The Number of Non-overlapping Edge Unfoldings in Convex Regular-faced Polyhedra

EuroCG 2024

◎ Takumi SHIOTA[†], Yudai ENOMOTO[‡],
Takashi HORIYAMA[‡], Toshiki SAITOH[†]

† Kyushu Institute of Technology, Japan
‡ Hokkaido University, Japan

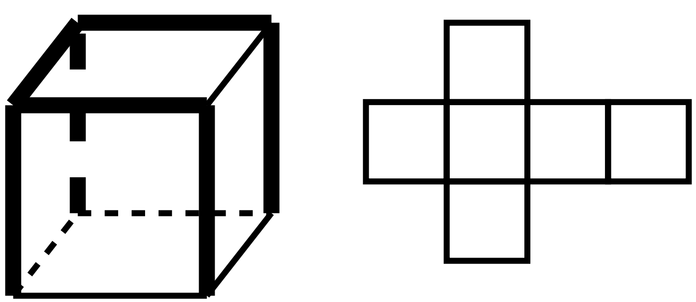
March 15, 2024

Edge unfoldings

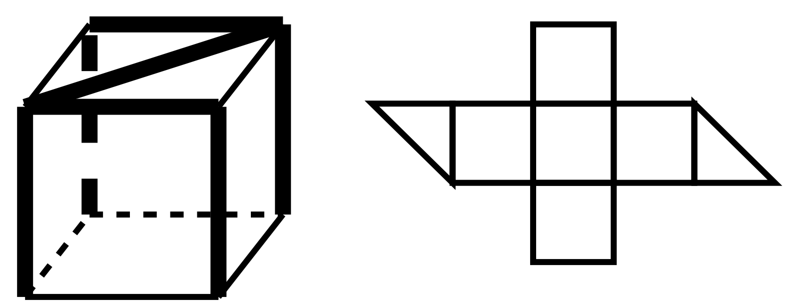
Definition 1 [R. Uehara, 2020]

An edge unfolding of the polyhedron is a flat polygon formed by cutting its edges and unfolding it into a plane.

Cutting along the thick line of each left cube …



(a) Edge unfolding



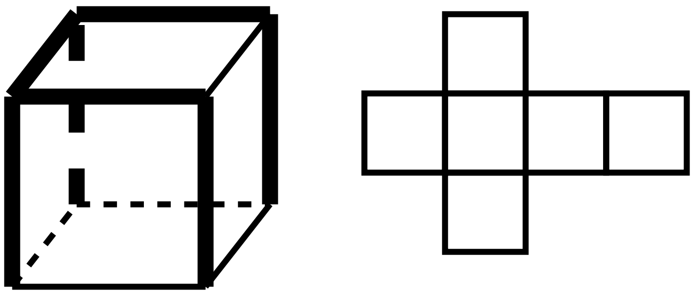
(b) Not edge unfolding

Edge unfoldings

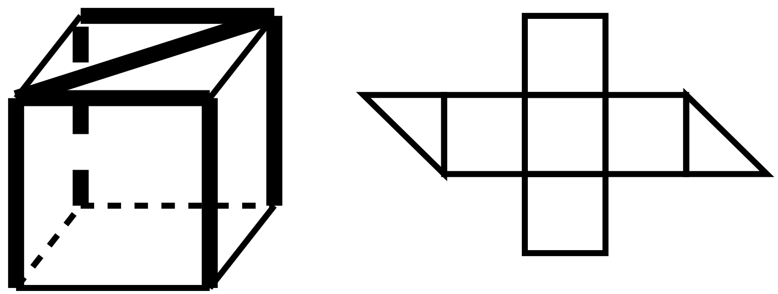
Definition 1 [R. Uehara, 2020]

An edge unfolding of the polyhedron is a flat polygon formed by cutting its edges and unfolding it into a plane.

Cutting along the thick line of each left cube …



(a) Edge unfolding



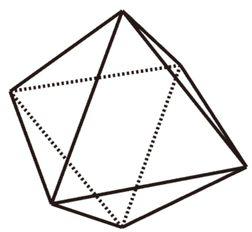
(b) Not edge unfolding

Convex regular-faced polyhedra

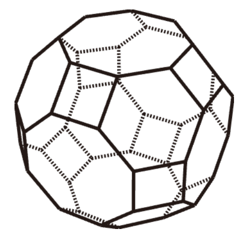
Definition 2

A convex regular-faced polyhedron is a convex polyhedron with all faces being regular polygons.

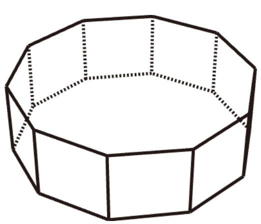
5 classes



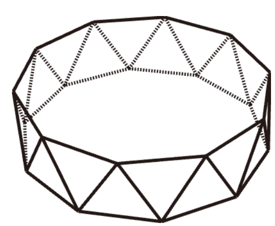
Platonic solids
(5 types)



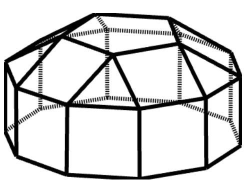
Archimedean solids
(13 types)



n-gonal Archimedean prisms ($n \geq 3$)



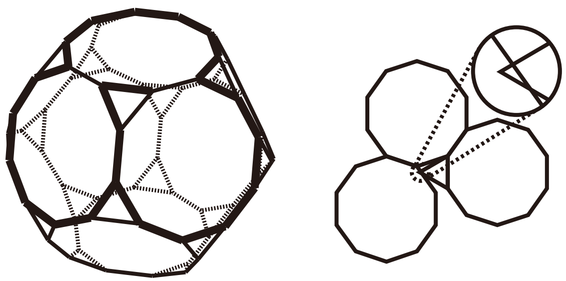
m-gonal Archimedean antiprisms ($m \geq 3$)



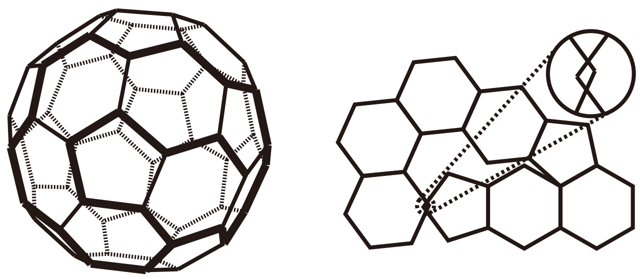
Johnson solids
(92 types)

Overlapping edge unfoldings

Overlapping edge unfoldings exist in some convex polyhedra.



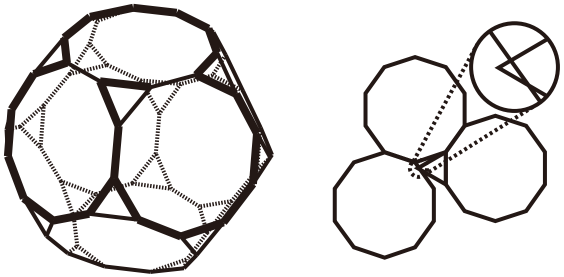
Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



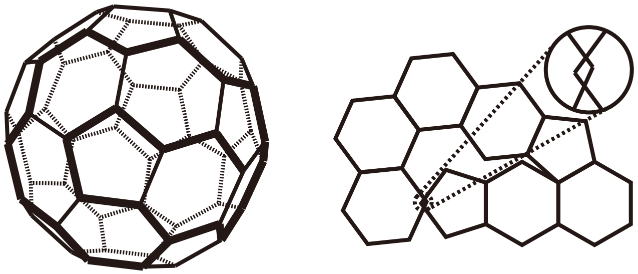
Truncated Icosahedron
[T. Horiyama and W. Shoji, 2011]

Overlapping edge unfoldings

Overlapping edge unfoldings exist in some convex polyhedra.

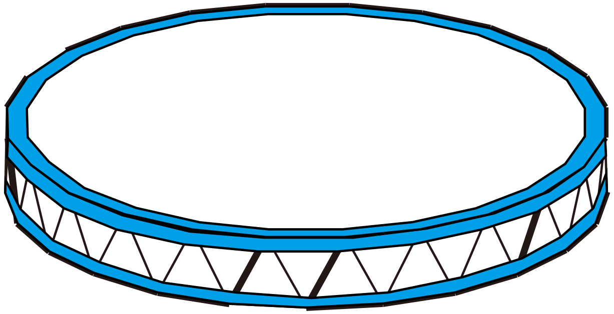


Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



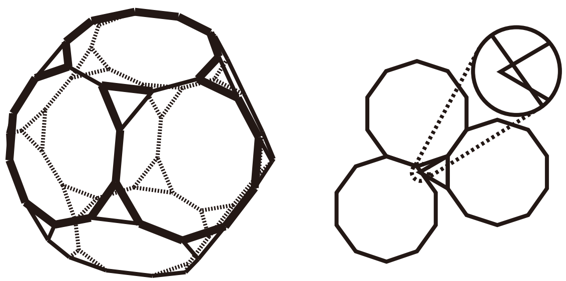
Truncated Icosahedron
[T. Horiyama and W. Shoji, 2011]

✓ Please peel off the tape and open the papercraft.

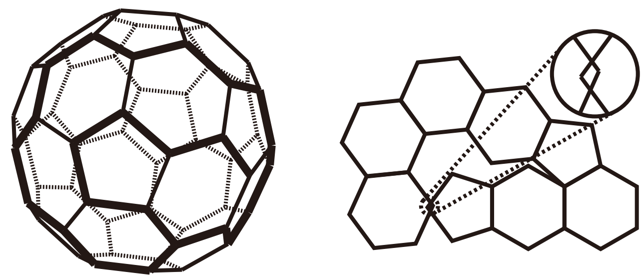


Overlapping edge unfoldings

Overlapping edge unfoldings exist in some convex polyhedra.

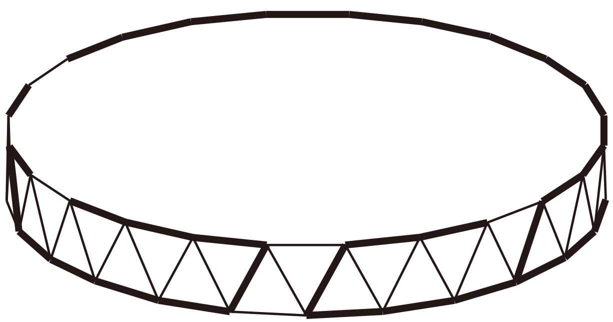


Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



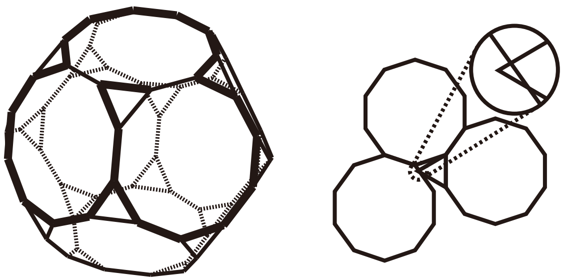
Truncated Icosahedron
[T. Horiyama and W. Shoji, 2011]

✓ Please peel off the tape and open the papercraft.

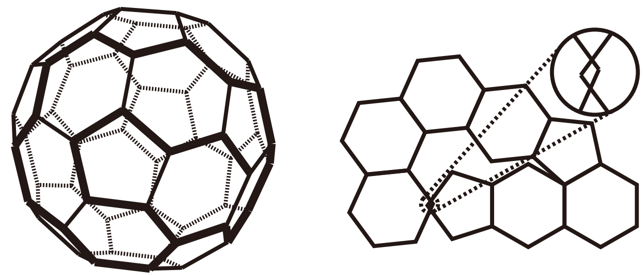


Overlapping edge unfoldings

Overlapping edge unfoldings exist in some convex polyhedra.

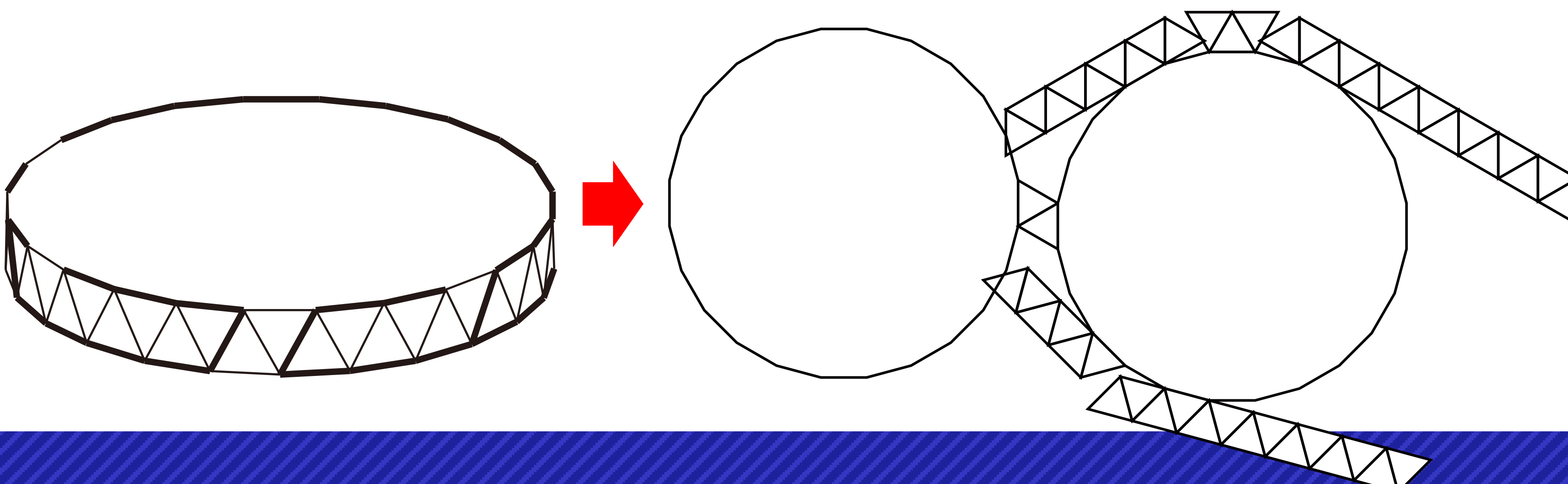


Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



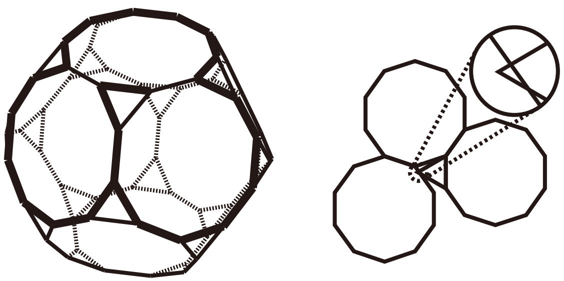
Truncated Icosahedron
[T. Horiyama and W. Shoji, 2011]

✓ Please peel off the tape and open the papercraft.

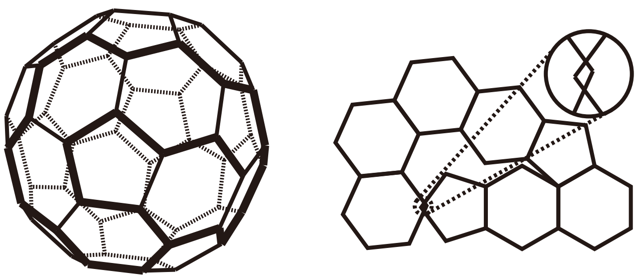


Overlapping edge unfoldings

Overlapping edge unfoldings exist in some convex polyhedra.

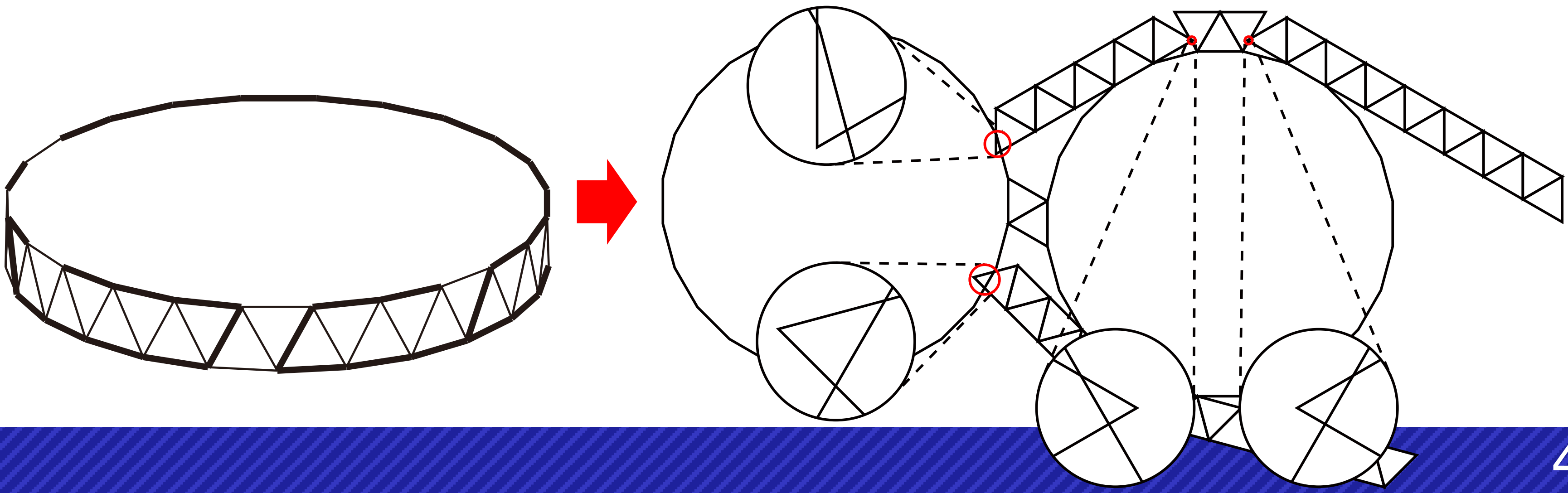


Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



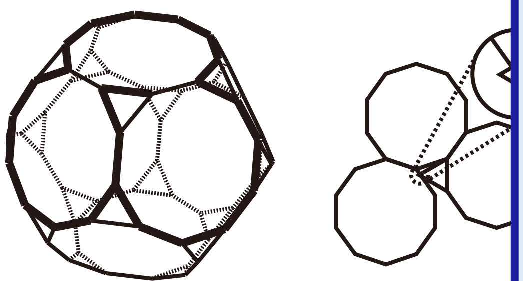
Truncated Icosahedron
[T. Horiyama and W. Shoji, 2011]

✓ Please peel off the tape and open the papercraft.



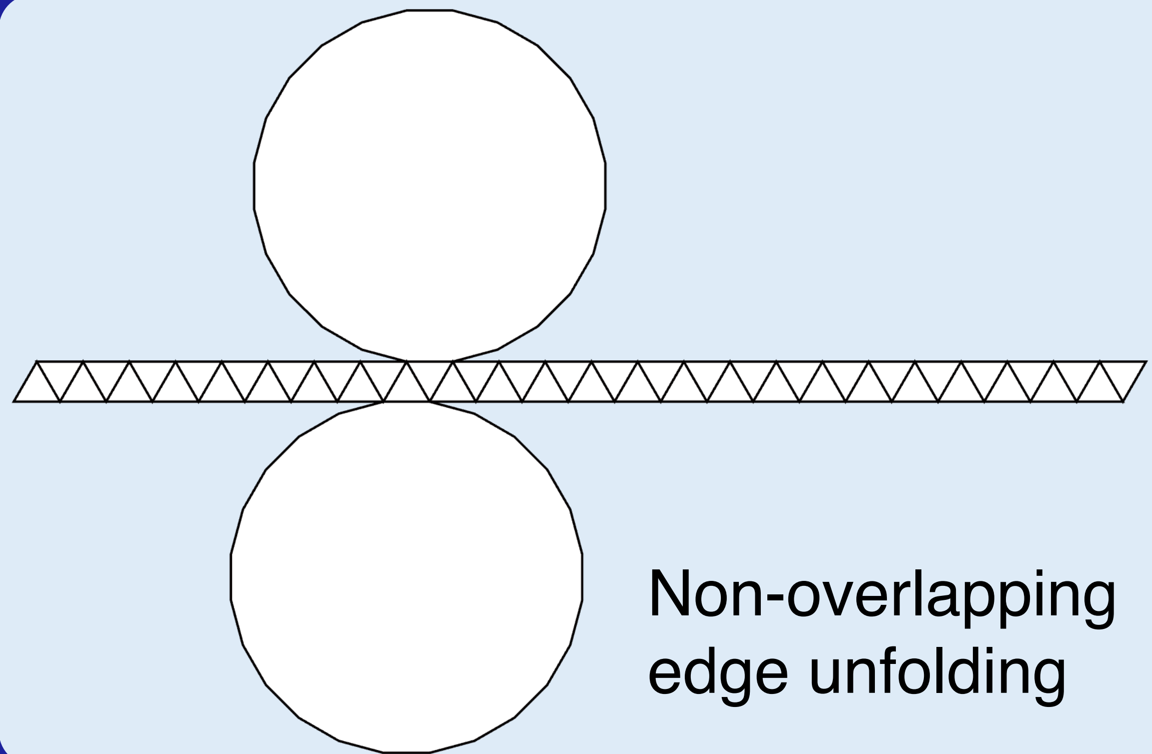
Overlapping e

Overlapping edge unfo

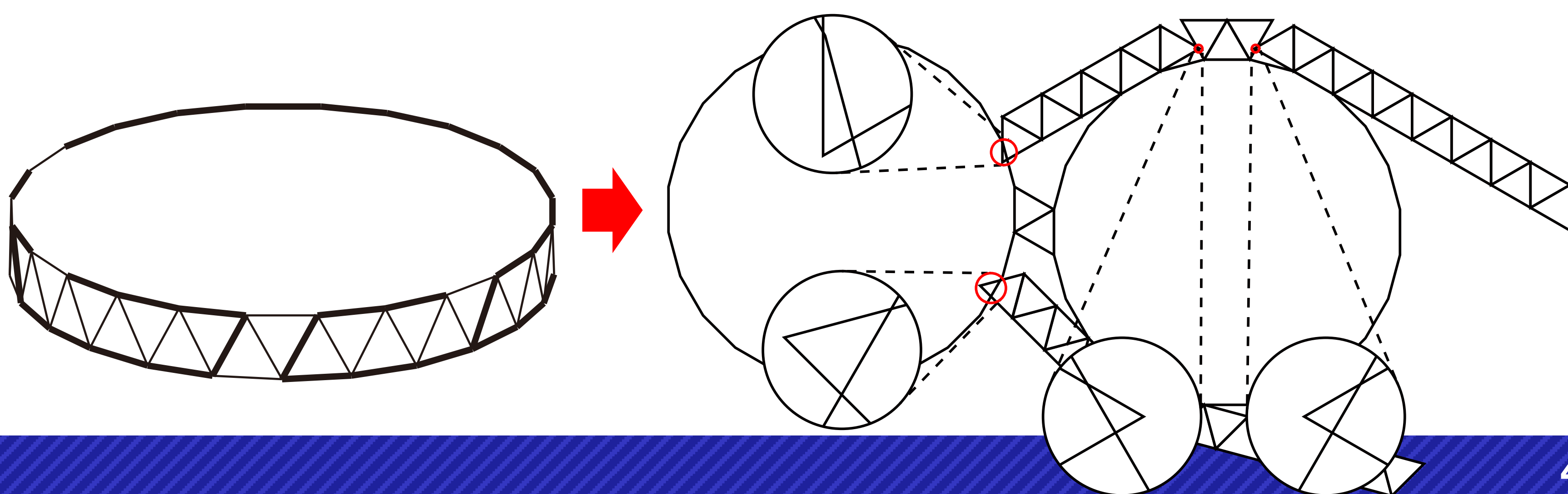


Truncated dodecahedr

[T. Horiyama and W. Shoji,

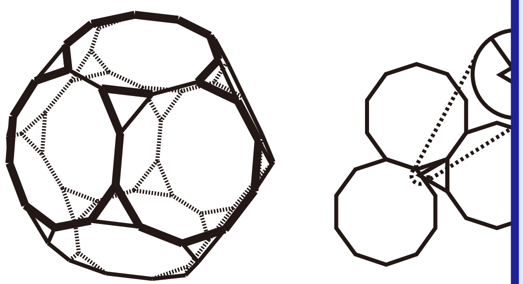


✓ Please peel off the tape and open the papercraft.



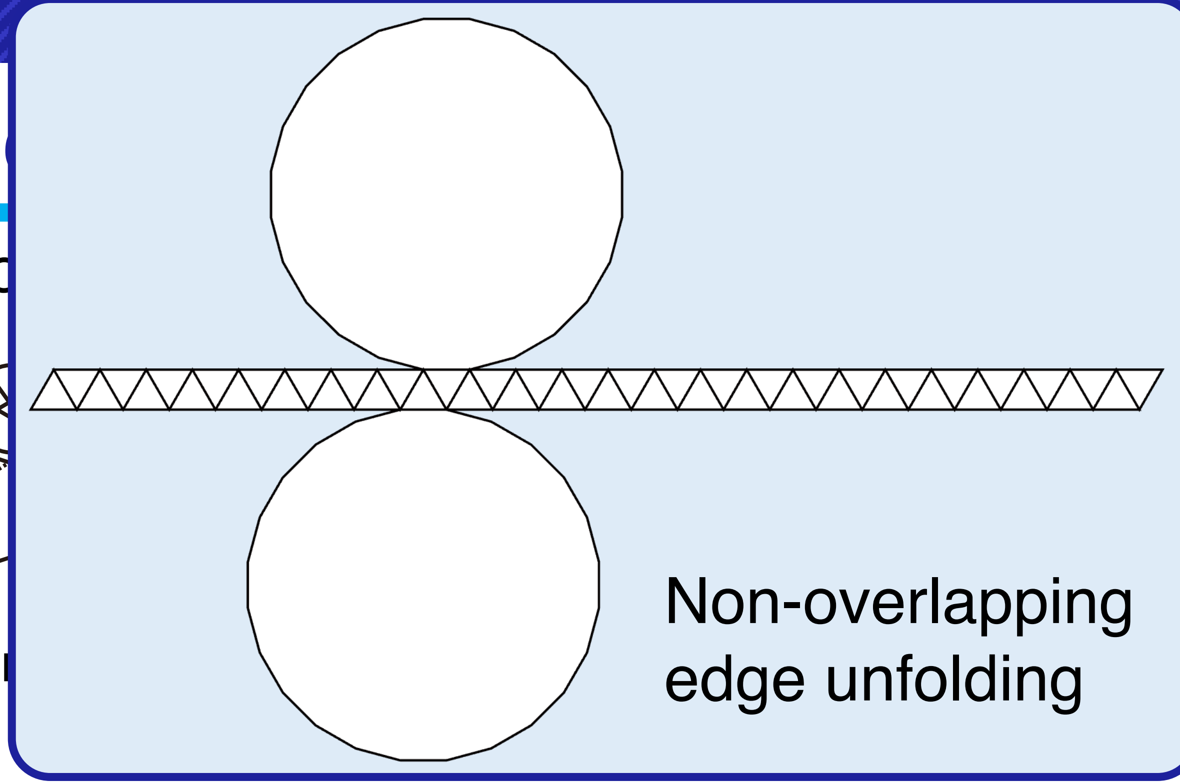
Overlapping e

Overlapping edge unfo



Truncated dodecahedr

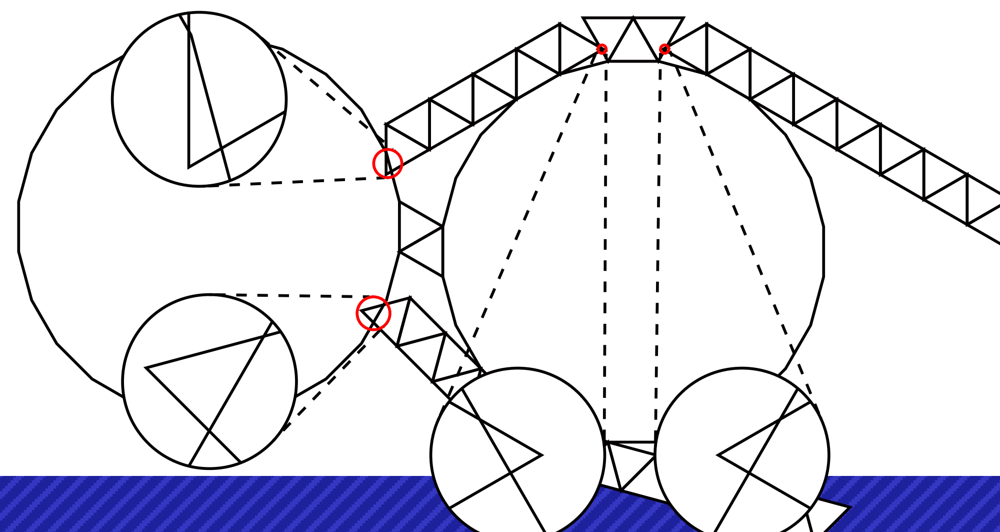
[T. Horiyama and W. Shoji,



✓ Please peel off the tape and open the papercraft.

We want to know …

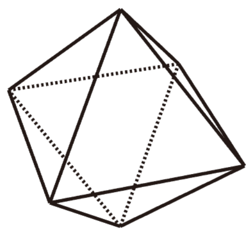
non-overlapping edge unfoldings and get their percentages among all edge unfoldings.



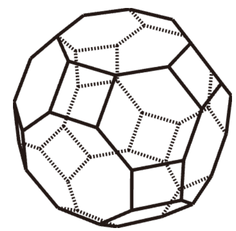
Our results

Our results

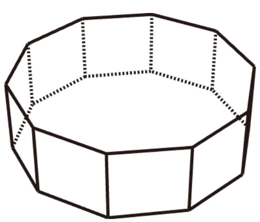
Count # non-overlapping edge unfoldings for some convex regular-faced polyhedra and get their percentages.



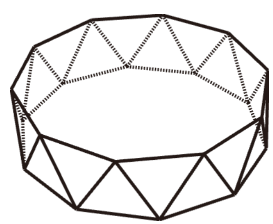
Platonic solids
(5 types)



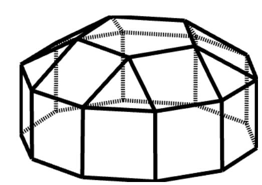
Archimedean solids
(13 types)



n-gonal Archimedean prisms ($n \geq 3$)



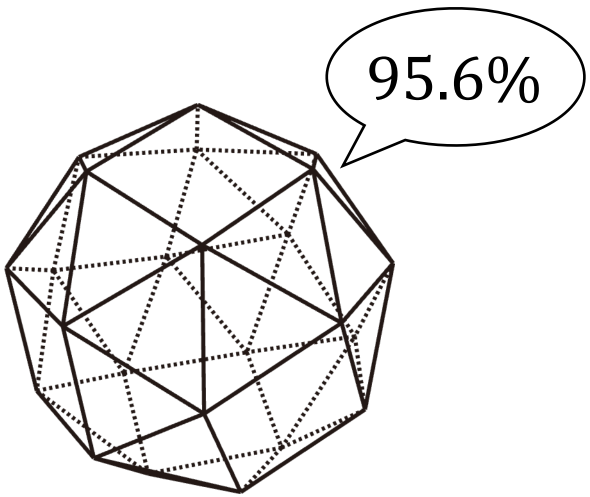
m-gonal Archimedean antiprisms ($m \geq 3$)



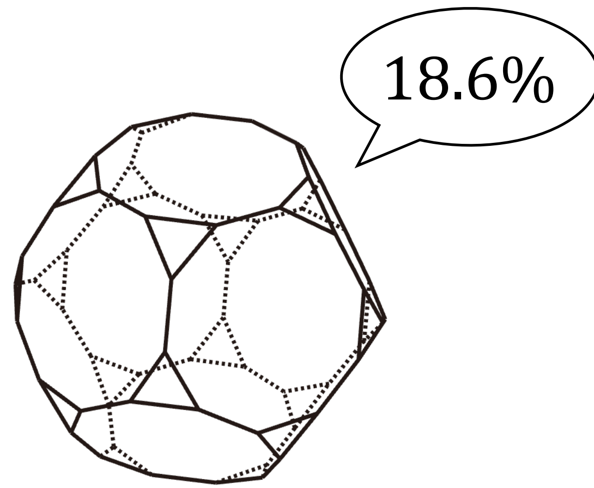
Johnson solids
(92 types)

Our results in Archimedean solids

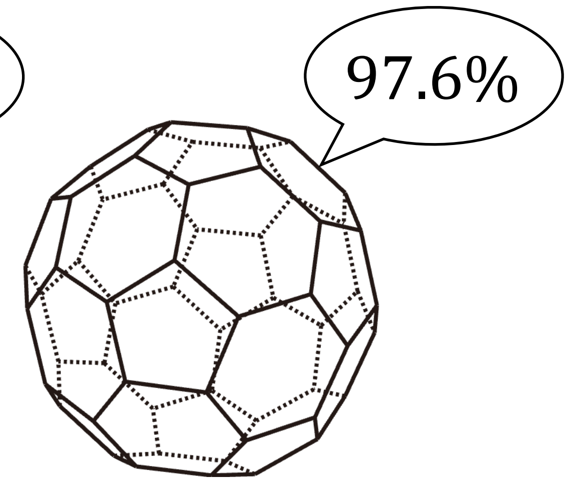
Archimedean solids	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
Sunb cube	89,904,012,853,248	85,967,688,920,076
Truncated dodecahedron	4,982,259,375,000,000,000	931,603,573,888,462,350
Truncated Icosahedron	375,291,866,372,898,816,000	366,359,657,802,290,909,354



Sunb cube



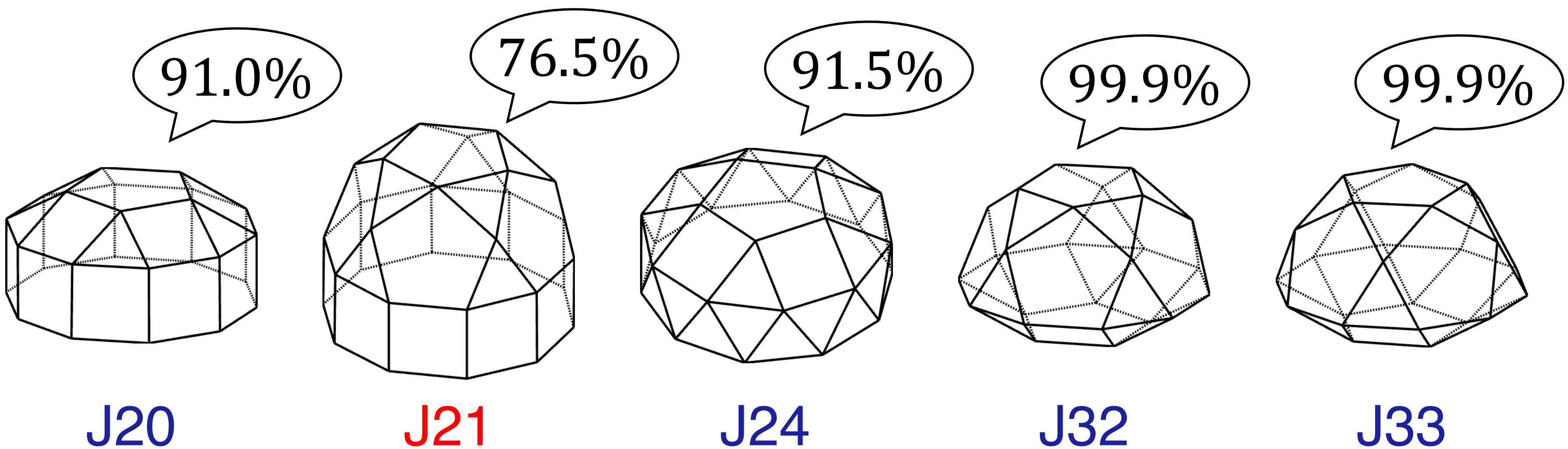
Truncated dodecahedron



Truncated Icosahedron

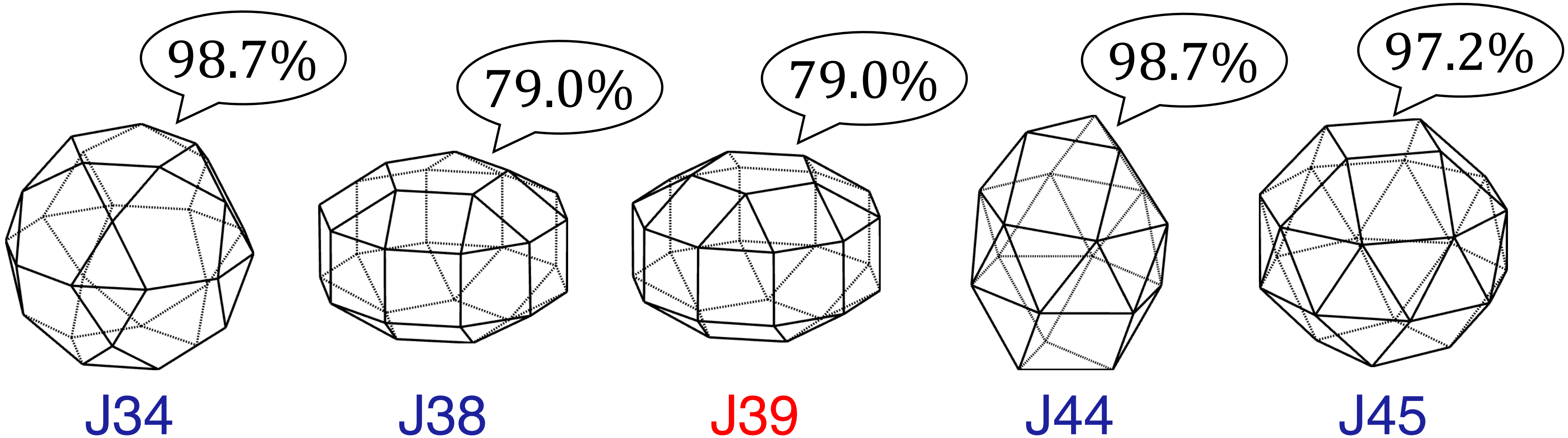
Our results in Johnson solids (1)

Johnson solids	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
J20	29,821,320,745	27,158,087,415
J21	8,223,103,375,490	6,297,186,667,720
J24	5,996,600,870,820	5,492,624,228,190
J32	699,537,024,120	699,433,603,320
J33	745,208,449,920	745,198,979,400



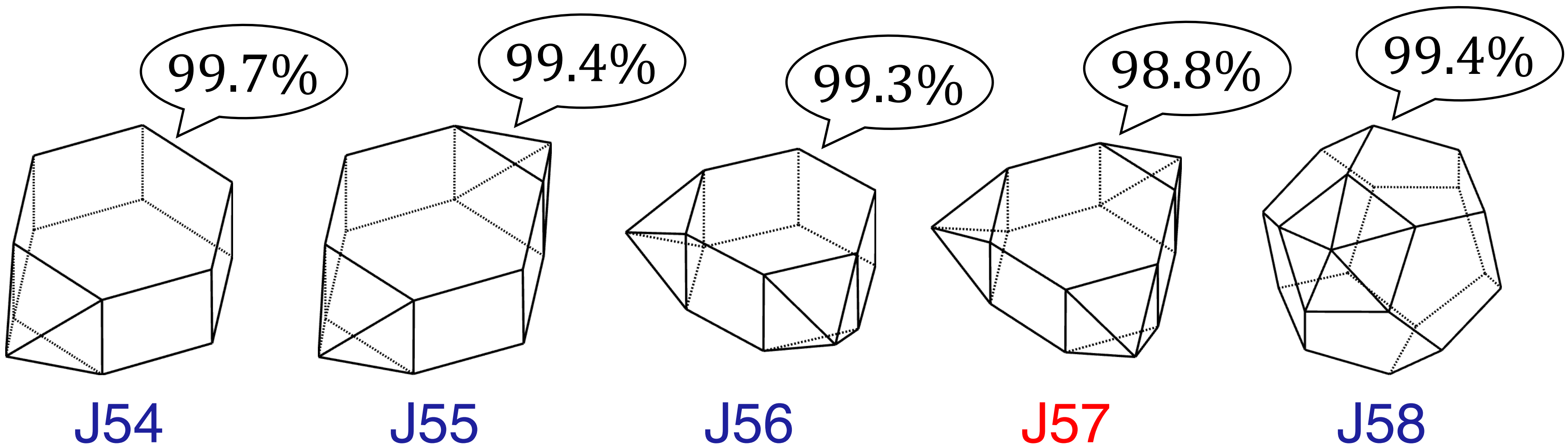
Our results in Johnson solids (2)

Johnson solids	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
J34	193,003,269,869,040	190,653,702,525,040
J38	270,745,016,304,350	214,085,775,357,270
J39	272,026,496,000,000	215,087,798,524,180
J44	5,295,528,588	5,231,781,954
J45	13,769,880,349,680	13,386,219,088,644



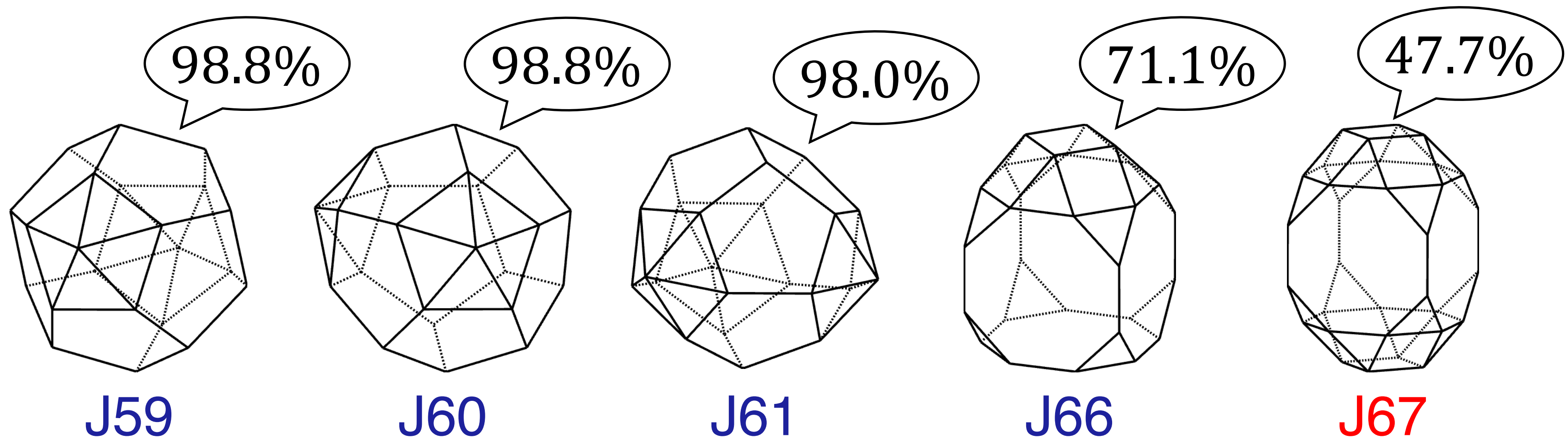
Our results in Johnson solids (3)

Johnson solids	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
J54	75,973	75,749
J55	709,632	705,144
J56	707,232	702,520
J57	6,531,840	6,457,860
J58	92,724,962	92,219,782



Our results in Johnson solids (4)

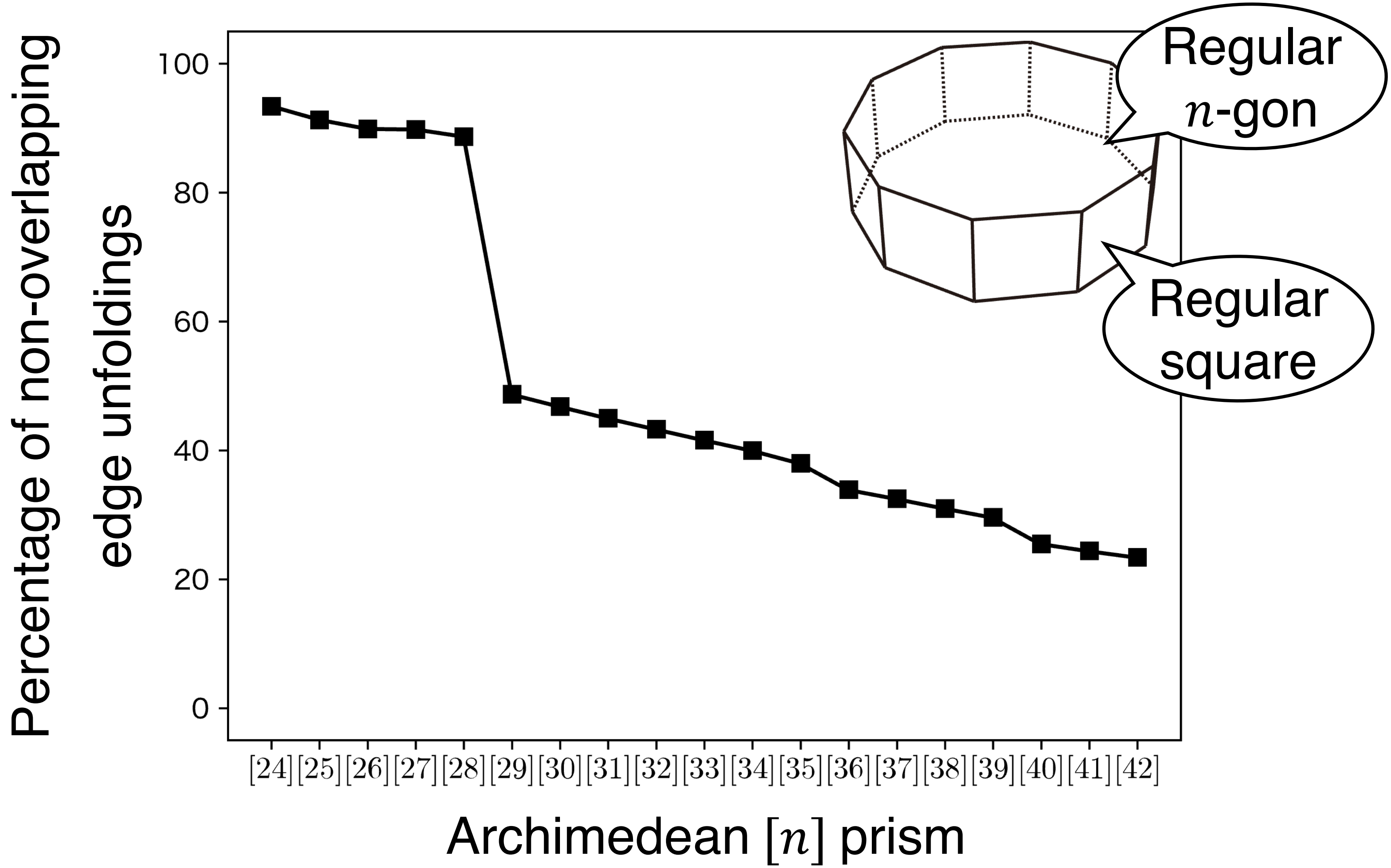
Johnson solids	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
J59	1,651,482,010	1,632,941,030
J60	1,641,317,568	1,621,738,522
J61	28,745,798,400	28,183,512,978
J66	54,921,311,280	39,055,563,000
J67	90,974,647,120,896	43,437,626,181,464



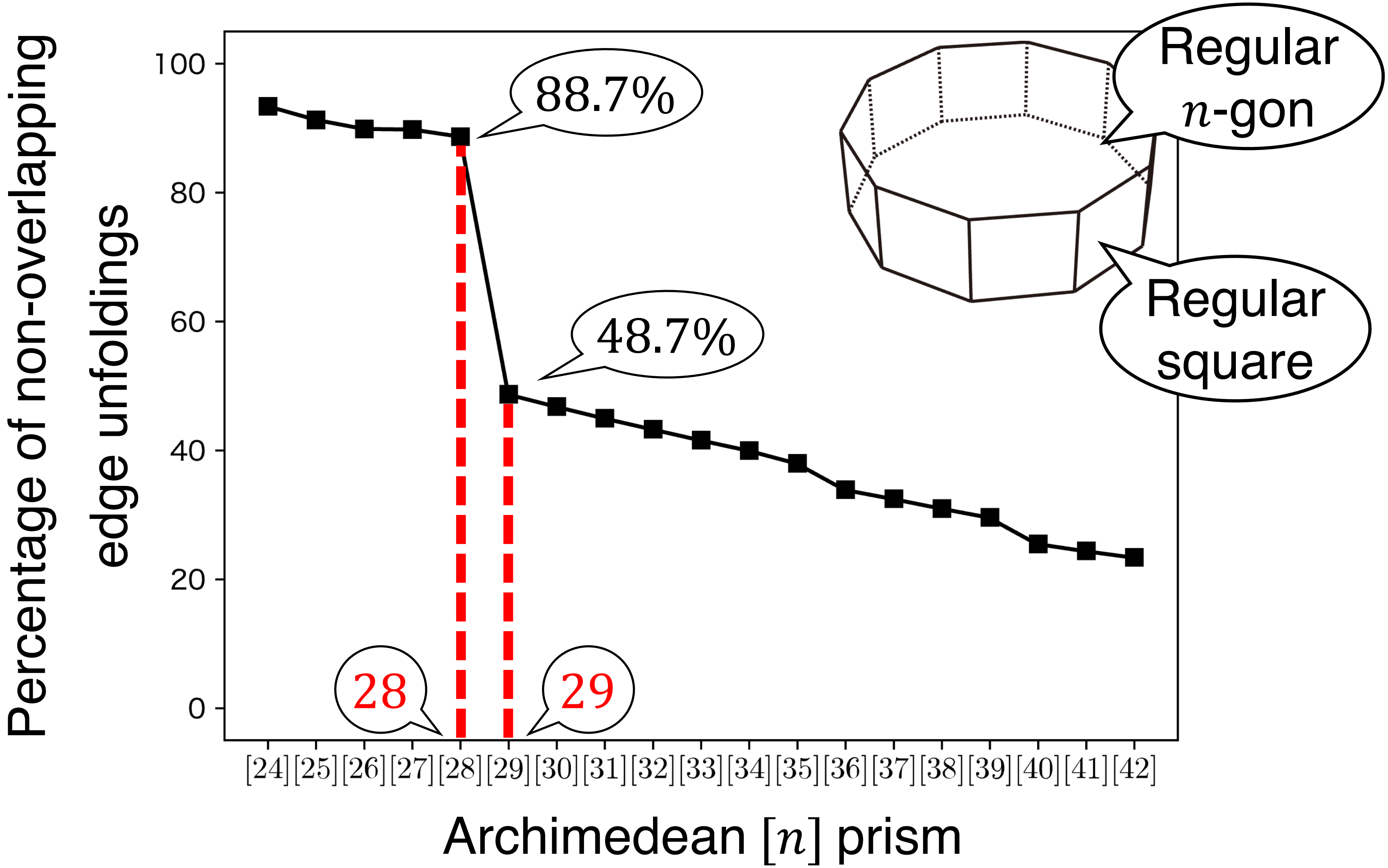
Our results in Archimedean prisms

Archimedean n prisms	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
24 prism	639,620,518,118,400	597,547,526,278,102
25 prism	2,486,558,615,814,025	2,270,951,013,426,530
26 prism	9,651,161,613,824,796	8,680,724,875,408,140
27 prism	37,403,957,244,654,675	33,593,039,475,394,300
28 prism	144,763,597,316,784,768	128,484,071,528,042,000
29 prism	559,560,282,425,278,229	273,052,412,937,434,000
30 prism	2,160,318,004,043,512,500	1,012,562,467,010,050,000
31 prism	8,331,163,769,982,715,231	3,755,308,489,795,020,000
32 prism	32,095,304,749,163,937,792	13,910,558,120,316,400,000
33 prism	123,524,473,883,545,449,825	51,464,102,399,119,800,000
34 prism	474,969,297,739,230,927,564	190,077,650,531,107,000,000
35 prism	1,824,745,126,233,358,110,635	694,876,093,525,600,000,000
36 prism	7,004,614,136,879,907,849,600	2,380,408,316,368,090,000,000
37 prism	26,867,730,730,869,118,775,917	8,734,608,096,670,700,000,000
38 prism	102,981,783,095,242,000,000,000	31,927,951,665,245,000,000,000
39 prism	394,447,279,575,099,000,000,000	117,143,971,138,672,000,000,000
40 prism	1,509,843,372,596,510,000,000,000	385,268,000,158,423,000,000,000
41 prism	5,775,682,482,451,350,000,000,000	1,409,268,044,697,380,000,000,000
42 prism	22,080,875,606,379,200,000,000,000	5,178,957,938,434,480,000,000,000

Our results in Archimedean prisms



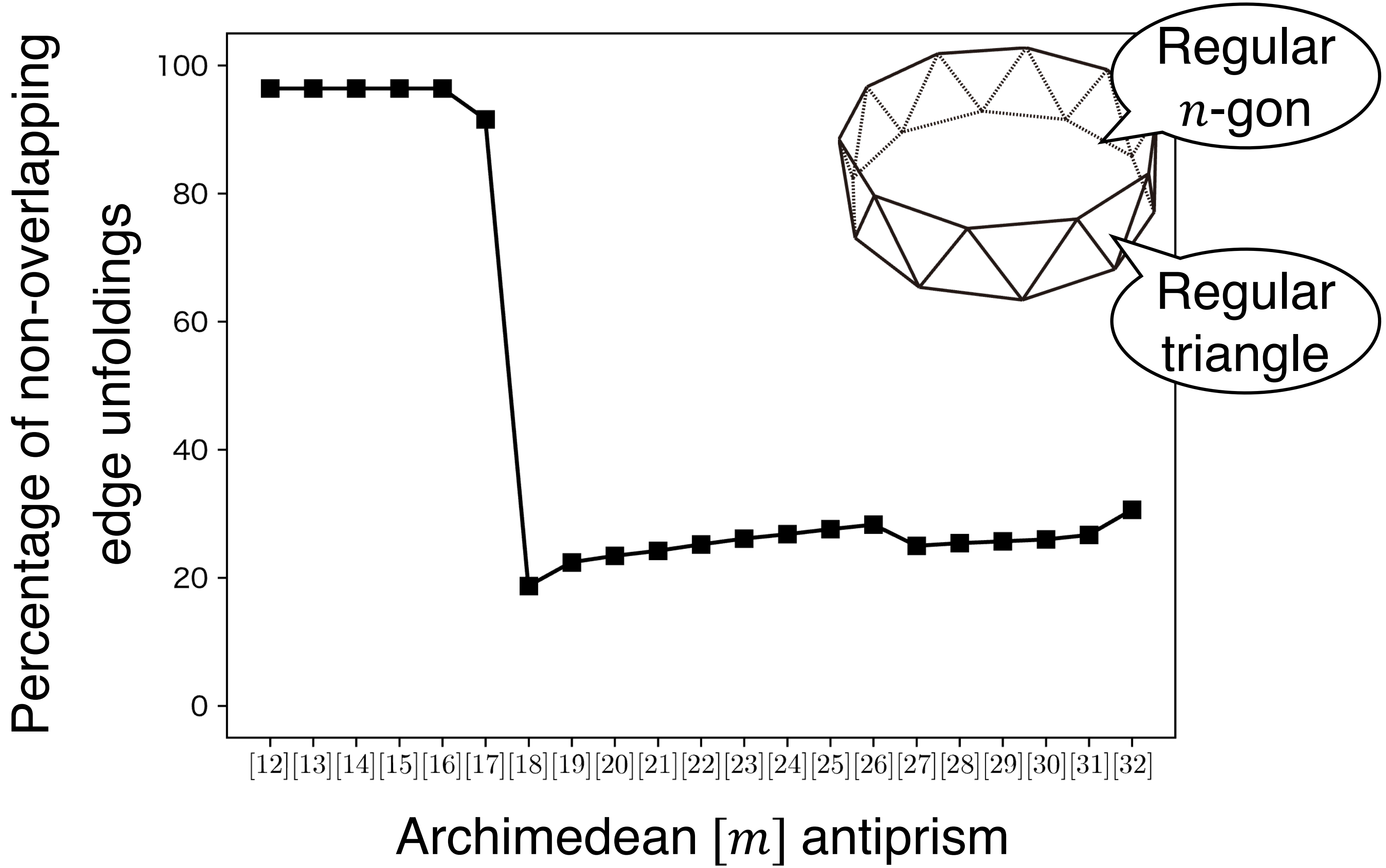
Our results in Archimedean prisms



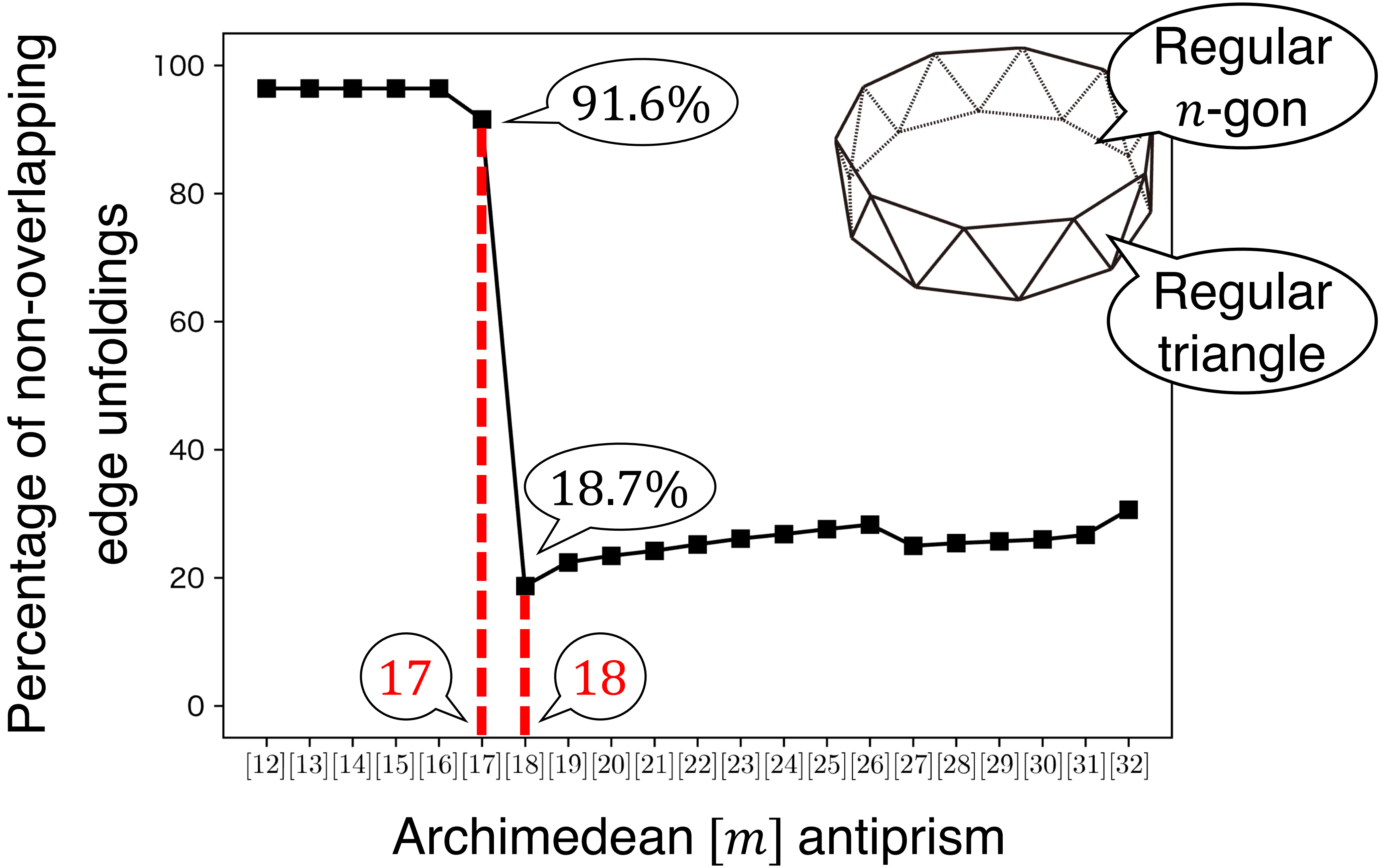
Our results in Archimedean antiprisms

Archimedean m antiprisms	# edge unfoldings [T. Horiyama et al., 2013]	# non-overlapping edge unfoldings
12 antiprism	51,599,794,176	49,743,531,024
13 antiprism	383,142,771,674	369,359,503,344
14 antiprism	2,828,107,288,188	2,726,368,290,352
15 antiprism	20,768,716,848,000	20,021,578,135,380
16 antiprism	151,840,963,183,392	146,378,600,602,880
17 antiprism	1,105,779,284,582,140	1,013,491,325,102,940
18 antiprism	8,024,954,790,380,540	1,501,154,452,845,900
19 antiprism	58,059,628,319,357,300	13,038,527,513,687,400
20 antiprism	418,891,171,182,561,000	98,027,112,294,661,100
21 antiprism	3,014,678,940,049,370,000	732,157,627,679,302,000
22 antiprism	21,646,865,272,061,200,000	5,463,662,878,677,080,000
23 antiprism	155,113,904,634,576,000,000	40,508,628,620,513,100,000
24 antiprism	1,109,391,149,998,440,000,000	298,293,520,418,401,000,000
25 antiprism	7,920,708,398,483,720,000,000	2,188,171,009,006,050,000,000
26 antiprism	56,460,916,728,463,100,000,000	15,982,421,259,908,100,000,000
27 antiprism	401,873,068,071,158,000,000,000	100,599,073,148,261,000,000,000
28 antiprism	2,856,496,726,273,360,000,000,000	725,756,982,845,834,000,000,000
29 antiprism	20,277,959,821,998,000,000,000,000	5,224,196,129,087,410,000,000,000
30 antiprism	143,779,866,504,299,000,000,000,000	37,518,568,275,655,300,000,000,000
31 antiprism	1,018,331,261,238,040,000,000,000,000	272,565,329,790,964,000,000,000,000
32 antiprism	7,204,899,406,395,020,000,000,000,000	2,207,488,168,172,480,000,000,000,000

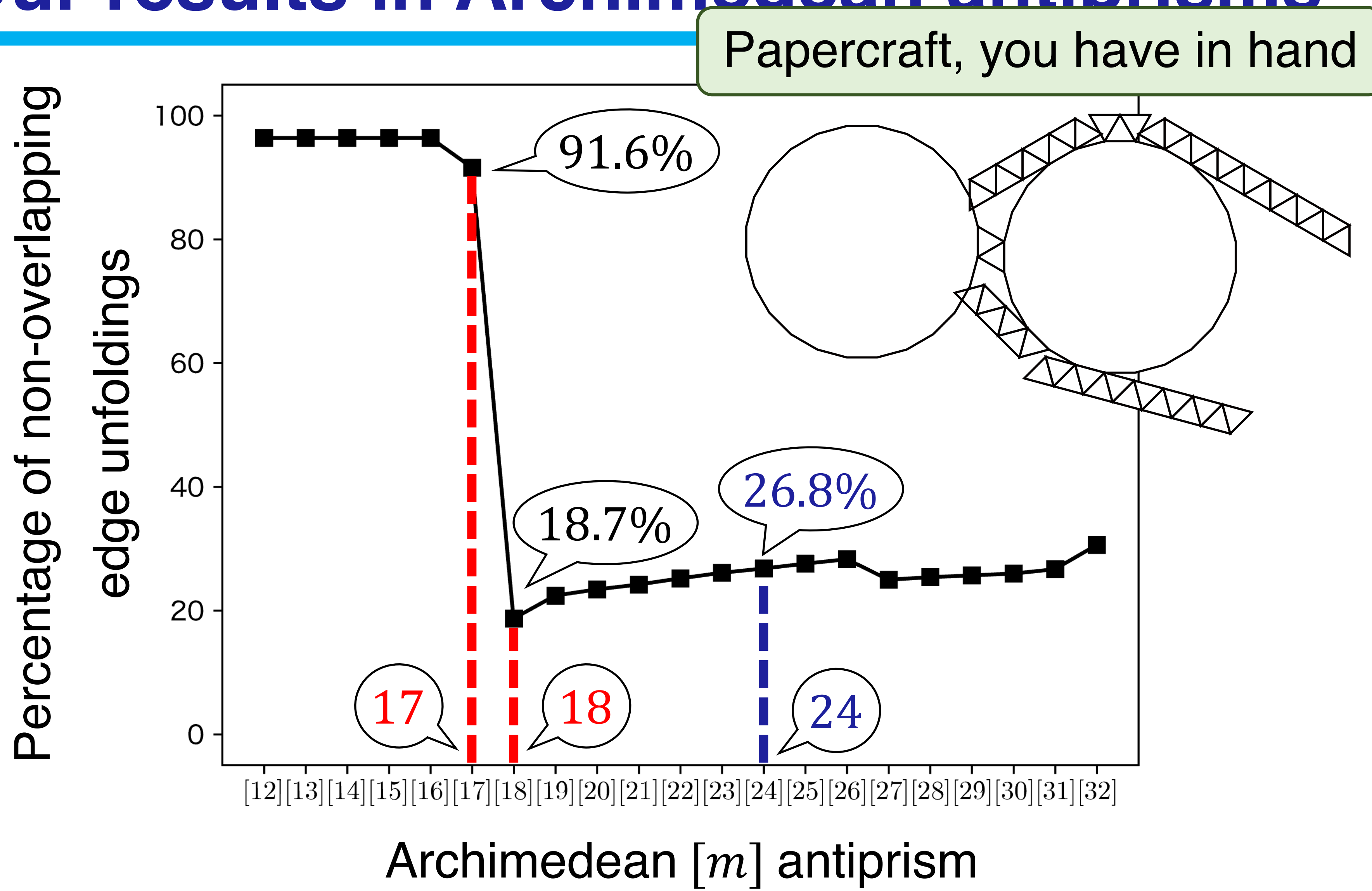
Our results in Archimedean antiprisms



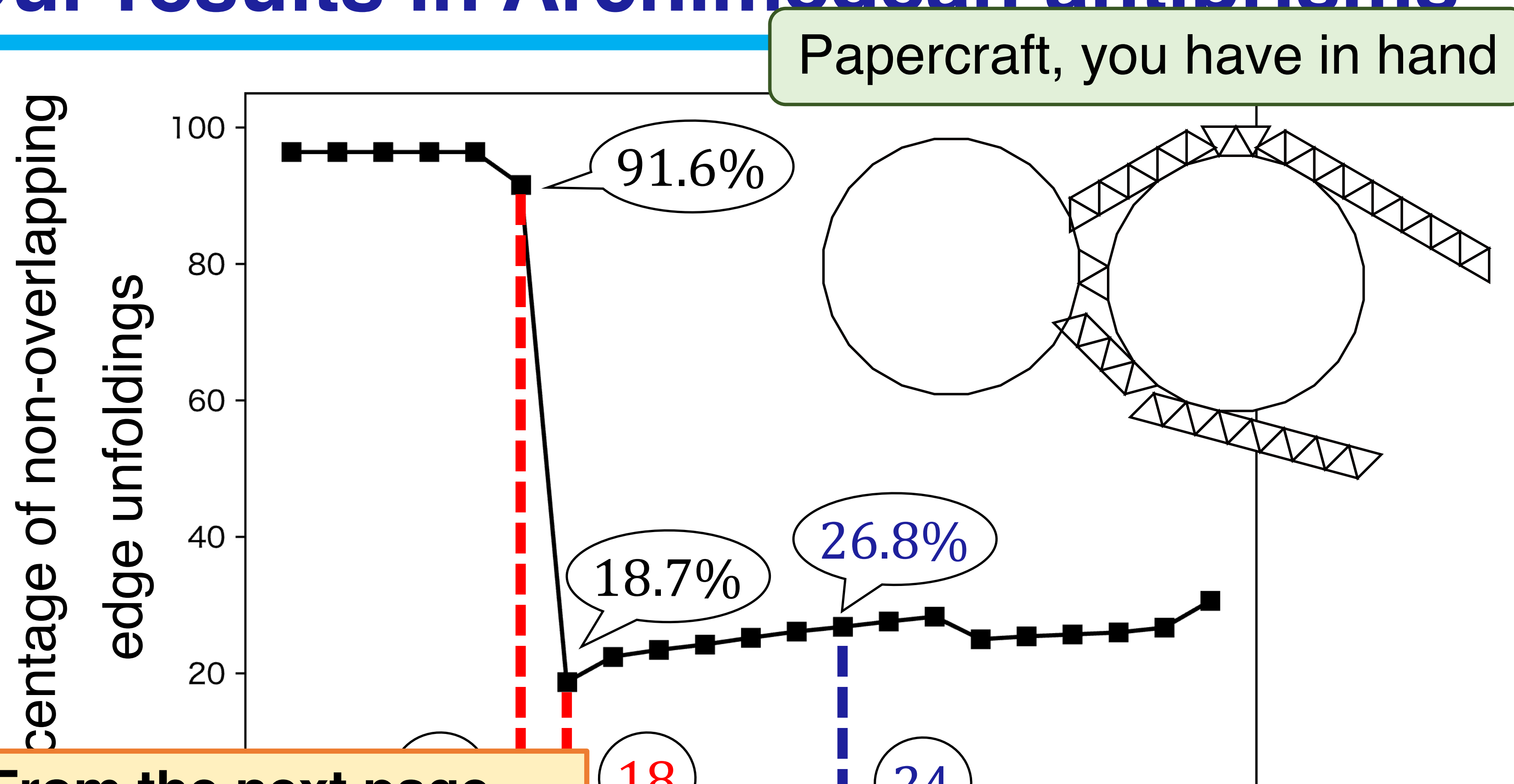
Our results in Archimedean antiprisms



Our results in Archimedean antiprisms



Our results in Archimedean antiprisms



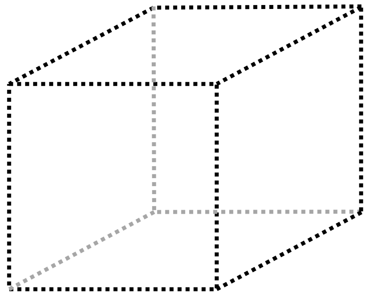
From the next page …

1. How to count # edge unfoldings (include overlapping ones)
2. How to count # non-overlapping edge unfoldings

edge unfoldings including overlaps

Theorem 3 (Dual) [R. Uehara, 2020]

For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.

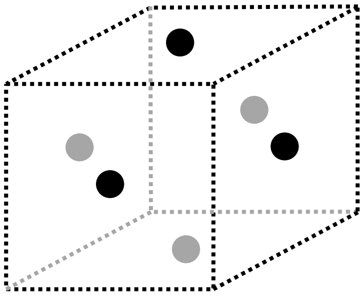


Polyhedron Q

edge unfoldings including overlaps

Theorem 3 (Dual) [R. Uehara, 2020]

For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.

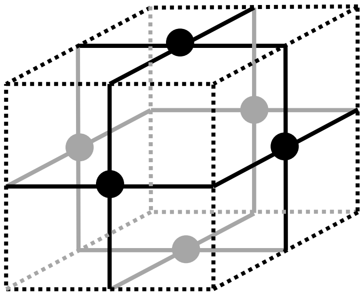


Polyhedron Q

edge unfoldings including overlaps

Theorem 3 (Dual) [R. Uehara, 2020]

For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.



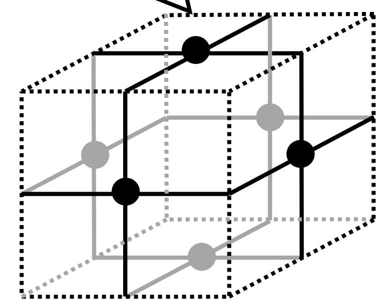
Polyhedron Q

edge unfoldings includ s

Graph
$G_Q = (V_Q, E_Q)$

Theorem 3 (Dual) [R. Uehara, 2020]

For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.



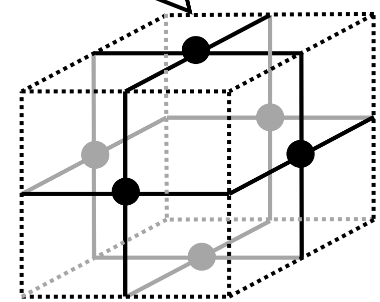
Polyhedron Q

edge unfoldings includ s

Graph
$G_Q = (V_Q, E_Q)$

Theorem 3 (Dual) [R. Uehara, 2020]

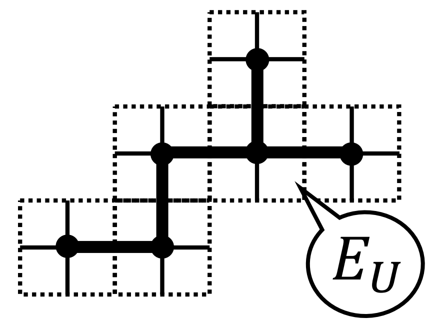
For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.



Polyhedron Q

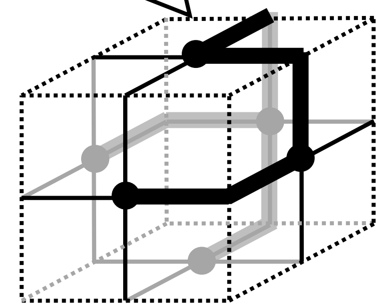
Theorem 4 [R. Uehara, 2020]

Let $E_U (\subset E_Q)$ be the set of edges that are not cut when unfolding Q. Then, E_U corresponds to a spanning tree in graph G_Q.



Edge unfolding U

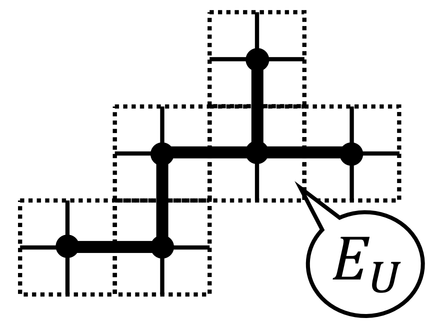
edge unfoldings includ s



Polyhedron Q

Theorem 3 (Dual) [R. Uehara, 2020]

For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.



Edge unfolding U

Theorem 4 [R. Uehara, 2020]

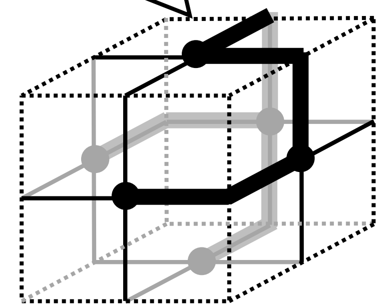
Let $E_U (\subset E_Q)$ be the set of edges that are not cut when unfolding Q. Then, E_U corresponds to a spanning tree in graph G_Q.

edge unfoldings includ… s

Graph
$G_Q = (V_Q, E_Q)$

Theorem 3 (Dual) [R. Uehara, 2020]

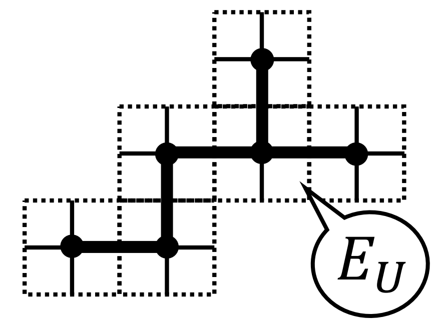
For a polyhedron Q, let V_Q be the set of faces, and E_Q be the set of edges connecting adjacent faces. Then Q can be seen as $G_Q = (V_Q, E_Q)$.



Polyhedron Q

Theorem 4 [R. Uehara, 2020]

Let $E_U (\subset E_Q)$ be the set of edges that are not cut when unfolding Q. Then, E_U corresponds to a spanning tree in graph G_Q.



Edge unfolding U

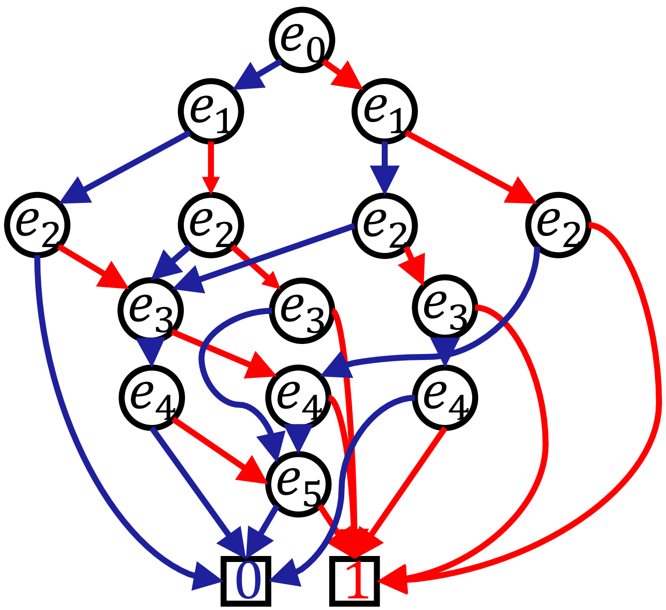
In short …

$\# \, Q$'s edge unfoldings $= \# \, G_Q$'s spanning trees

edge unfoldings including overlaps

G_Q's spanning tree can be counted using ZDD

[J. Kawahara et al, 2017]

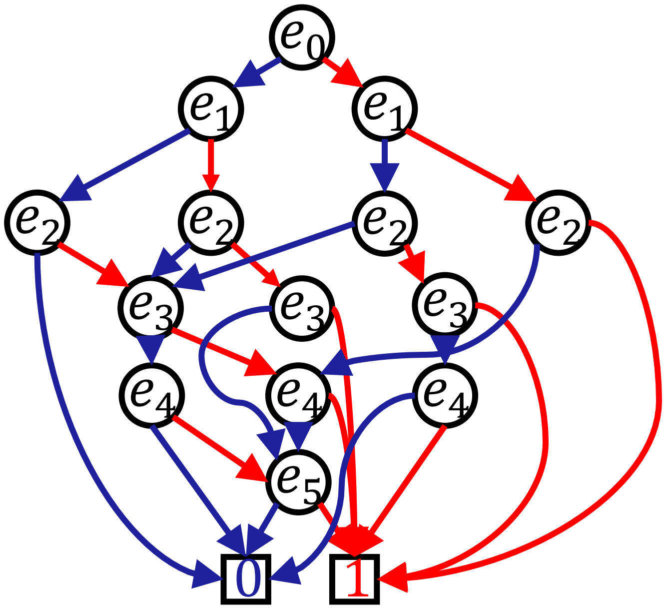


e… erlaps

A data structure that compactly represents a family of sets using a directed acyclic graph.

\# G_Q's spanning tree can be counted using ZDD

[J. Kawahara et al, 2017]

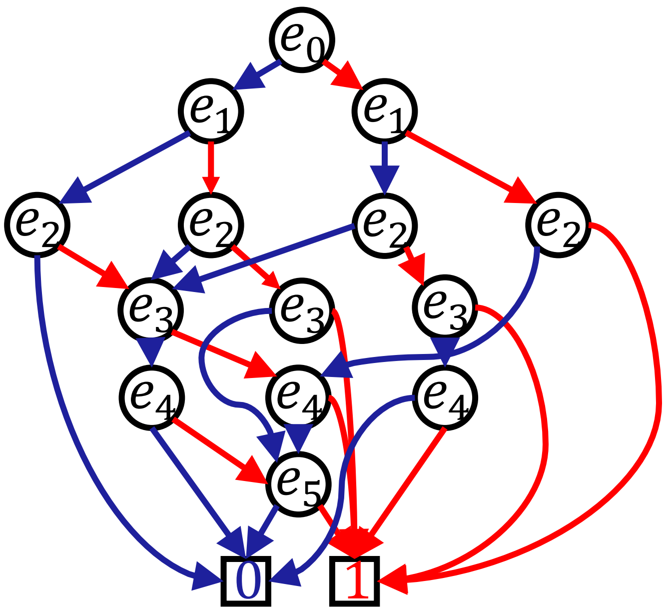


e erlaps

A data structure that compactly represents a family of sets using a directed acyclic graph.

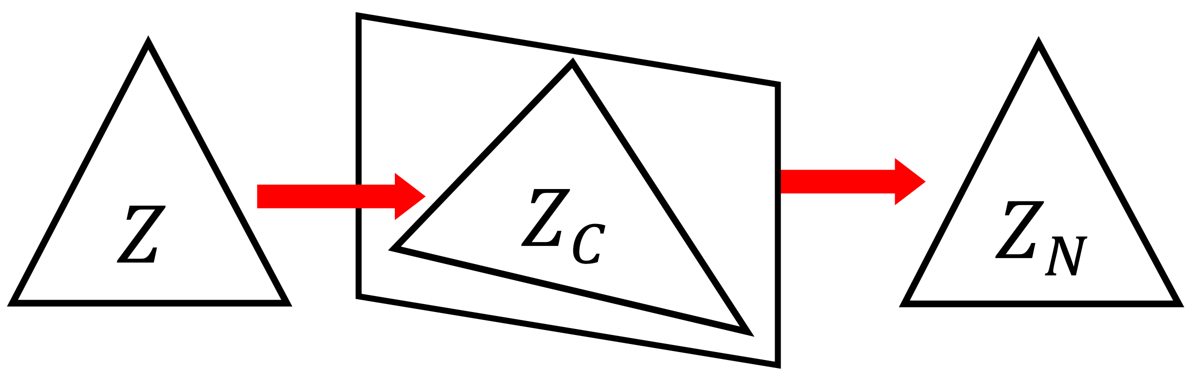
G_Q's spanning tree can be counted using ZDD

[J. Kawahara et al, 2017]



Subsetting method [H. Iwashita et al., 2013]

The ZDD operation generates a new ZDD Z_N by extracting family of sets satisfying a constraint C from a ZDD Z.

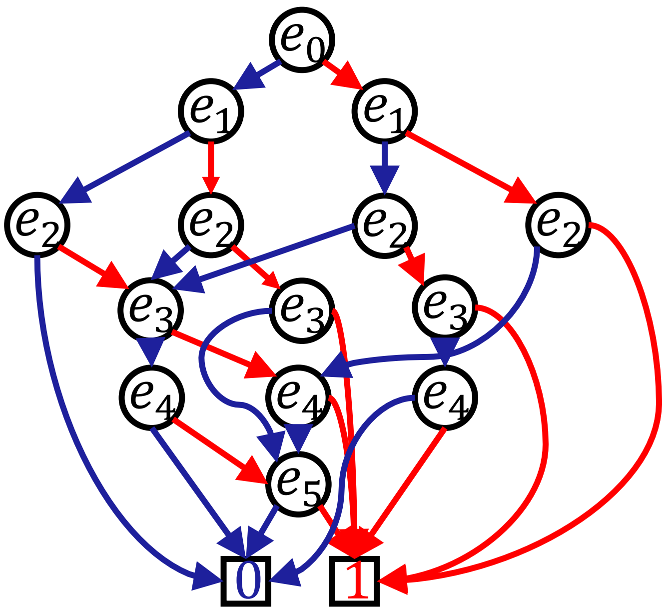


e… erlaps

A data structure that compactly represents a family of sets using a directed acyclic graph.

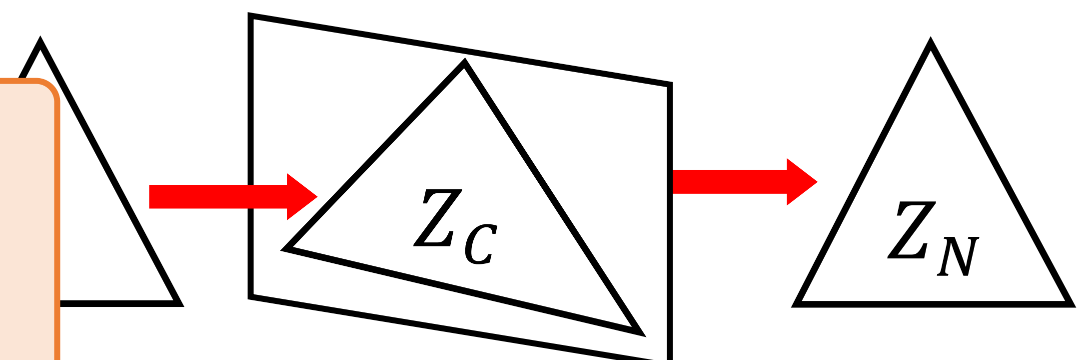
\# G_Q's spanning tree can be counted using ZDD

[J. Kawahara et al, 2017]



Subsetting method [H. Iwashita et al., 2013]

The ZDD operation generates a new ZDD Z_N by extracting family of sets satisfying a constraint C from a ZDD Z.



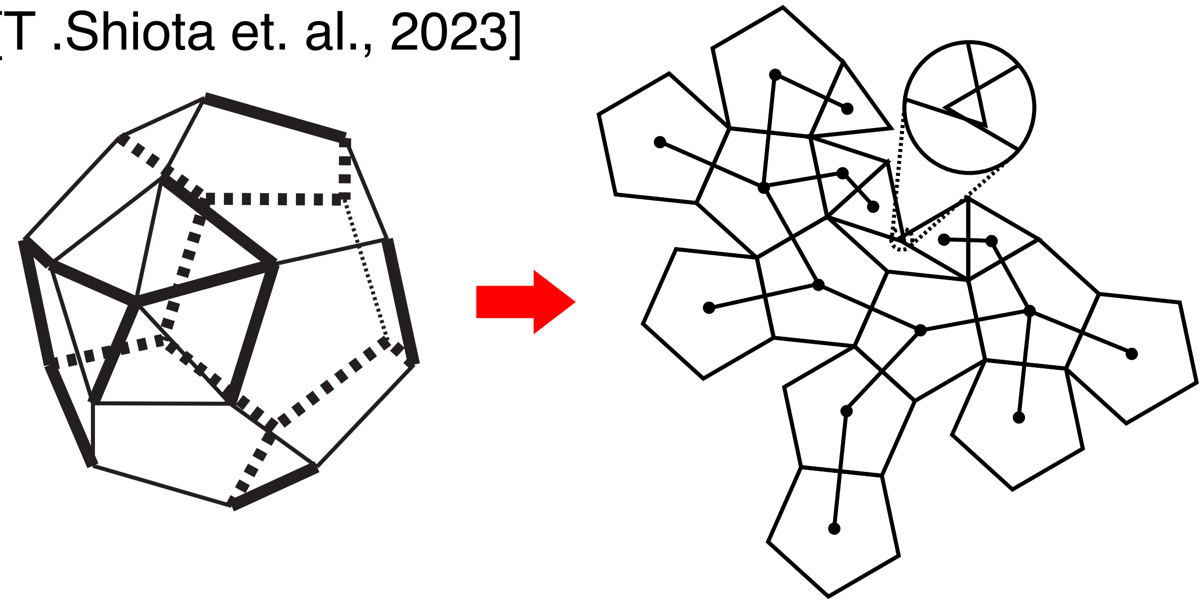
In this study …

Count # non-overlapping edge unfoldings using the subsetting method.

non-overlapping edge unfoldings

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]



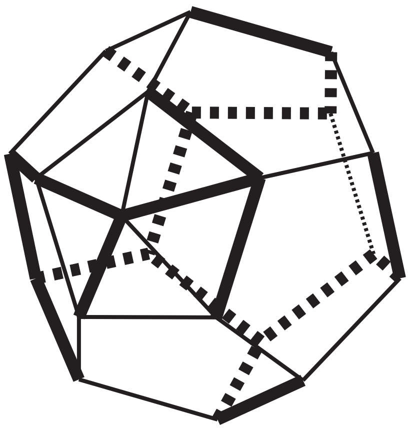
Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8, \cdots, e_{21}\}$

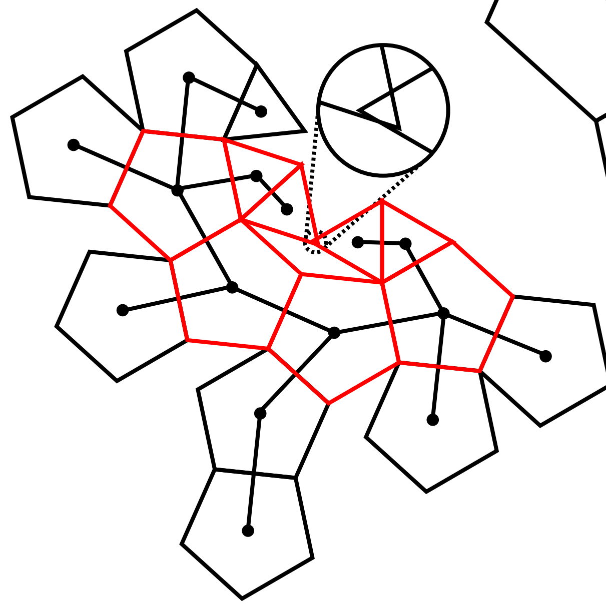
no

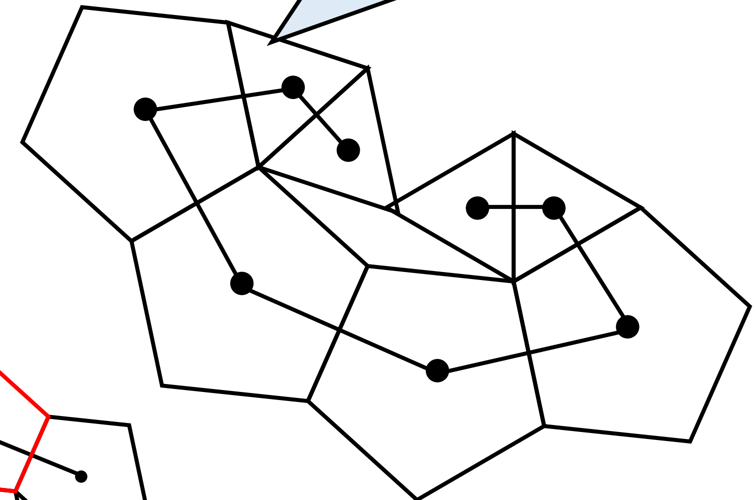
Minimal overlapping partial edge unfolding (**MOPE**) C_1

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]



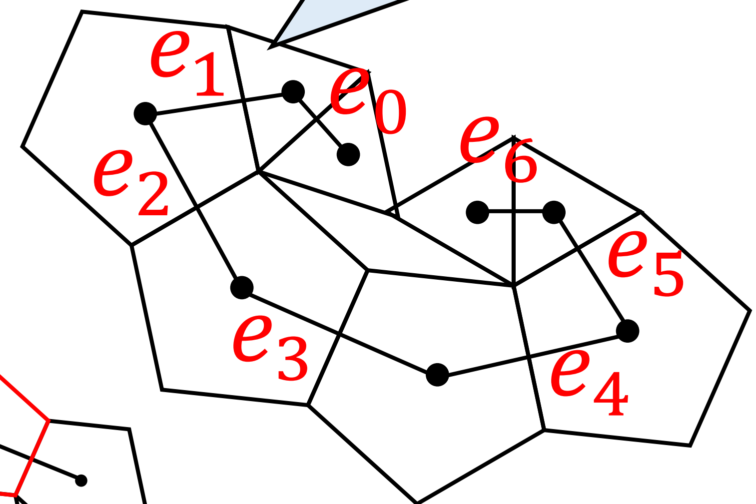




Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8, \cdots, e_{21}\}$

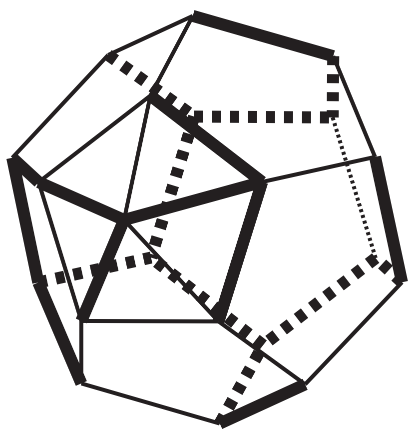
no

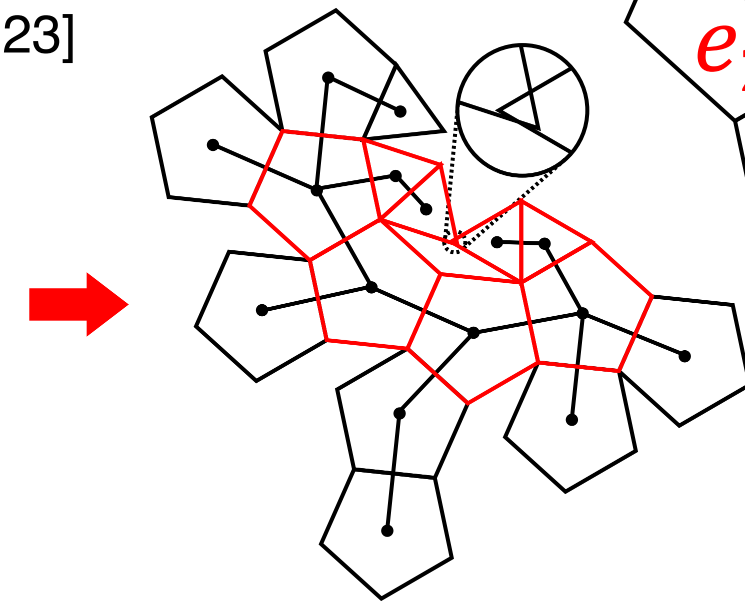
Minimal overlapping partial edge unfolding (**MOPE**) C_1



[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]





Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8, \cdots, e_{21}\}$

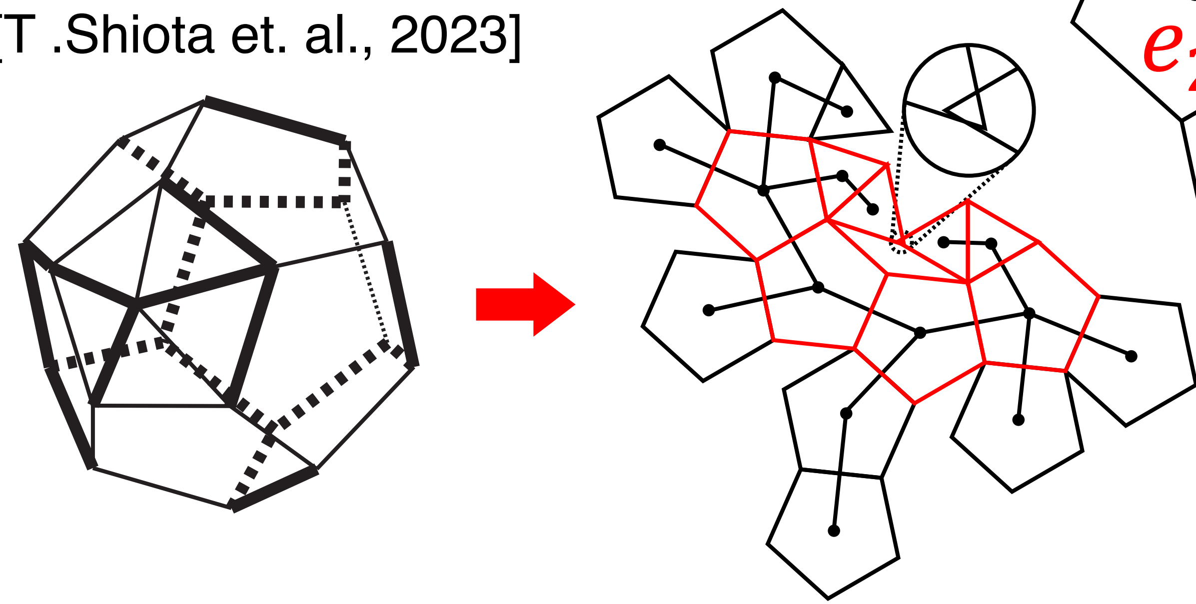
MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

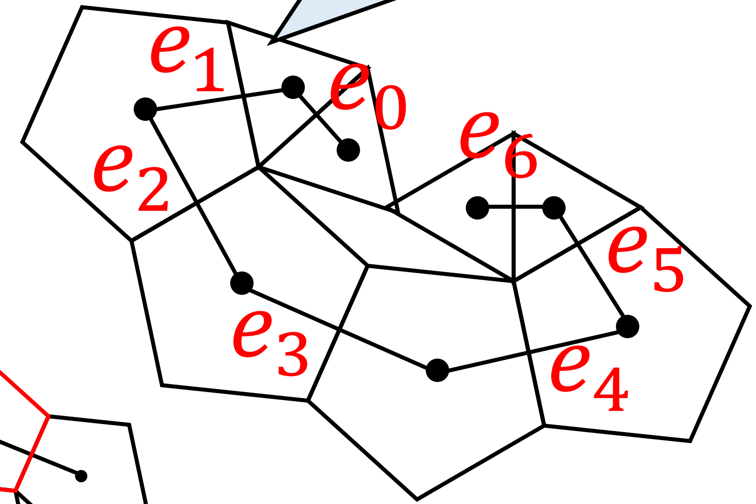
no

Minimal overlapping partial edge unfolding (**MOPE**) C_1

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]





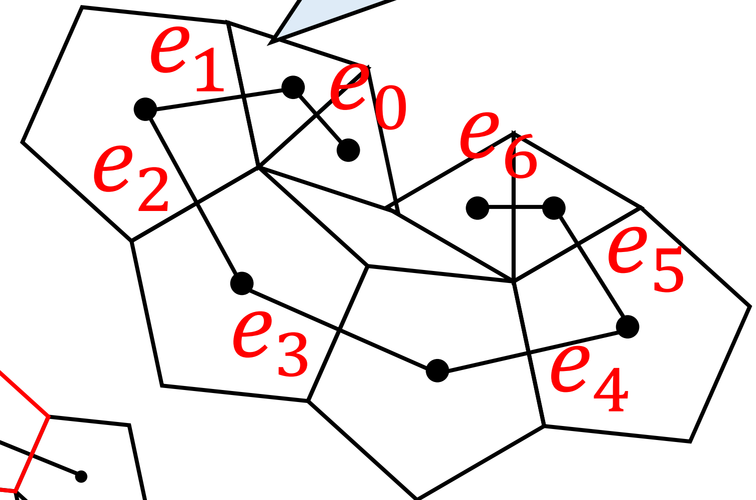
Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8, \cdots, e_{21}\}$

MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

➢ $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

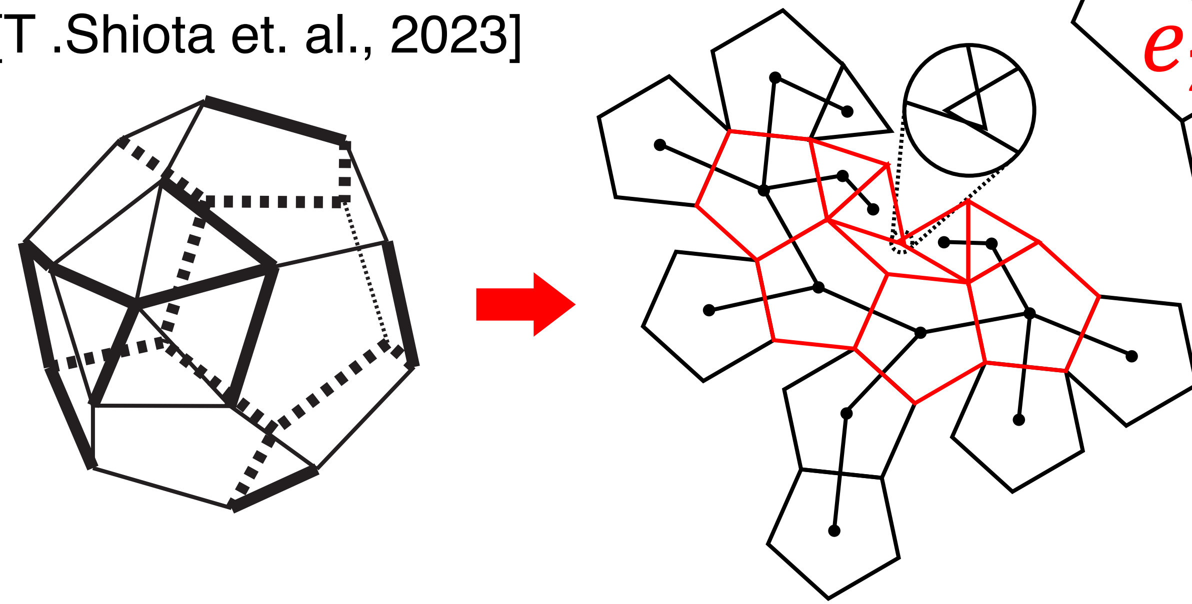
no

Minimal overlapping partial edge unfolding (**MOPE**) C_1



[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]



Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8, \cdots, e_{21}\}$

MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

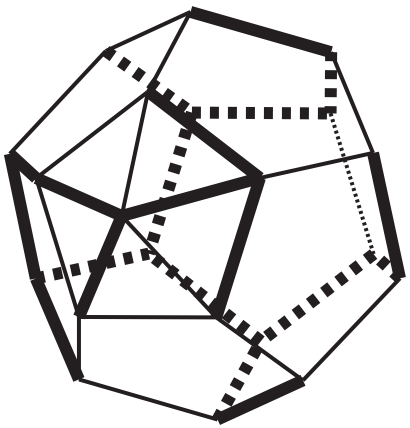
➢ $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

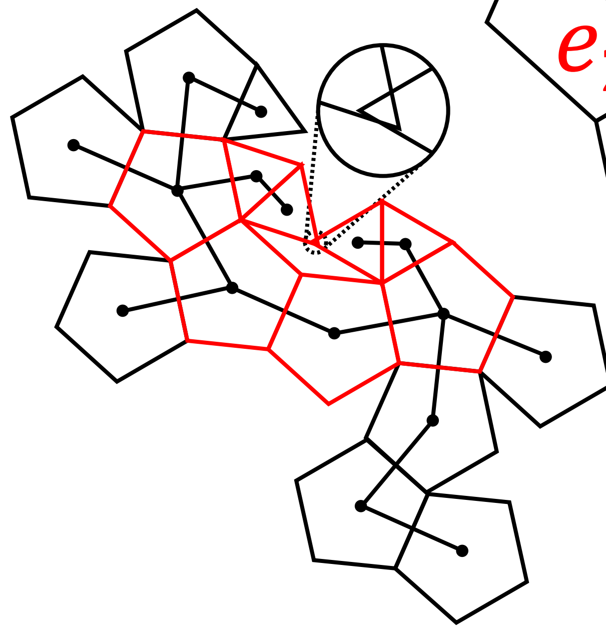
no

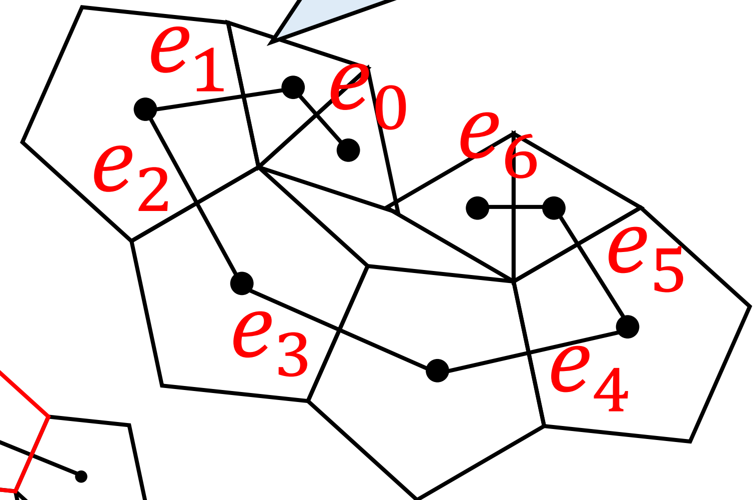
Minimal overlapping partial edge unfolding (**MOPE**) C_1

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]







Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, a, \cdots, x\}$

MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

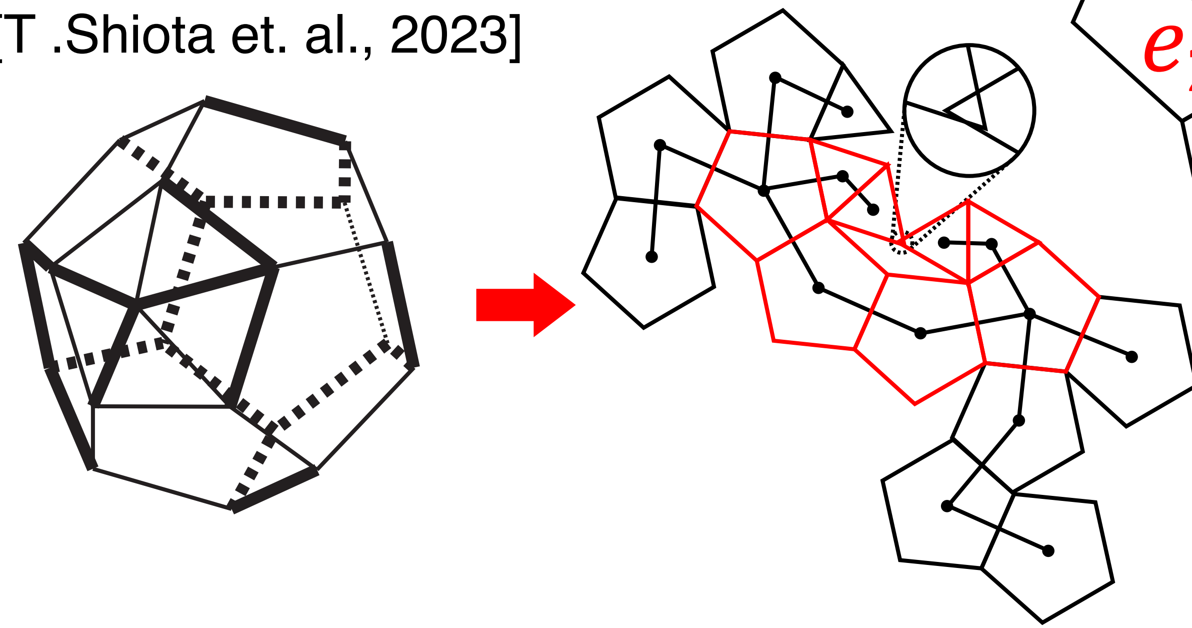
- $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

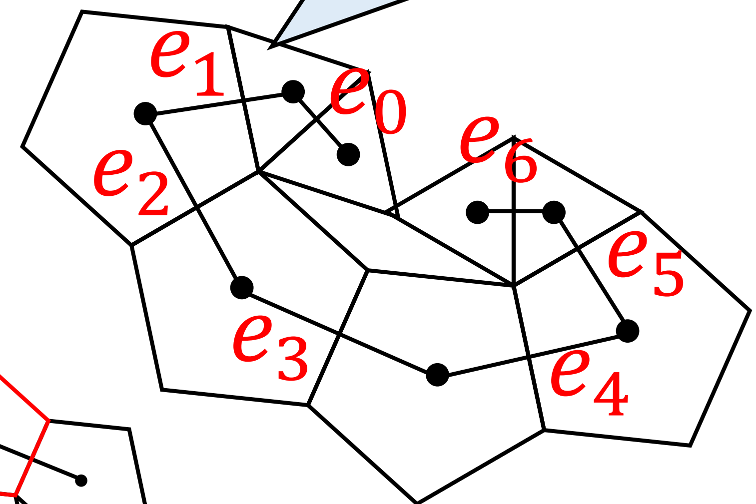
no

Minimal overlapping partial edge unfolding (**MOPE**) C_1

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]





Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, a, \cdots, y\}$

MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

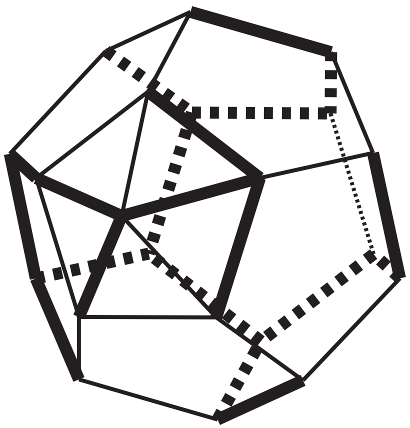
- $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

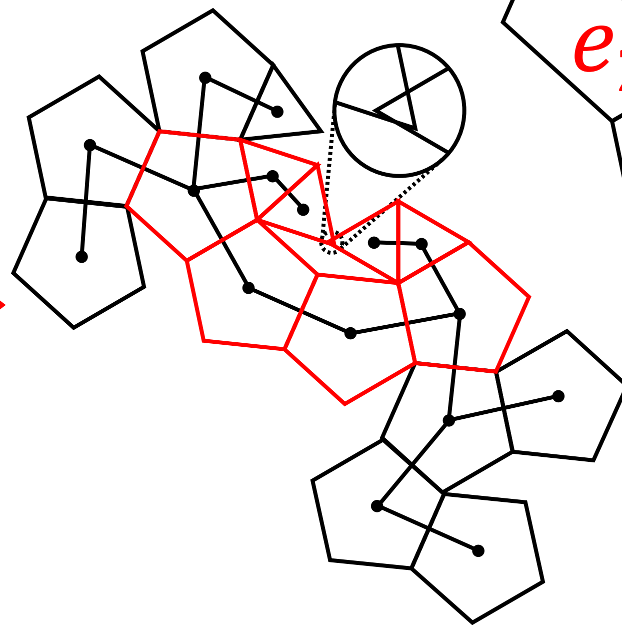
no

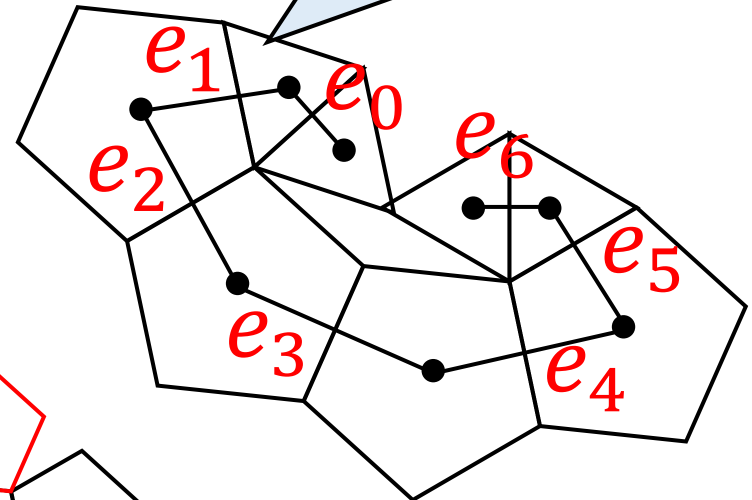
Minimal overlapping partial edge unfolding (**MOPE**) C_1

[Ex.] Overlapping edge unfoldings

[T .Shiota et. al., 2023]







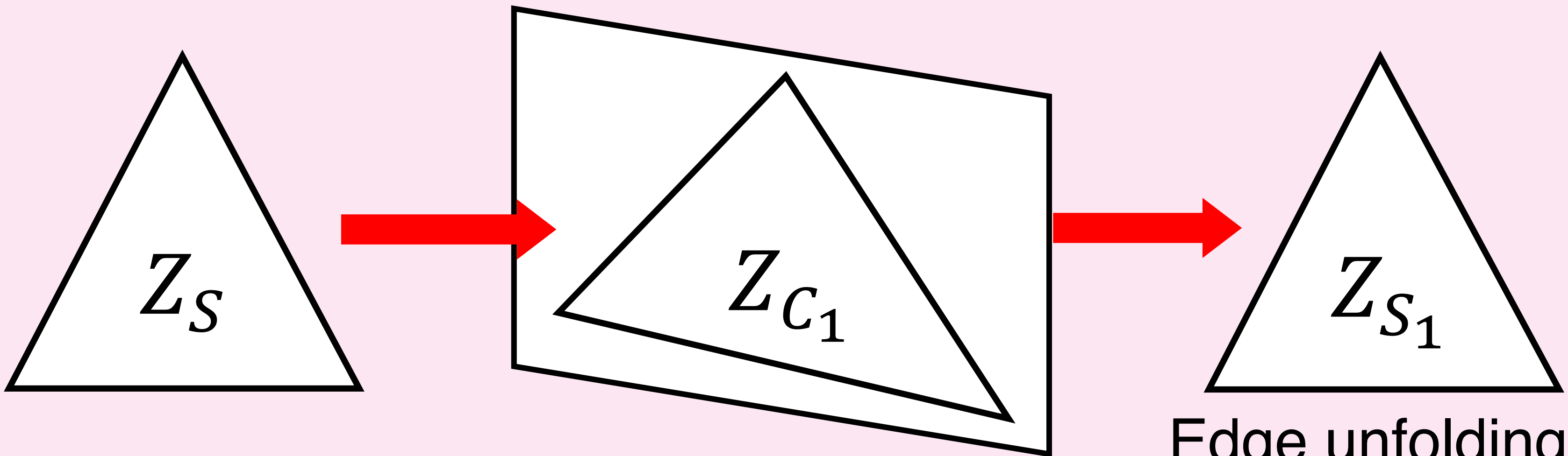
Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, b, \cdots, y\}$

MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

➢ $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

Minimal overlapping partial edge unfolding (**MOPE**) C_1

Subsetting method



All edge unfoldings

Filter to remove edge unfoldings that include C_1

Edge unfoldings excluding those that include C_1

Spanning tree's edge set $E_T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, b, \cdots, y\}$

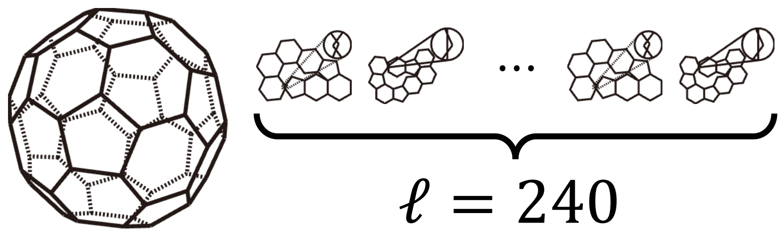
MOPE C_1's edge set $E_{C_1} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$

- $E_{C_1} \subset E_T$ ➡ Edge unfoldings always have overlap

non-overlapping edge unfoldings

- MOPE C_k can be enumerated by using rotational unfolding [T. Shiota et. al., 2023]

  ℓ : # MOPEs in a polyhedron Q

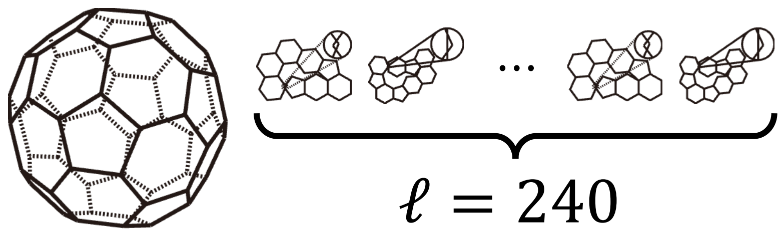


Truncated icosahedron

non-overlapping edge unfoldings

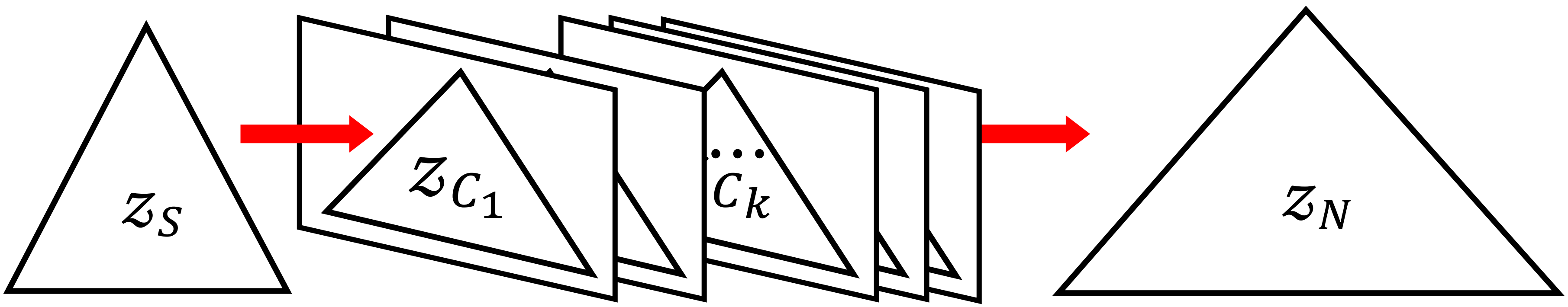
- MOPE C_k can be enumerated by using rotational unfolding [T. Shiota et. al., 2023]

  ℓ : # MOPEs in a polyhedron Q



Truncated icosahedron

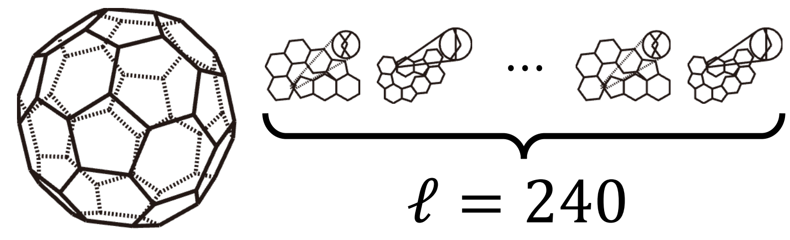
- Apply the subsetting method to MOPE C_k $(1 \leq k \leq \ell)$



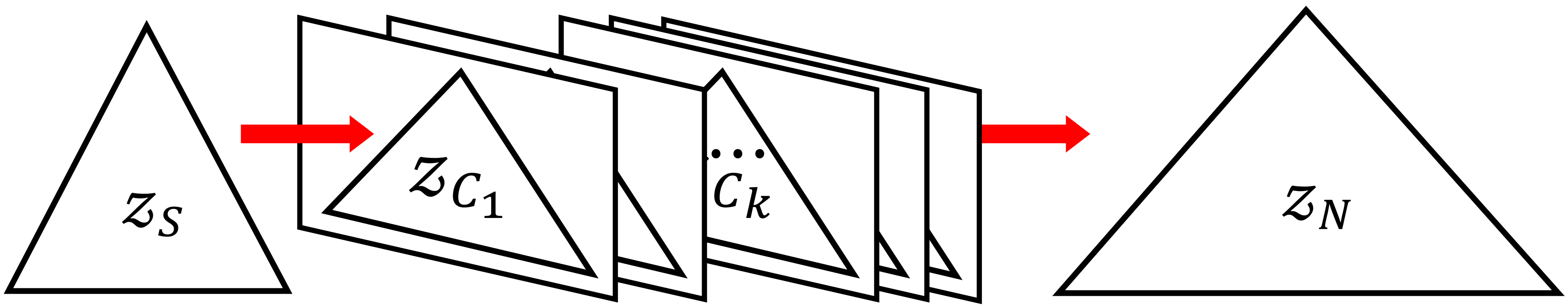
Edge unfoldings excluding those that include $C_1 \sim C_\ell$

non-overlapping edge unfoldings

- MOPE C_k can be enumerated by using rotational unfolding [T. Shiota et. al., 2023]
  ℓ : # MOPEs in a polyhedron Q



- Apply the subsetting method to MOPE C_k $(1 \leq k \leq \ell)$



Non-overlapping edge unfolding

Edge unfoldings excluding those that include $C_1 \sim C_\ell$

Summary

- We developed an algorithm for counting the number of non-overlapping edge unfoldings in convex polyhedra.
  - ✓ Focus on MOPEs
  - ✓ Using the subsetting method (the ZDD operation)

Summary

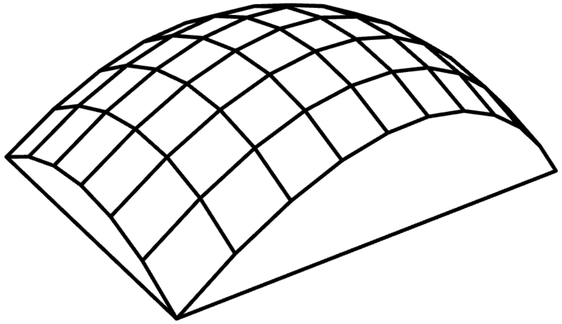
- We developed an algorithm for counting the number of non-overlapping edge unfoldings in convex polyhedra.
  - ✓ Focus on MOPEs
  - ✓ Using the subsetting method (the ZDD operation)
- We applied this algorithm for 63 types convex regular-faced polyhedra.

Summary

- We developed an algorithm for counting the number of non-overlapping edge unfoldings in convex polyhedra.
  - ✓ Focus on MOPEs
  - ✓ Using the subsetting method (the ZDD operation)
- We applied this algorithm for 63 types convex regular-faced polyhedra.

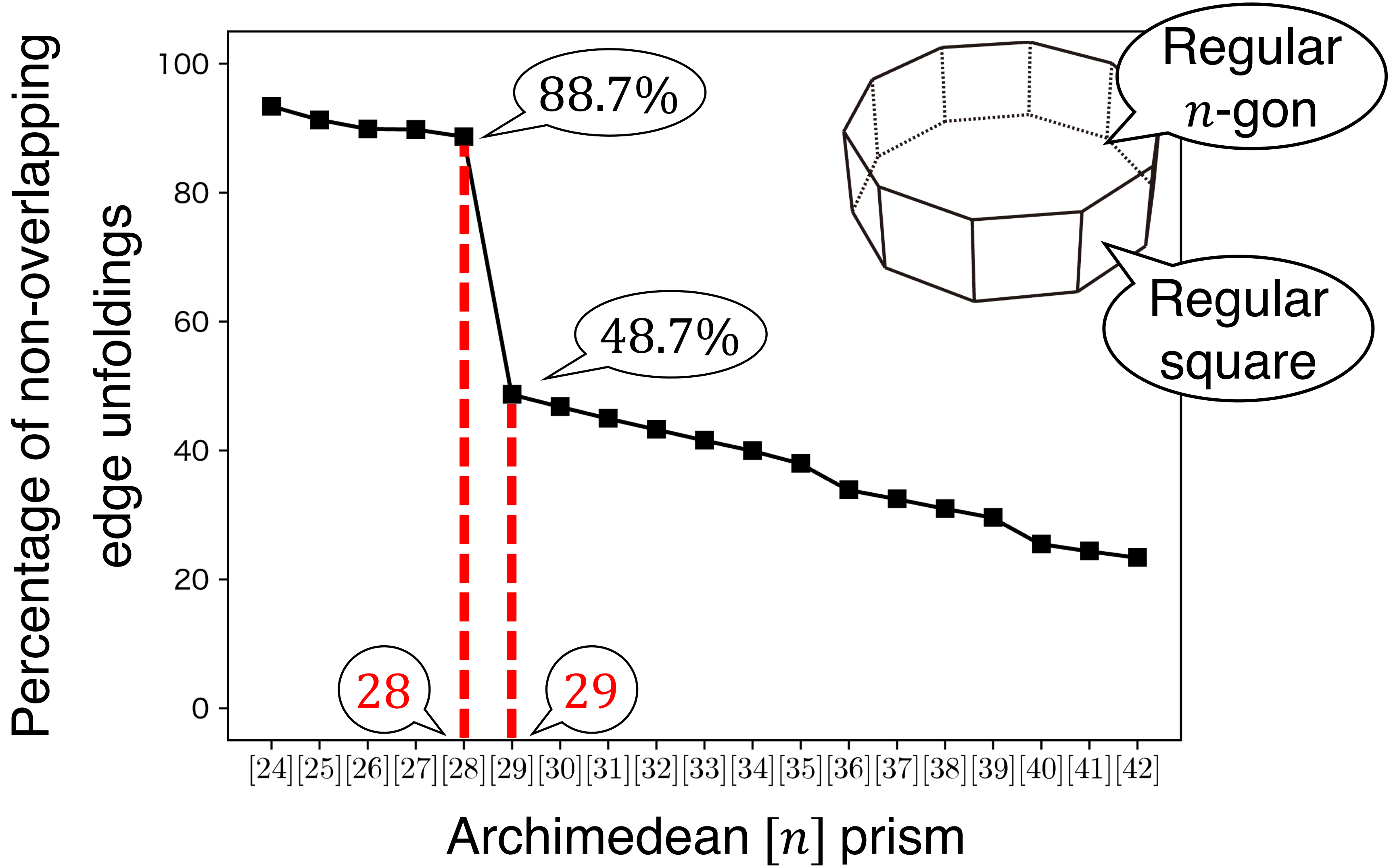
Future works

Apply this algorithm for other convex regular-faced polyhedra and more general polyhedra.



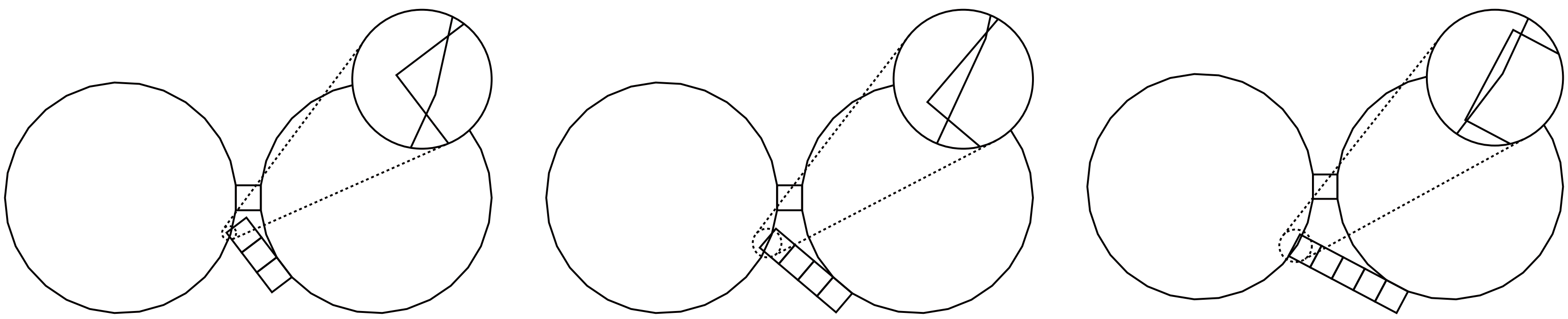
Turtle
[W. Schlickenrieder, 1997]

Why did the percentage go down?



Why did the percentage go down?

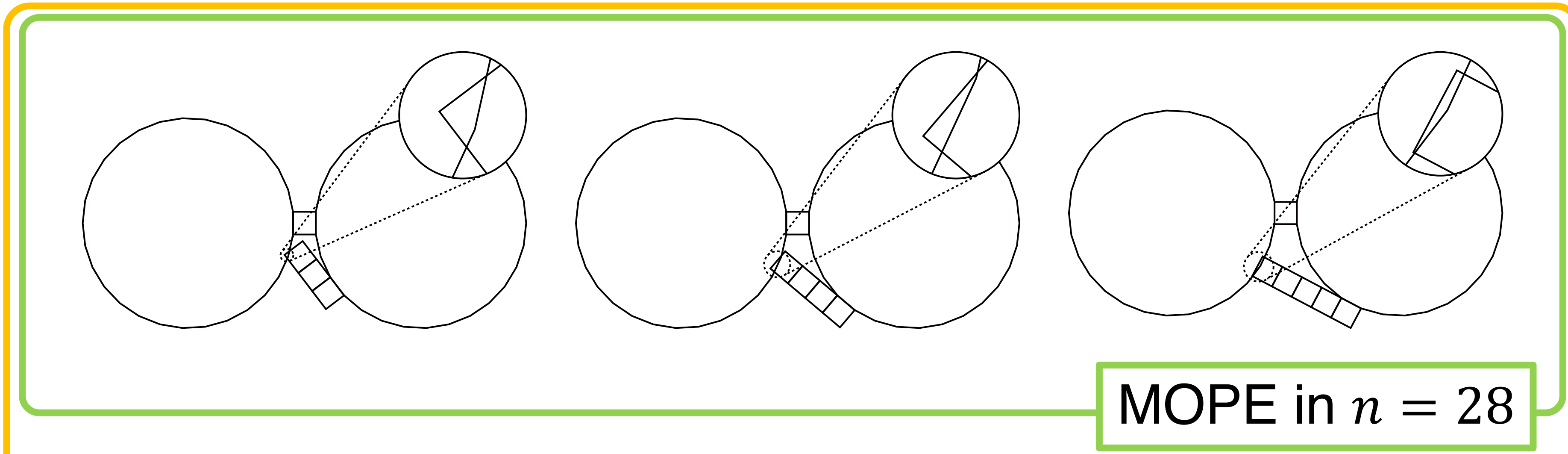
MOPEs in n-gonal Archimedean prisms

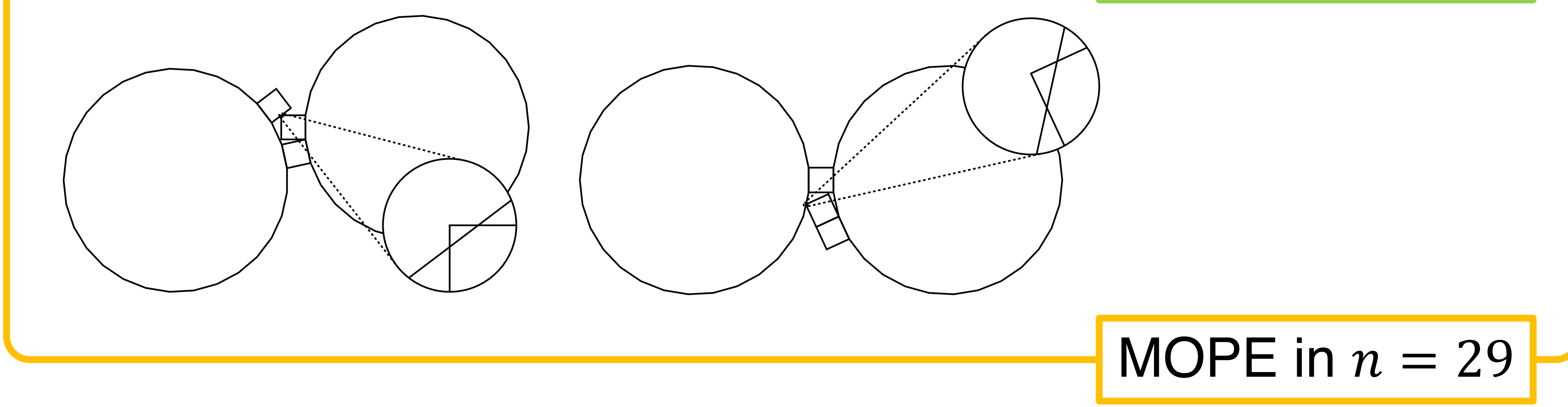


MOPE in $n = 28$

Why did the percentage go down?

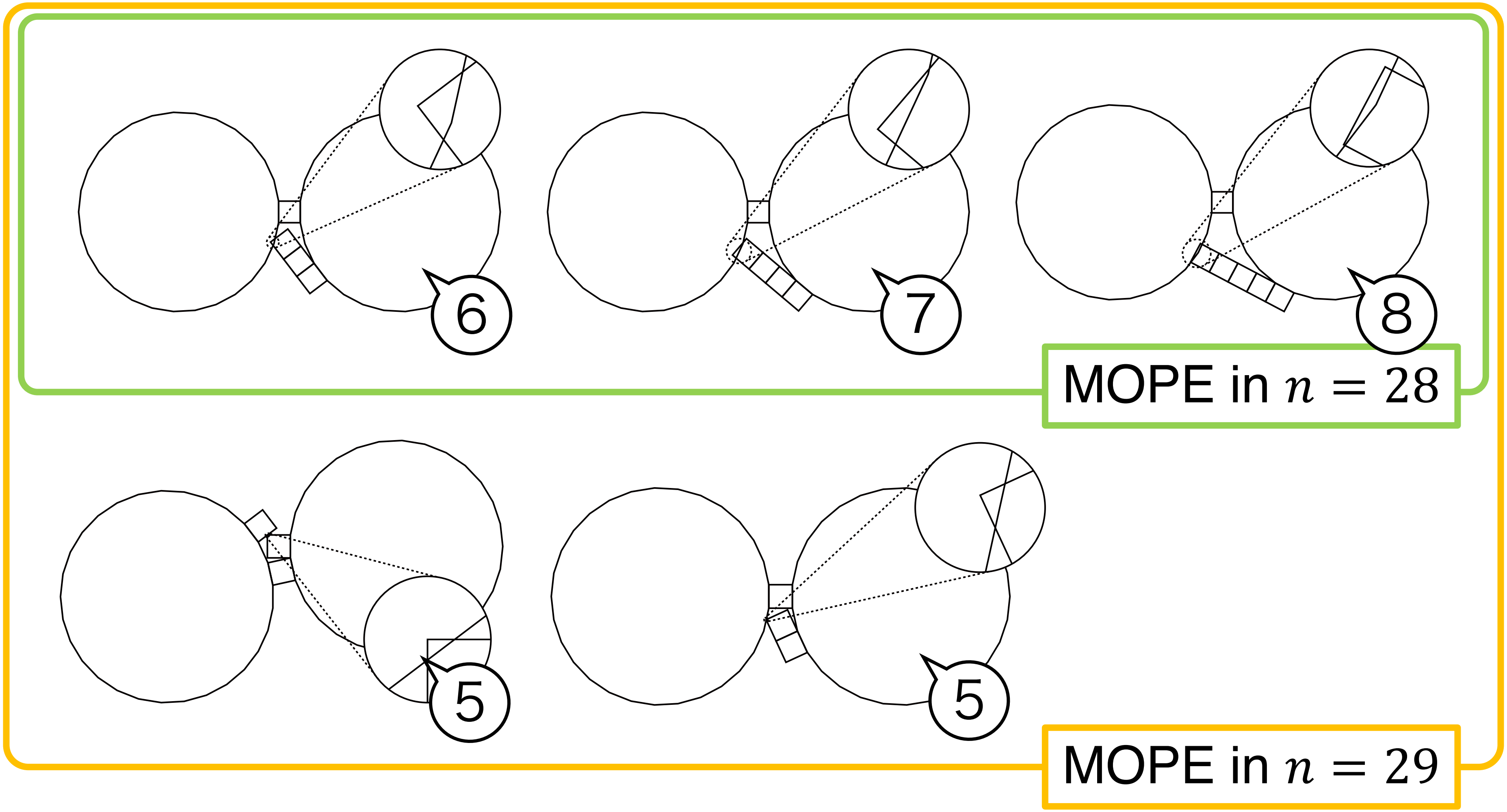
MOPEs in n-gonal Archimedean prisms



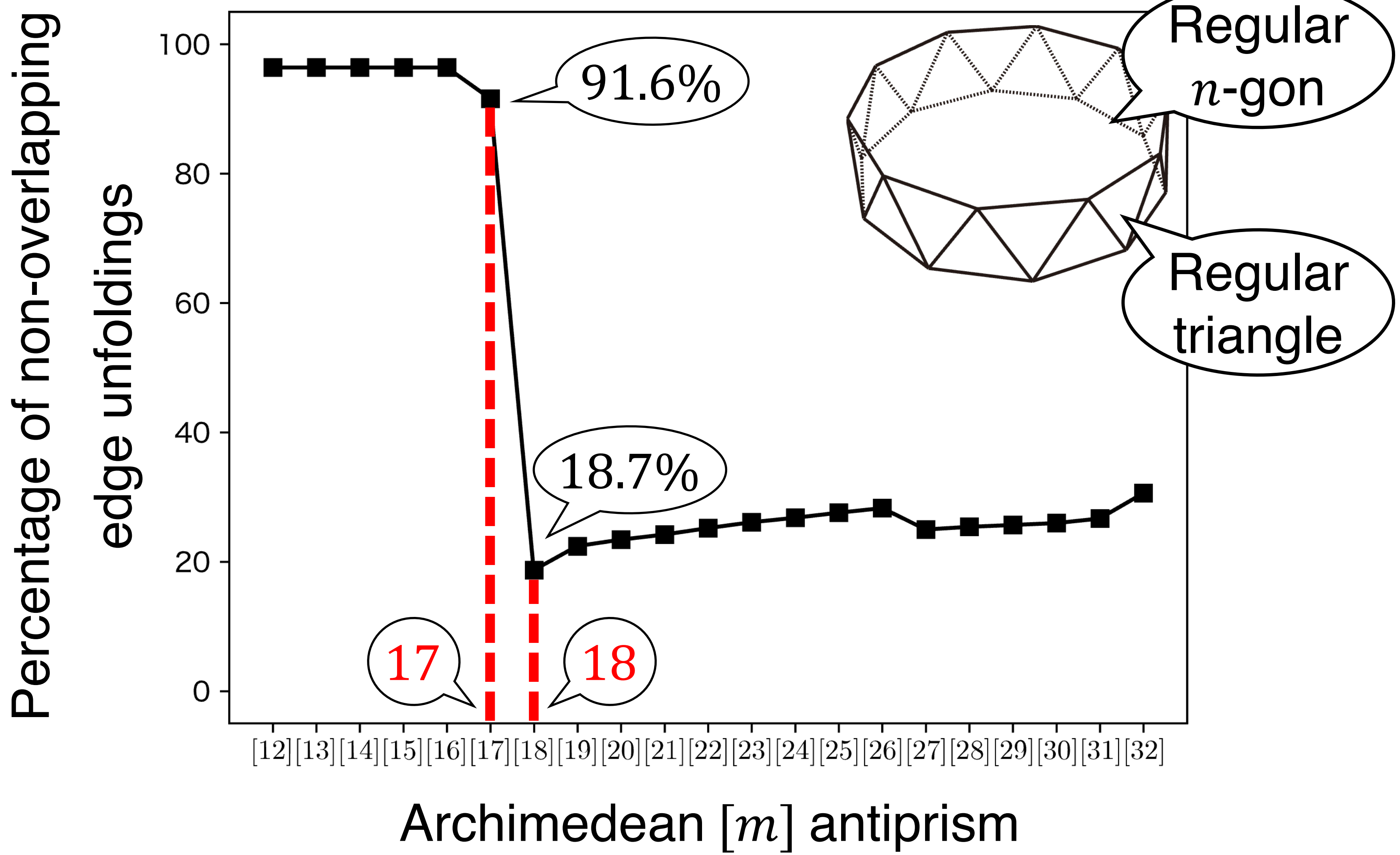


Why did the percentage go down?

MOPEs in n-gonal Archimedean prisms



Why did the percentage go down?



Why did the percentage go down?

MOPEs in m-gonal Archimedean antiprisms