Computing an ε-net of a closed hyperbolic surface

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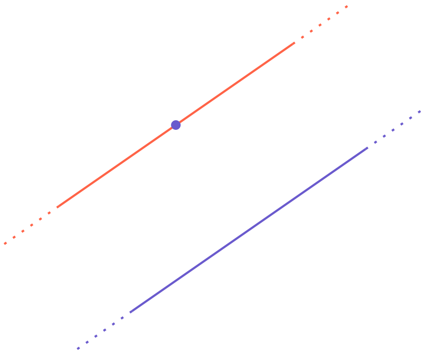
Euclidean geometry

Axioms of Euclidean geometry:

1. There is one and only one line segment between any two given points.
2. Any line segment can be extended continuously to a line.
3. There is one and only one circle with any given center and any given radius.
4. All right angles are congruent to one another.
5. (Parallel postulate) Given a line and a point not on the line, there is **exactly one** line through the point that is parallel to the given line.



Hyperbolic geometry

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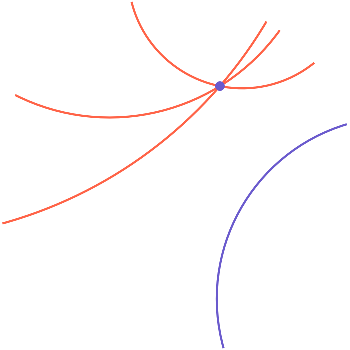
3. There is one and only one circle with any given center and any given radius.
4. All right angles are congruent to one another.
5. Given a line and a point not on the line, there are **infinitely many** lines through the point that are parallel to the given line.



The Poincaré disk model of the hyperbolic plane

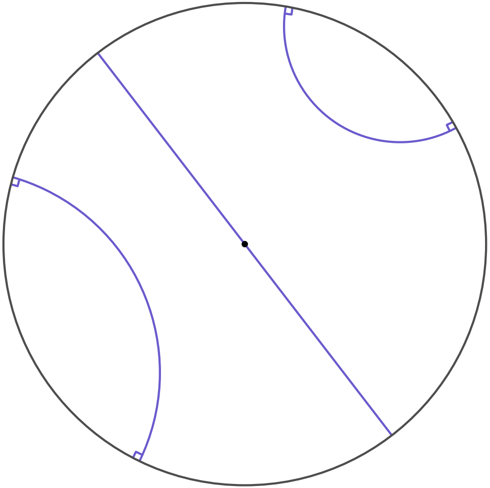
Model = metric space satisfying the axioms

Definition

The Poincaré disk model of the hyperbolic plane is

$$\mathbb{D} = \left\{ u + iv \in \mathbb{C} : u^2 + v^2 < 1 \right\}$$

equipped with the metric $ds^2 = \frac{4(du^2+dv^2)}{(1-(u^2+v^2))^2}$.



Geodesics in the Poincaré disk model.

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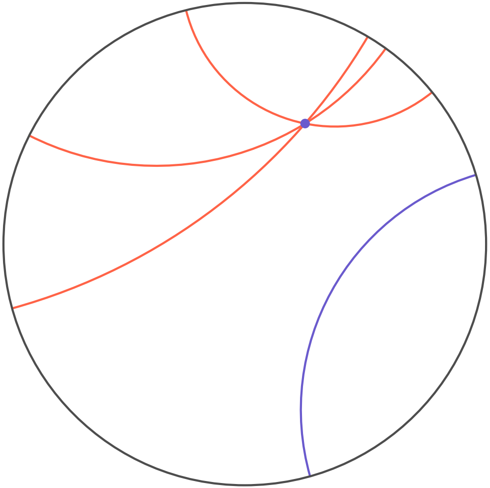

Illustration of the 5th axiom.

Hyperbolic surface

- Surface = 2-dimensional compact & connected manifold without boundary.
- Hyperbolic surface = surface + metric s.t. it is locally isometric to the hyperbolic plane $\mathbb{H}^2$.

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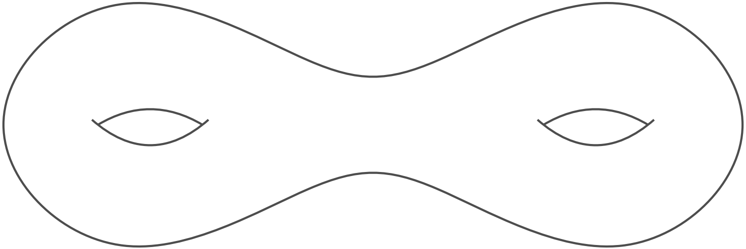
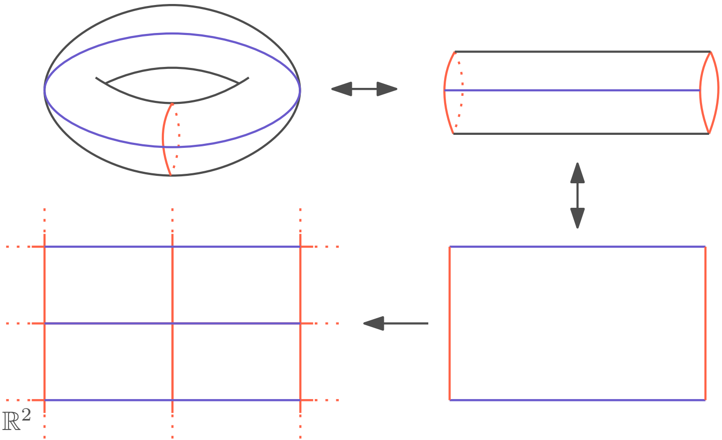
- Surface = 2-dimensional compact & connected manifold without boundary.
- Hyperbolic surface = surface + metric s.t. it is locally isometric to the hyperbolic plane $\mathbb{H}^2$.
- Any surface with genus $g \geqslant 2$ (number of handles) admits a hyperbolic metric.



A genus 2 surface.

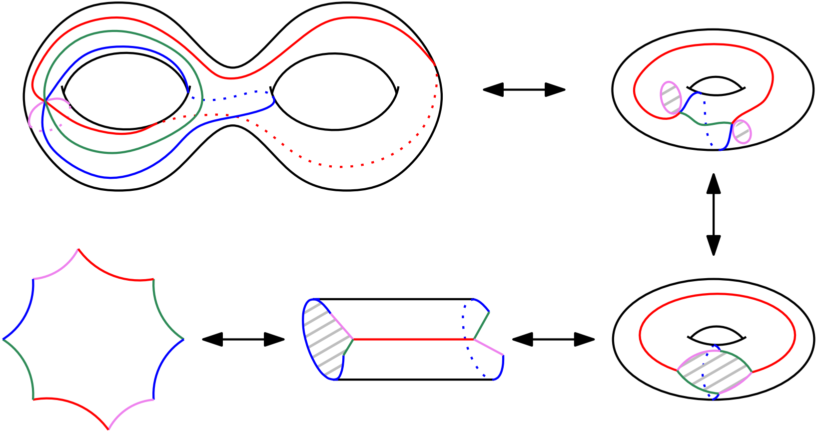
Fundamental domain

Fundamental domain for the flat torus ($g = 1$, Euclidean metric):



Fundamental domain

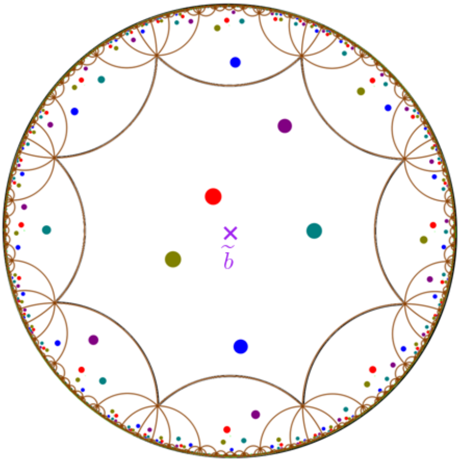
Fundamental domain for a genus 2 hyperbolic surface:



(Iordan Iordanov)

Dirichlet domain

Hyperbolic surface $S = \mathbb{H}^2/\Gamma$
Γ = group of orientation-preserving isometries



Ex: the Bolza surface. (CGAL documentation.)

Dirichlet domain

Hyperbolic surface $S = \mathbb{H}^2/\Gamma$
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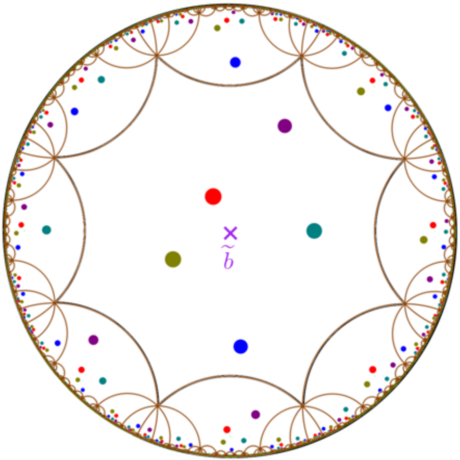
Dirichlet domain $\mathcal{D}_{\widetilde{b}}$ of a point $\widetilde{b} \in \mathbb{H}^2$ = Voronoi cell of $\widetilde{b}$ in the Voronoi diagram of $\Gamma\widetilde{b}$



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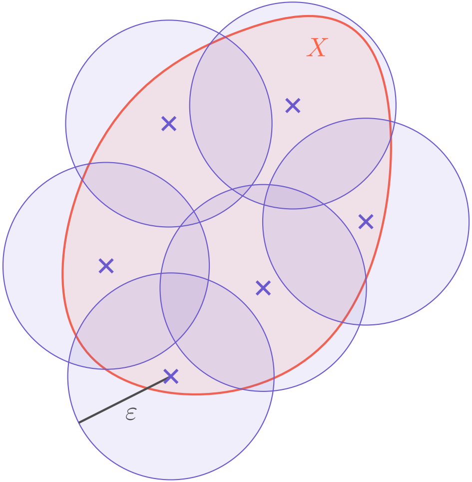
ε-net

(X, d) a metric space
$\varepsilon > 0$
A subset $P \subset X$ is an ε-net if:

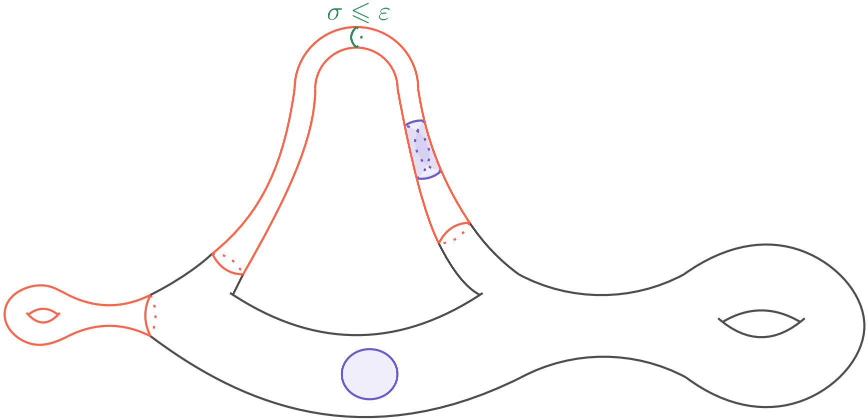
- the closed balls $\{x \in X \mid d(x, p) \leqslant \varepsilon\}_{p \in P}$ cover X,
- if $p \neq q \in P$ then $d(p, q) \geqslant \varepsilon$.



ε-net of a metric space.

Upper bound on the size

Hyperbolic surface $S = \mathbb{H}^2/\Gamma$ of genus g and systole σ



Area: $\mathcal{A}(S) = 4\pi(g-1)$

Upper bound on the size

Hyperbolic surface $S = \mathbb{H}^2/\Gamma$ of genus g and systole σ
$N :=$ number of points of an ε-net of S

Proposition

$$N \leqslant 16(g-1)\left(\frac{1}{\varepsilon^2} + \frac{1}{\sigma^2}\right).$$

If $\varepsilon < \sigma$, then $N \leqslant \dfrac{16(g-1)}{\varepsilon^2}$.

Algorithm overview

Input:

- DT of S with a single vertex $b \in S$,
- Dirichlet domain $\mathcal{D}_{\widetilde{b}}$ of a repres. $\widetilde{b}$ of b, (Despré, Kolbe, Parlier, Teillaud, 2023)
- group Γ.

Output: ε-net P_N and Delaunay triangulation $DT(P_N)$.

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Key idea: (Shewchuck, 2002)

- insert circumcenter of a Delaunay triangle with circumradius $> \varepsilon$
- update the DT with flip algo
- repeat until all triangles have circumradius $\leqslant \varepsilon$.

Data structure: contains a repres. of each vertex in $\mathcal{D}_{\widetilde{b}}$.

Details of the algorithm

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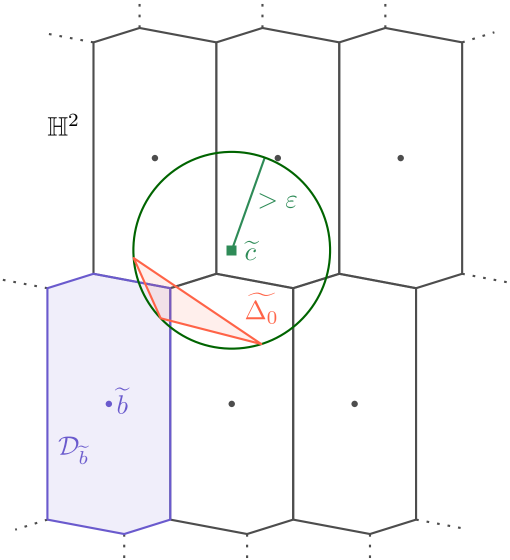

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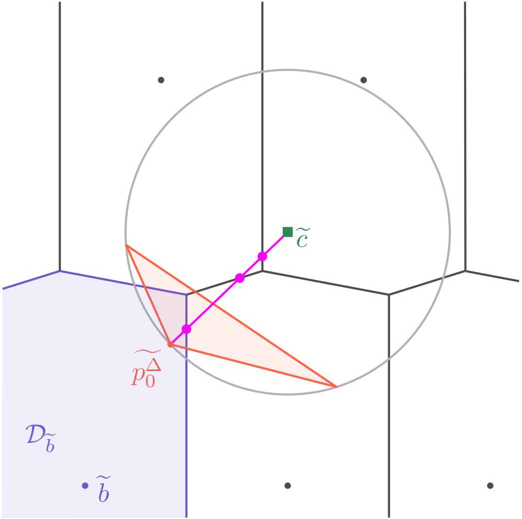

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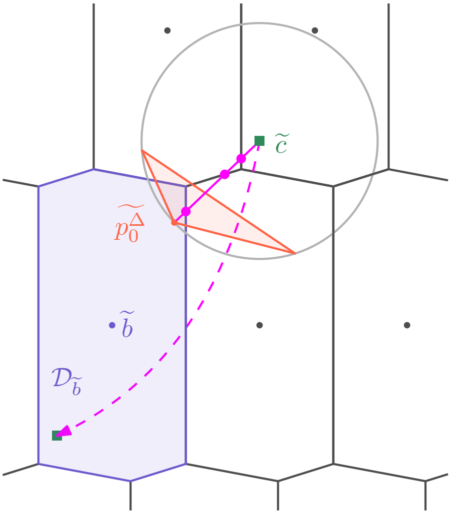

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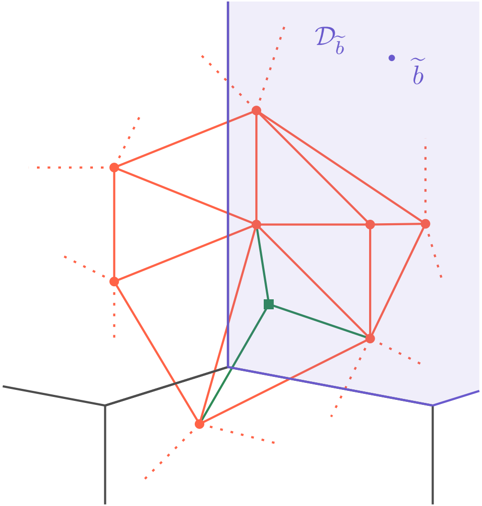

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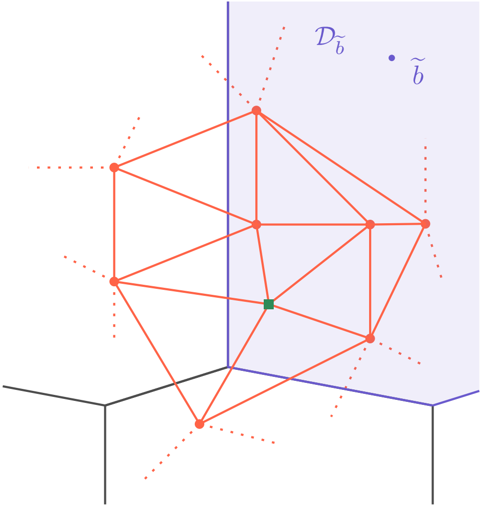

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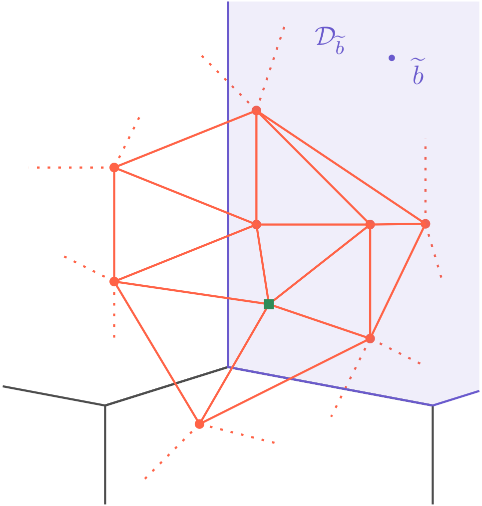

Complexity of the algorithm

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Proposition

This algorithm computes an ε-net using at most

$$\left(10 + C'_h \operatorname{Diam}(S)^{6g-4}\right) N^2 + (N-1)(144g^2 - 104g + 35) - 10$$

elementary operations.
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The complexity depends on the complexity of the flip algorithm.

Thank you!