The Complexity of the Lower Envelope of Collections of Various Geometric Shapes

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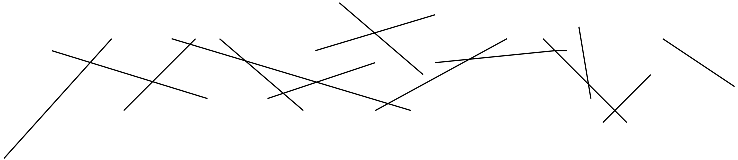
The lower envelope of a collection of segments

S = set of n (no necessarily disjoint) line segments in the plane



The complexity of the lower envelope of S is $\Theta(n\alpha(n))$

- the lower bound $\Omega(n\alpha(n))$ was proved by Wiernik and Sharir (1988)

- the upper bound $O(n\alpha(n))$ was proved by Hart and Sharir (1986)

- the lower envelope can be computed in $O(n \log n)$ time and $O(n\alpha(n))$ space, Hershberger (1989)

Motivation

Under which geometric properties of a given collection of n geometric objects

we can ensure that their lower envelope has a tight complexity, e.g.,

- linear

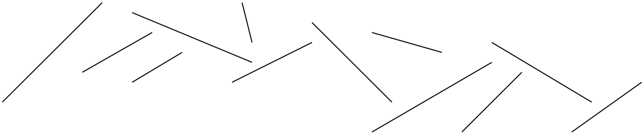
- $\Omega(n\alpha(n))$

- $\cdots$

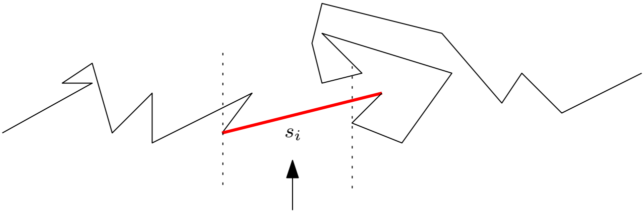
The lower envelope of a particular collections of segments

$S =$ set of n segments in the plane

- disjoint segments: the complexity of the lower envelope is $\Theta(n)$



- a simple polygonal chain: the complexity of the lower envelope is $\Theta(n)$



The lower envelope of collections of objects

- Rays
- Segments with unit length
- Unit squares
- Convex polygons
- Circles
- Ellipses
- ...

The lower envelope of collections of objects

Observation (1)

- Let S_1 and S_2 be two sets of n_1 and n_2 planar geometric objects whose lower envelopes have complexity $O\big(f_1(n_1)\big)$ and $O\big(f_2(n_2)\big)$, respectively.

- If any pair of objects in the set $S_1 \cup S_2$ intersect at most $O(1)$ times, then the lower envelope of $S_1 \cup S_2$ has complexity $O\big(f_1(n) + f_2(n)\big)$, where $n = n_1 + n_2$.

The lower envelope of rays

S is a set of n rays in the plane

$R =$ subset of rays that have no point to the left of their starting point
$L =$ subset of remaining rays

Lower bound:

- n rays to the right with starting points $(1, n), (2, n-1), \ldots, (n, 1)$

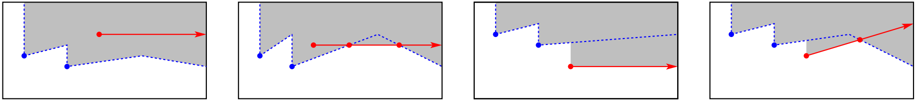
- the lower envelope has vertices at each integer x-coordinate from 1 to n

The complexity of the lower envelope of R is $\Omega(n)$

The lower envelope of rays

Upper bound:

- two rays in R that intersect, the ray that lies above the other after their intersection point will never again be included in the lower envelope after that point



The complexity of the lower envelope of R is $O(n)$, and therefore $\Theta(n)$

- Analogously for L. Using Observation (1) we get:

▶ The complexity of the lower envelope of S is $\Theta(n)$

This result was stated (without a proof) by Sharir and Agarwal (1995)

The lower envelope of segments with unit length

$S =$ set of n segments with unit length in the plane

Lower bound:

Consider the lower envelope of the n unit length segments

$[(1,0),(2,0)],[(3,0),(4,0)],\dots,[(2n-1,0),(2n,0)]$

the lower bound has complexity $\Omega(n)$.

The lower envelope of segments with unit length

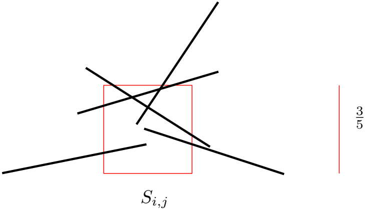
Upper bound:

Consider a square grid covering the plane whose cells have side length 3/5

$S_{i,j}$ is the grid cell at row i and column j.

$L_{i,j}$ is the set of segments in $S \cap S_{i,j}$,

$n_{i,j}$ is the number of segments in $L_{i,j}$.



The lower envelope of $L_{i,j}$ has complexity $O(n_{i,j})$

The lower envelope of segments with unit length

- $S_j^{(k)} = \bigcup_{i=-\infty}^{\infty} S_{3i+k,j}$, for $k \in \{0, 1, 2\}$, is the union of the cells in a grid column that are three cells apart.

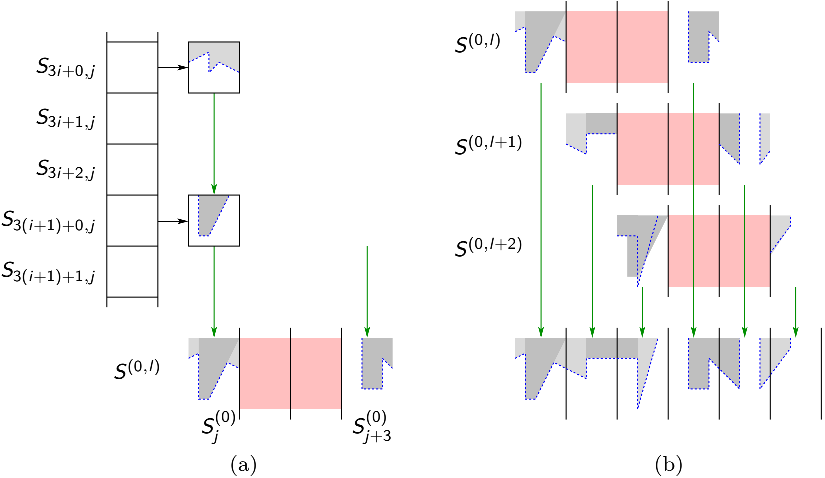
- $S_j^{(k)}$ is the k^{th} *sub-strip* of the j^{th} grid column.

- $L_j^{(k)} = \bigcup_{i=-\infty}^{\infty} L_{3i+k,j}$ is the set of segments in $S \cap S_j^{(k)}$.

- Let the number of segments of each set $L_j^{(k)}$ be $n_j^{(k)}$.

- No two grid cells $S_{3i+k,j}$ and $S_{3i'+k,j}$, with $i \neq i'$, in sub-strip $S_j^{(k)}$, contain a common segment since they are more than one unit apart vertically.

The lower envelope of segments with unit length



Figure

The lower envelope of segments with unit length

- The lower envelope of $S_j^{(k)}$ is the lower envelope of the lower envelopes for the squares included in $S_j^{(k)}$

- The complexity of the lower envelope of $S_j^{(k)}$ is:

$$\sum_{i=-\infty}^{\infty} O(n_{3i+k,j}) = O\left(n_j^{(k)}\right)$$

The lower envelope of segments with unit length

Analogous for the lower envelope of sub-strips that are three squares apart horizontally:

- $S^{(k,l)} = \bigcup_{j=-\infty}^{\infty} S_{3j+l}^{(k)}$, for $l \in \{0, 1, 2\}$
- $L^{(k,l)} = \bigcup_{j=-\infty}^{\infty} L_{3j+l}^{(k)}$ is the set of segments in $S \cap S^{(k,l)}$
- The number of segments in $L^{(k,l)}$ is $n^{(k,l)}$
- $L_{3j+l}^{(k)}$ and $L_{3j'+l}^{(k)}$, $j \neq j'$, have empty intersection since $S_{3j+l}^{(k)}$ and $S_{3j'+l}^{(k)}$ are a unit apart
- The complexity of lower envelope of the segments in $S^{(k,l)}$ is:

$$\sum_{j=-\infty}^{\infty} O\left(n_{3j+l}^{(k)}\right) = O\left(n^{(k,l)}\right) \subseteq O(n)$$

The lower envelope of segments with unit length

Upper bound:
The complexity of the lower envelope of $\bigcup_{\substack{0 \leq k \leq 2 \\ 0 \leq l \leq 2}} L^{(k,l)}$, i.e., the whole domain, is then

$$\sum_{\substack{0 \leq k \leq 2 \\ 0 \leq l \leq 2}} O\left(n^{(k,l)}\right) \subseteq \sum_{\substack{0 \leq k \leq 2 \\ 0 \leq l \leq 2}} O(n) = O(n) \tag{1}$$

using Observation (1) since we are summing over nine linear sized subsets

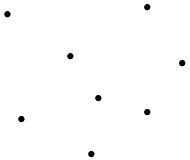
$S^{(0,0)}$	$S^{(0,1)}$	$S^{(0,2)}$
$S^{(1,0)}$	$S^{(1,1)}$	$S^{(1,2)}$
$S^{(2,0)}$	$S^{(2,1)}$	$S^{(2,2)}$

- The complexity of the lower envelope of S is $\Theta(n)$

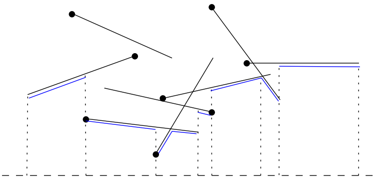
Segments Traced by Moving Points with Constant Speed

$P =$ set of n points, each moving at the same constant speed along a different line.

The points start moving at $t = 0$, at $t > 0$ the points have traced a set L_t.



(a)



(b)

Figure

- For any fixed $t > 0$, the lower envelope of L_t has a complexity of $\Theta(n)$ and can be computed in $O(n \log n)$ time and $O(n)$ space.

Collections of Unit Squares Under Linear Transformations

Consider n copies of a unit square (e.g., with corners at $[0, 0], [0, 1], [1, 0]$, and $[1, 1]$), and apply a subset of the transformations to these n unit squares.

The complexity of the lower envelope for the combinations of these transformations is:

Case	Rotation	Translation	Scaling	Complexity
1	$\times$	$\times$	$\times$	$\Theta\big(n\alpha(n)\big)$
2	$\times$	$\times$		$\Theta(n)$
3	$\times$		$\times$	$\Theta(n)$
4		$\times$	$\times$	$\Theta(n)$
5	$\times$			$\Theta(n)$
6		$\times$		$\Theta(n)$
7			$\times$	$\Theta(1)$

Table

Collections of Unit Squares Under Linear Transformations

Theorem (Case 1)

The lower envelope of a set of n unit squares that can be rotated, translated, and scaled has complexity $\Theta\big(n\alpha(n)\big)$.

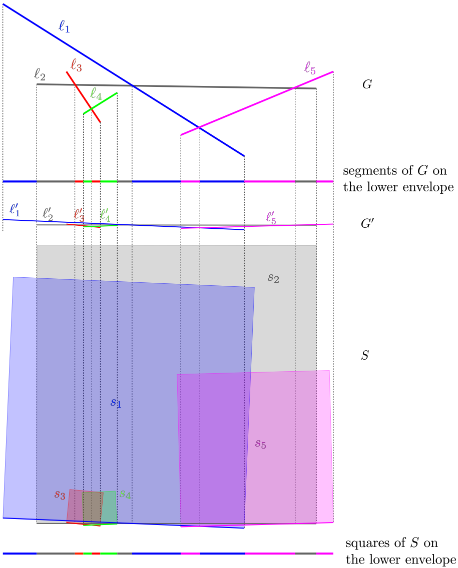
Upper bound:

- For n segments, Hart and Sharir showed that the complexity of the lower envelope can be at most $O\big(n\alpha(n)\big)$.

- For each of the n squares at most two line segments appear on the lower envelope.

- An upper bound of $O\big(n\alpha(n)\big)$ on the complexity of the lower envelope of the squares.

Lower bound:


ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5
G
segments of G on the lower envelope
ℓ'_1
ℓ'_2
ℓ'_3
ℓ'_4
ℓ'_5
G'
s_2
S
s_1
s_5
s_3
s_4
squares of S on the lower envelope

Other objects

- For collection of circles the lower envelope has $\Theta(n)$ complexity
- For collection of ellipses: very thin ellipses are almost like segments
- So what about other objects?

Thank you for your attention!!