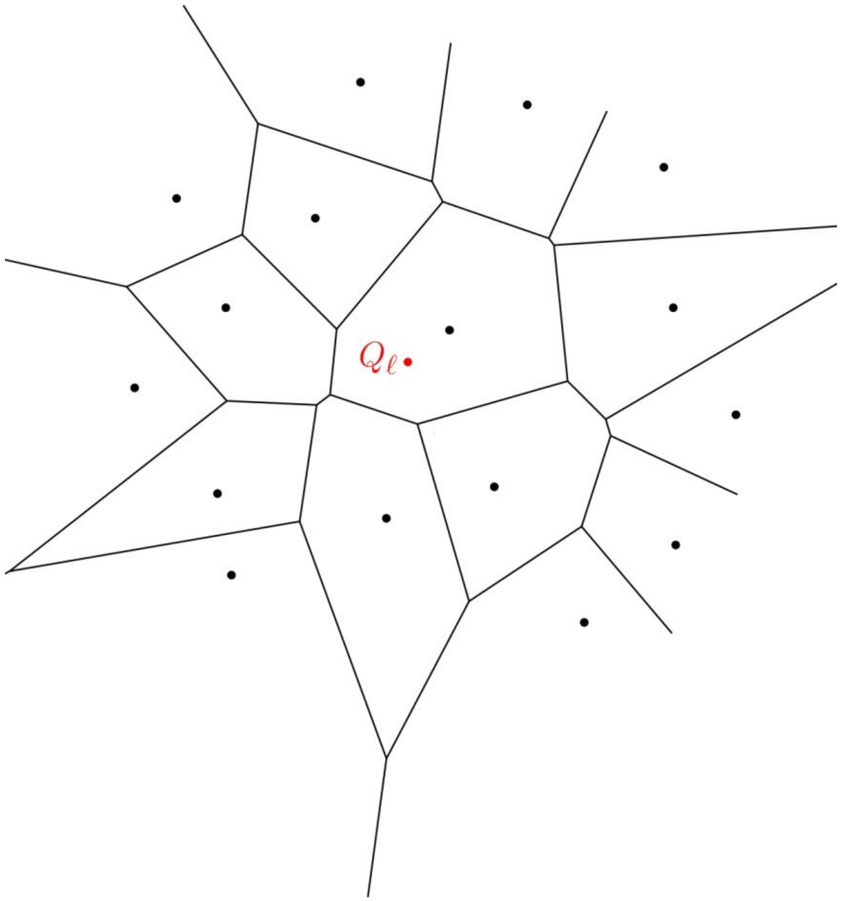
For a bounded cell $f(\{Q_l\})$of $V_1(S)$,

$$Q_l = \sum_{j \neq l} \frac{\sigma\left(f(\{Q_l, Q_j\}) \cap f(\{Q_l\})\right)}{\sigma\left(f(\{Q_l\})\right)} Q_j$$

Where σ denote the Lebesgue measure on $\mathbb{R}^d$.

Natural Neighbour Interpolation

Given a set of points and a function, this interpolation method provides a smooth approximation of new points to the function.



Natural Neighbor Interpolation

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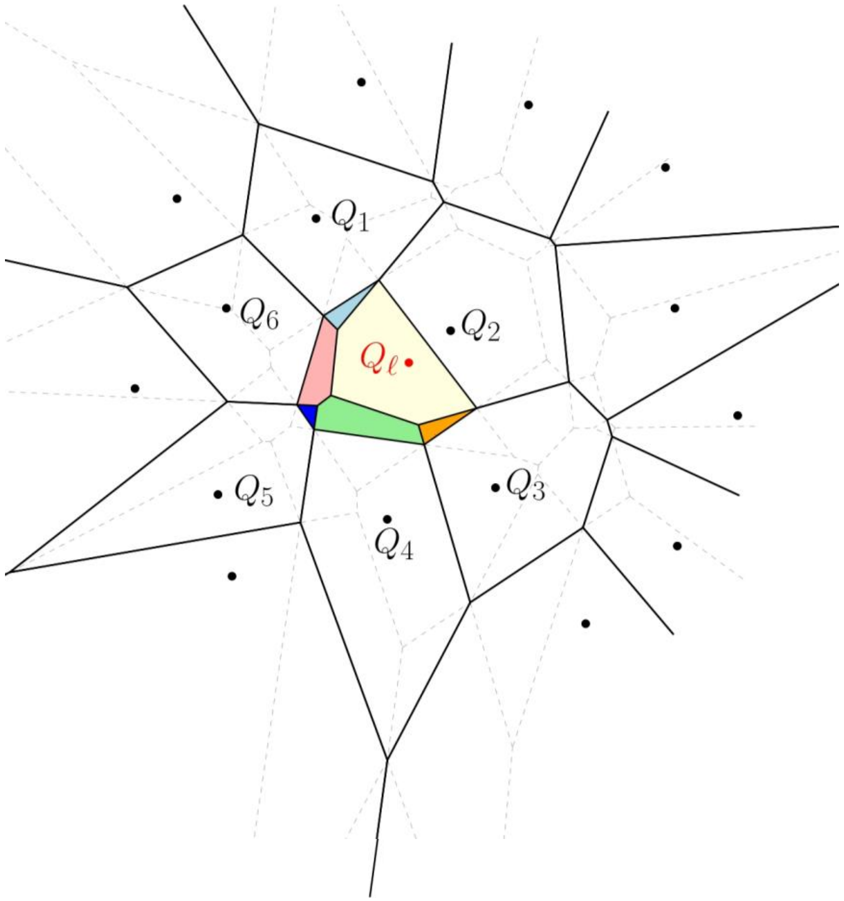
Sibson's algorithm uses the closest subset of the input set $S \setminus \{Q_l\}$, that we call its **natural neighbours**, to interpolate the function value of a query point, Q_l, and applies weights based on the ratios of volumes provided by Sibson's formula.



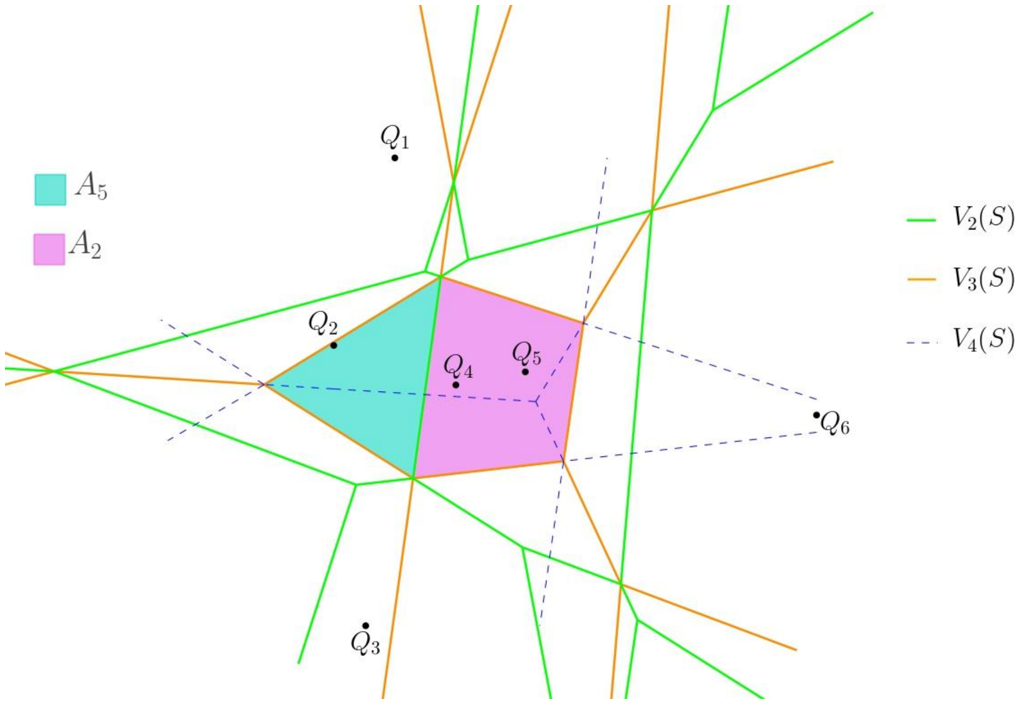
Aurenhammer's formula

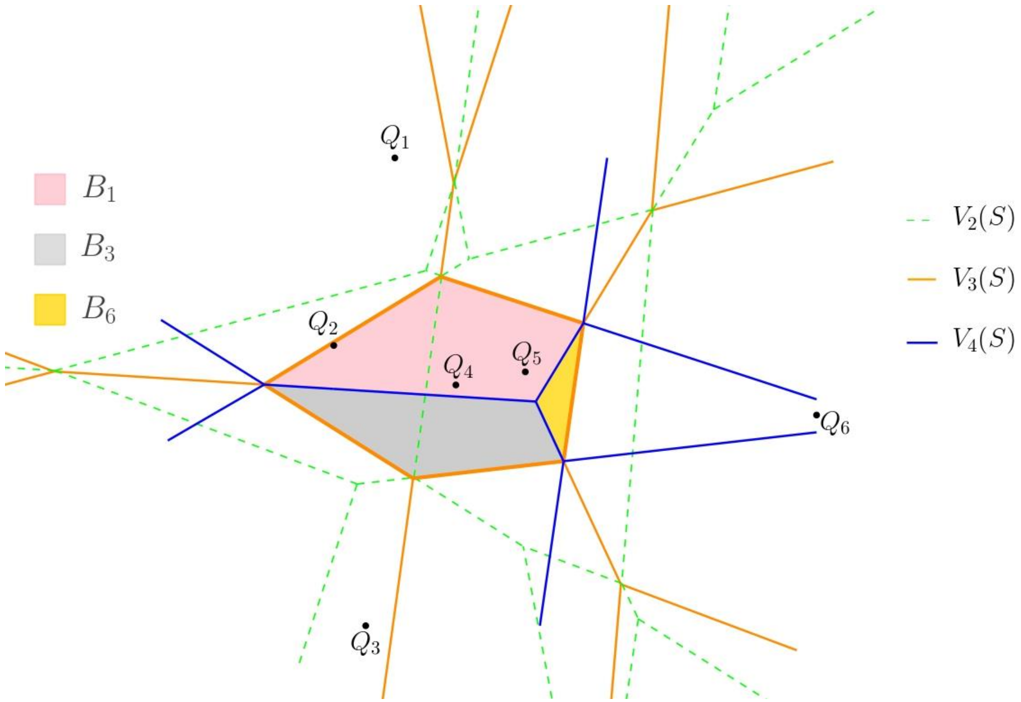
THEOREM

For a bounded cell $f(P_k)$ of $V_k(S)$,

$$\sum_{\substack{f(P_{k-1})\in V_{k-1}(S) \\ Q_i \in P_k \setminus P_{k-1}}} \sigma\big(f(P_{k-1}) \cap f(P_k)\big) Q_i = \sum_{\substack{f(P_{k+1})\in V_{k+1}(S) \\ Q_j \in P_{k+1} \setminus P_k}} \sigma\big(f(P_{k+1}) \cap f(P_k)\big) Q_j$$

Aurenhammer's formula





Geometric Interpretation

Each side of the equation describes a point H that is a convex combination of points from S.

$$H = \sum_{\substack{f(P_{k-1}) \in V_{k-1}(S) \\ Q_i \in P_k \setminus P_{k-1}}} \frac{\sigma\big(f(P_{k-1}) \cap f(P_k)\big)}{\sigma\big(f(P_k)\big)} Q_i = \sum_{\substack{f(P_{k+1}) \in V_{k+1}(S) \\ Q_j \in P_{k+1} \setminus P_k}} \frac{\sigma\big(f(P_{k+1}) \cap f(P_k)\big)}{\sigma\big(f(P_k)\big)} Q_j$$

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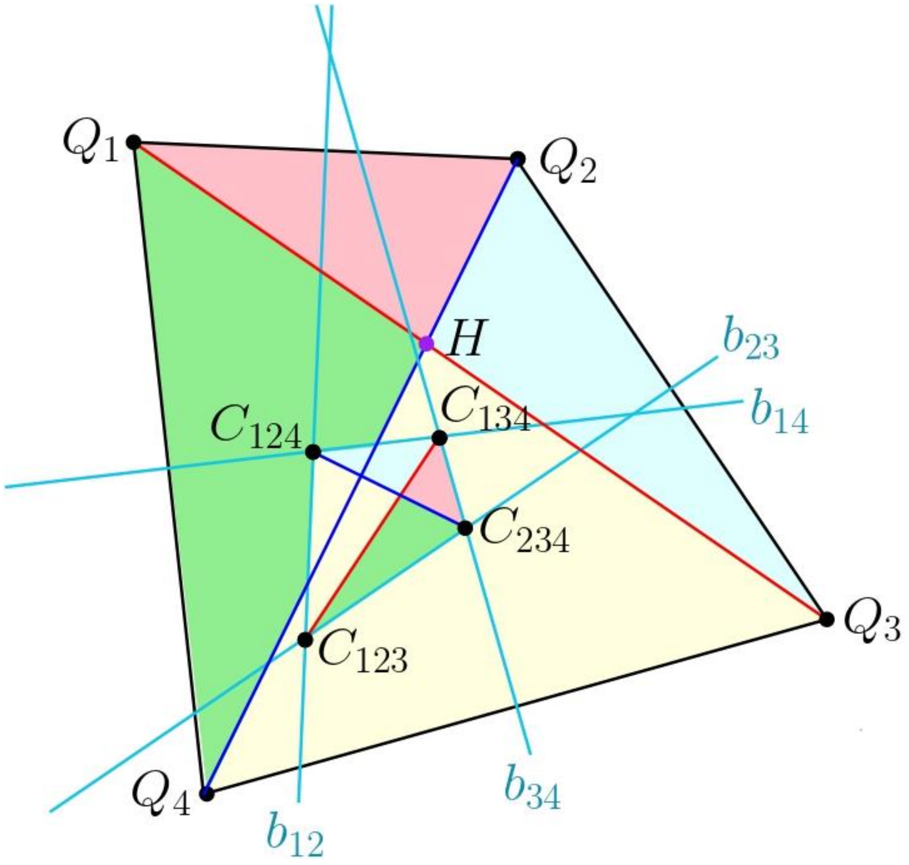
What can we say about this point H?

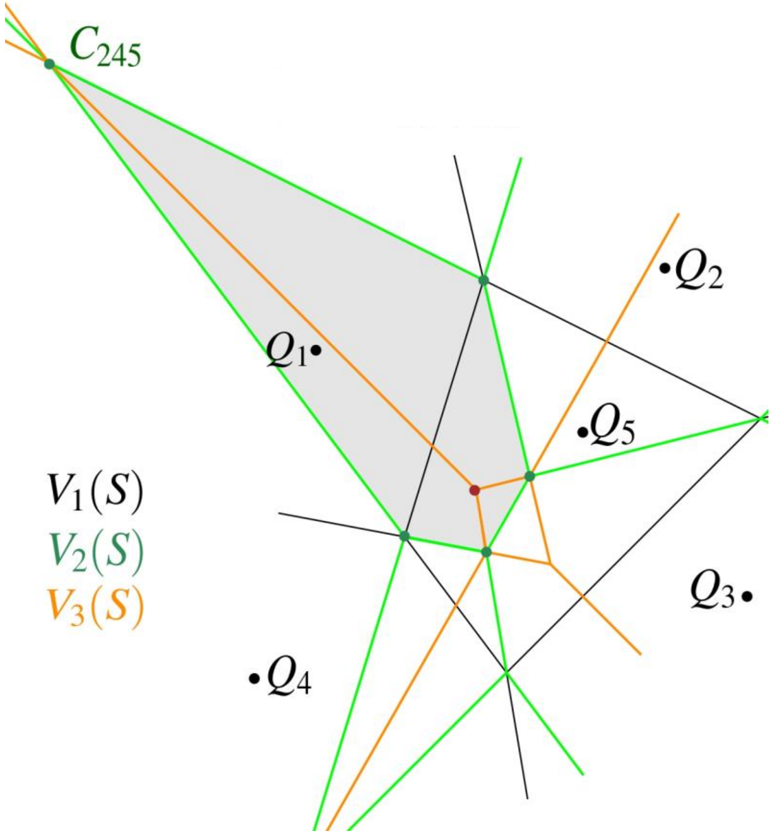
Geometric Interpretation

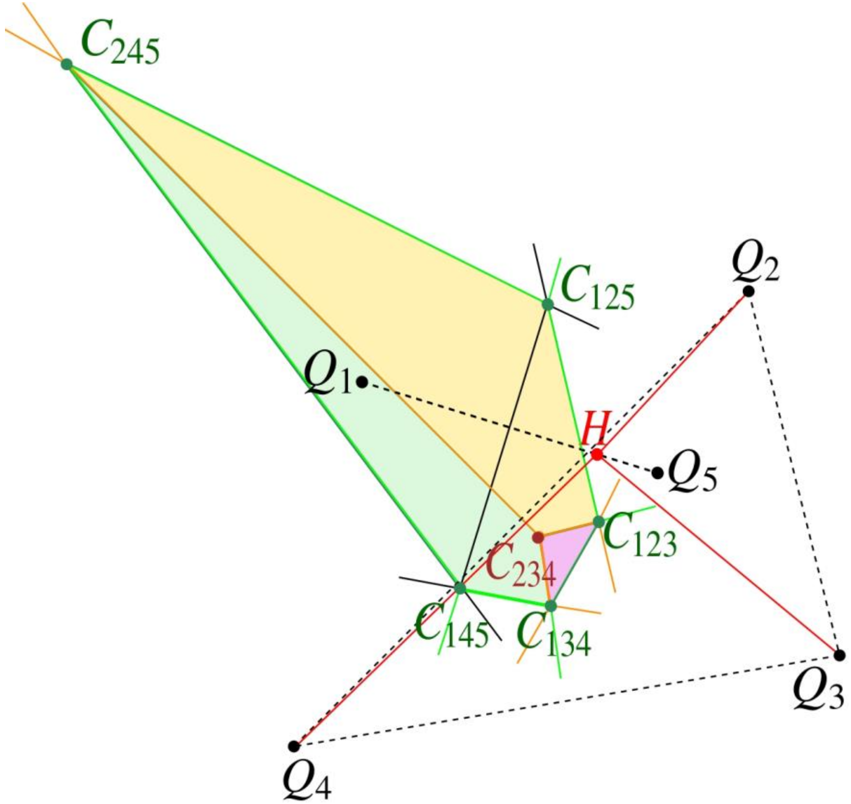
PROPERTY

Given a quadrilateral cell □$(C_{123}C_{124}C_{134}C_{234})$ of $V_k(S)$, the four corresponding points Q_1, Q_2, Q_3, Q_4 of S that participate in the perpendicular bisectors that define □$(C_{123}C_{124}C_{134}C_{234})$, also form a convex quadrilateral, □(Q_1, Q_2, Q_3, Q_4) .



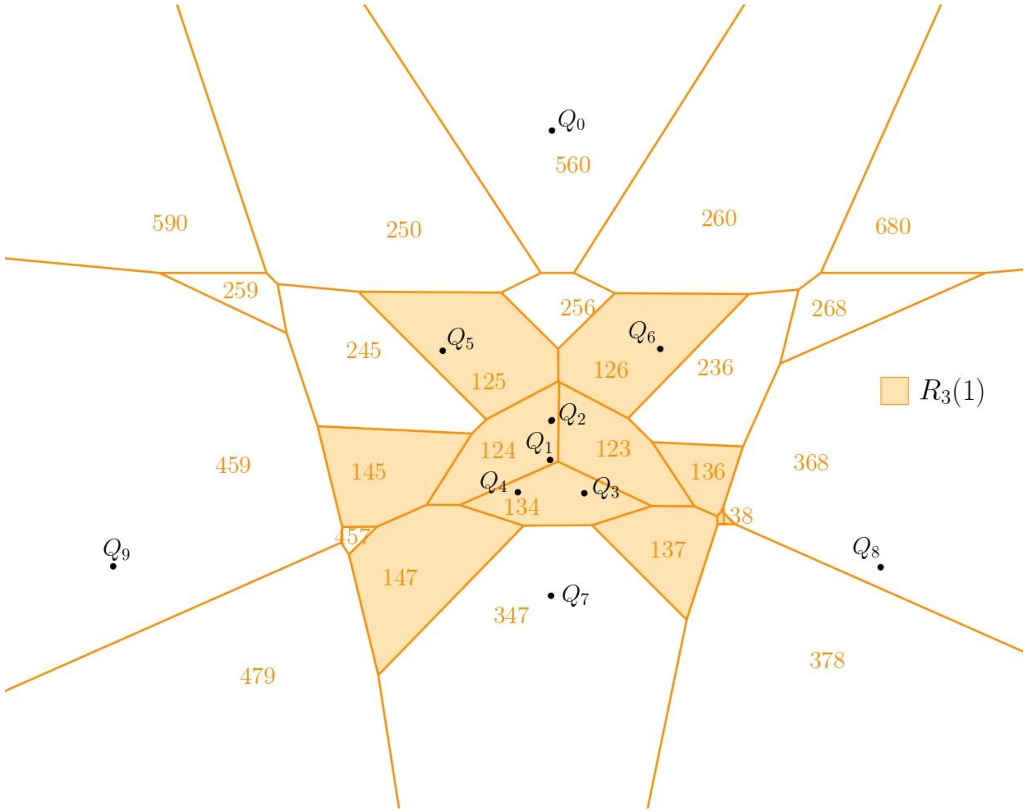
Geometric Interpretation





The regions $R_k(l)$

The region $R_k(l)$ consists of all the points of the space that have point $Q_l \in S$ as one of their k nearest neighbors from S.



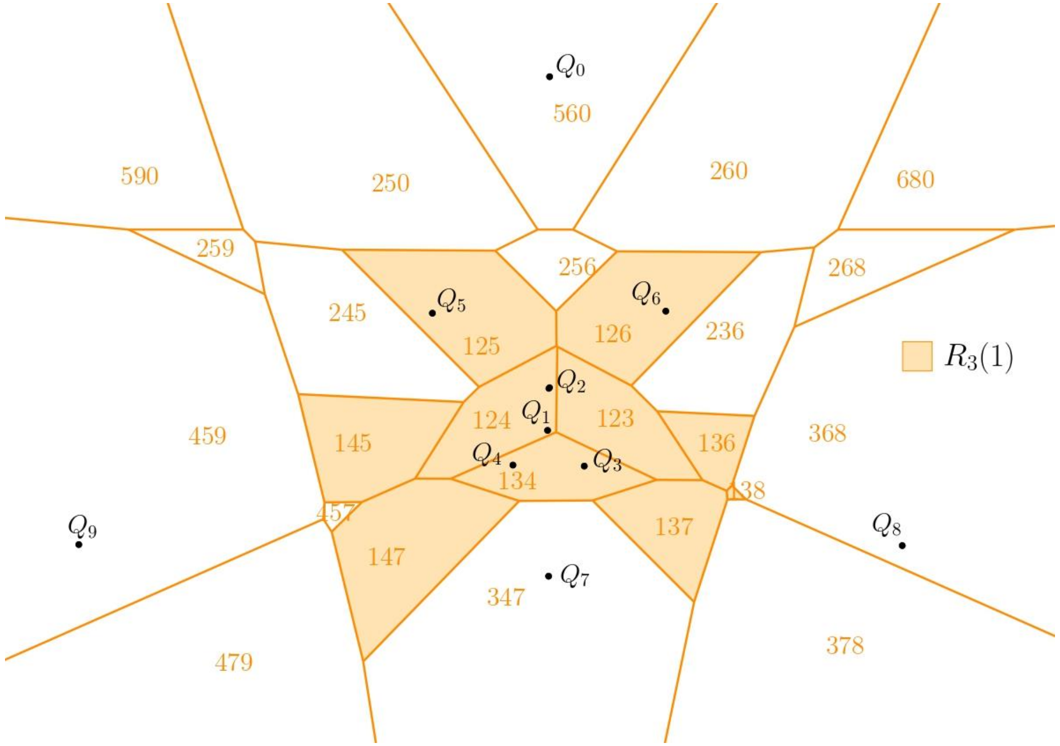
Sibson's formula for higher order

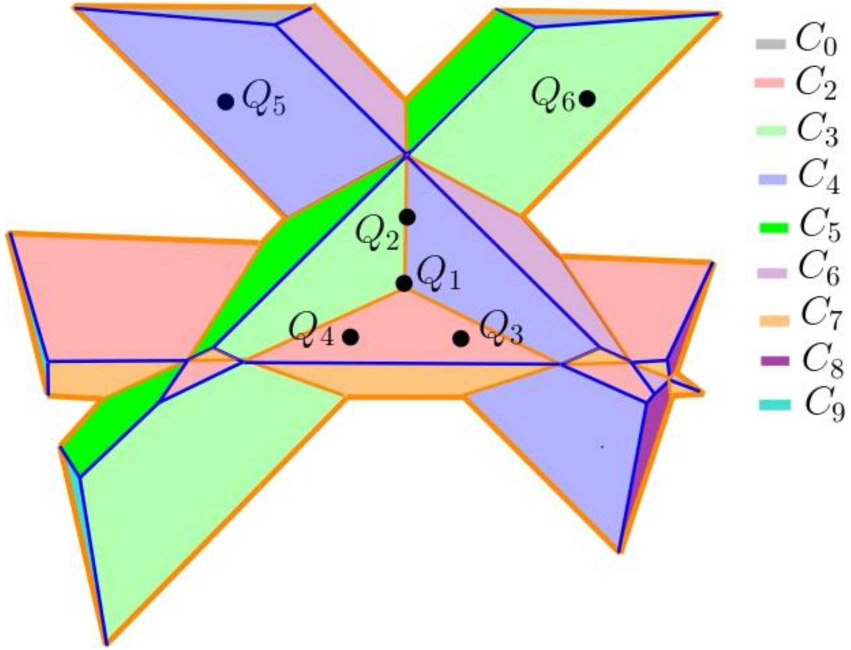
THEOREM

Let $1 \leq k \leq n-2$ and let $R_k(l)$ be a bounded región. Then,

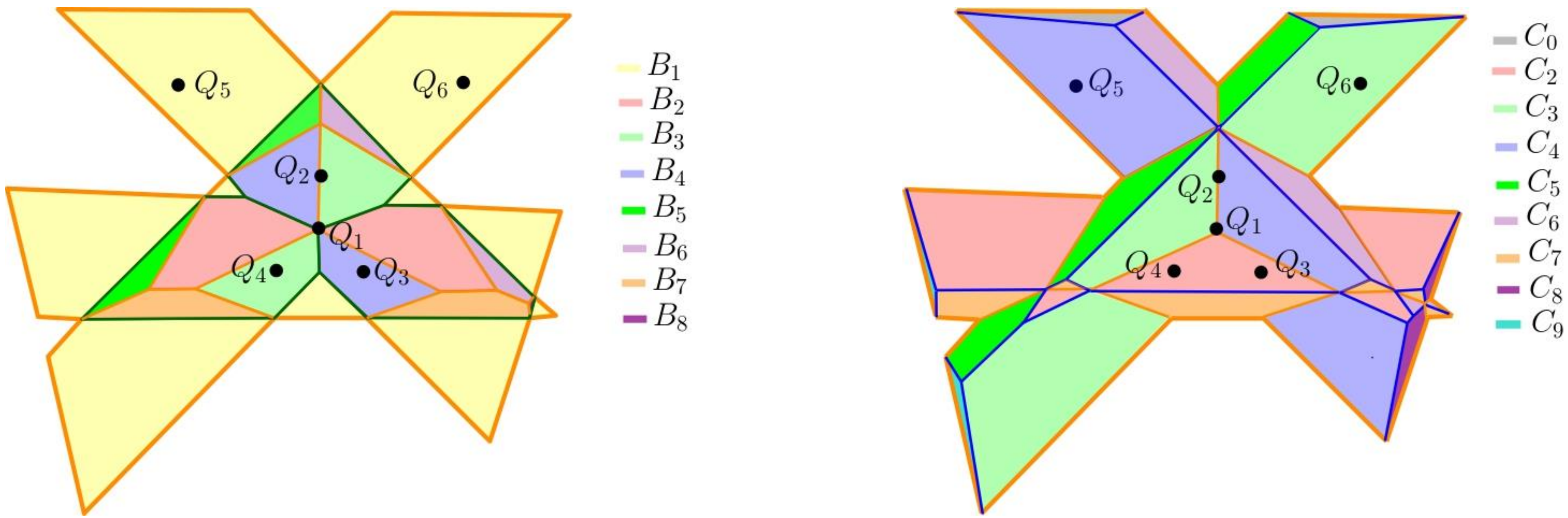
$$Q_l = \sum_{f(P_k) \in R_k(l)} \sum_{\substack{f(P_{k+1}) \in V_{k+1}(S) \\ Q_j \in P_{k+1} \setminus P_k}} \frac{\sigma\left(f(P_{k+1}) \cap f(P_k)\right)}{\sigma\left(R_k(l)\right)} Q_j$$

Sibson's formula for higher order



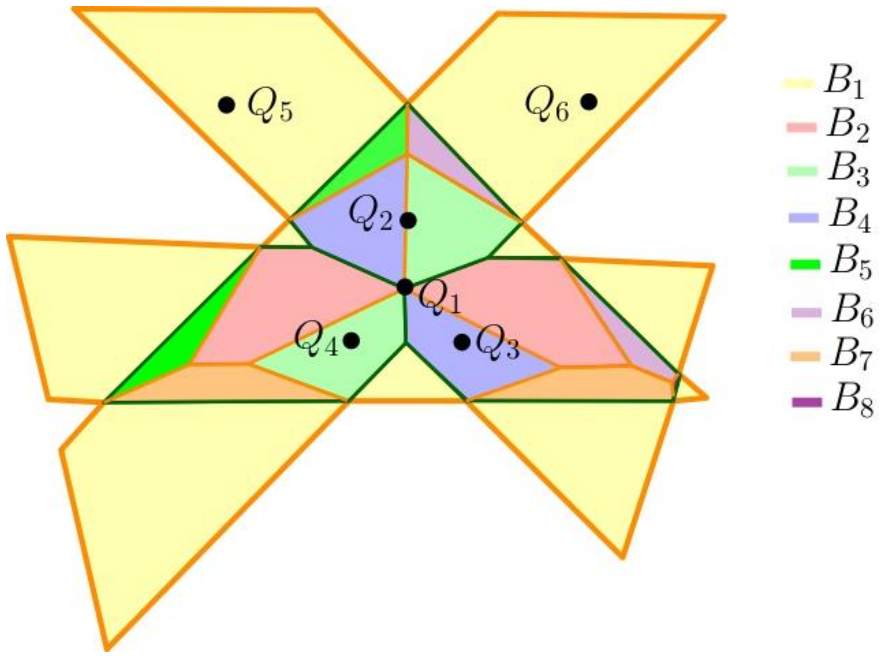


Proof Idea

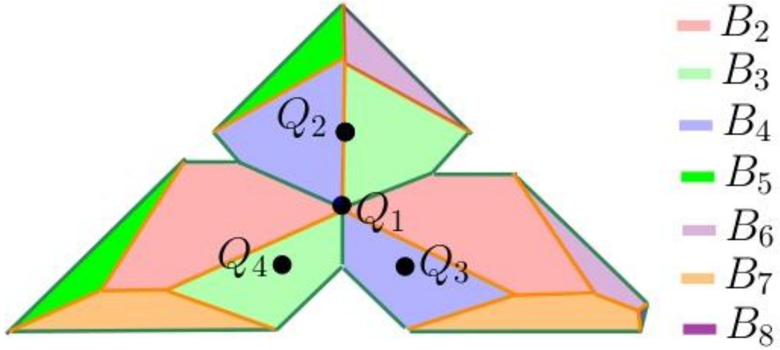


Q_5
Q_6
Q_2
Q_1
Q_4
Q_3
B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8
Q_5
Q_6
Q_2
Q_1
Q_4
Q_3
C_0
C_2
C_3
C_4
C_5
C_6
C_7
C_8
C_9

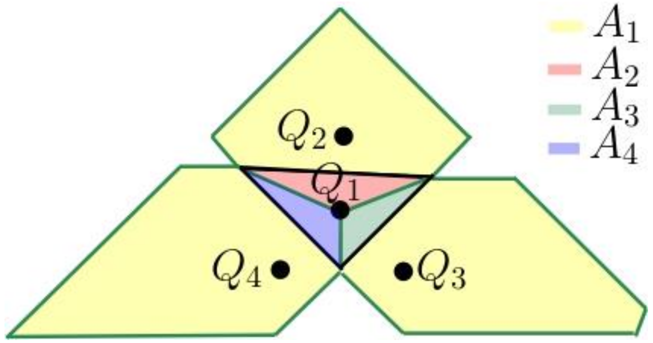
Proof Idea



Q_5
Q_6
Q_2
Q_1
Q_4
Q_3
B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8



Q_2
Q_1
Q_4
Q_3
B_2
B_3
B_4
B_5
B_6
B_7
B_8



Q_2
Q_1
Q_4
Q_3
A_1
A_2
A_3
A_4

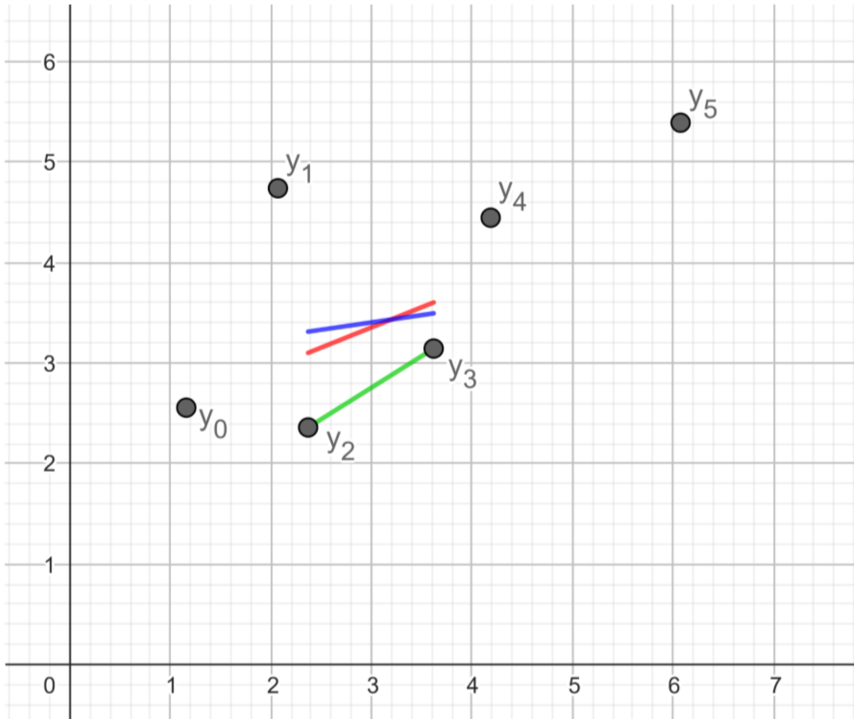
Interpolation

Sibson’s algorithm use the natural neighbours of the query point in the Voronoi diagram of order 1.

↓

Use the neighbours of the query point in the Voronoi diagram of order k.

Interpolation Example



In green, Sibson's original interpolation, used only $R_1(x)$. Only two points are used.

The blue segment shows the interpolation using $R_2(x)$. Four points are used.

The red segment shows the interpolation using $R_3(x)$. The six points are used.

Thanks For Your Attention