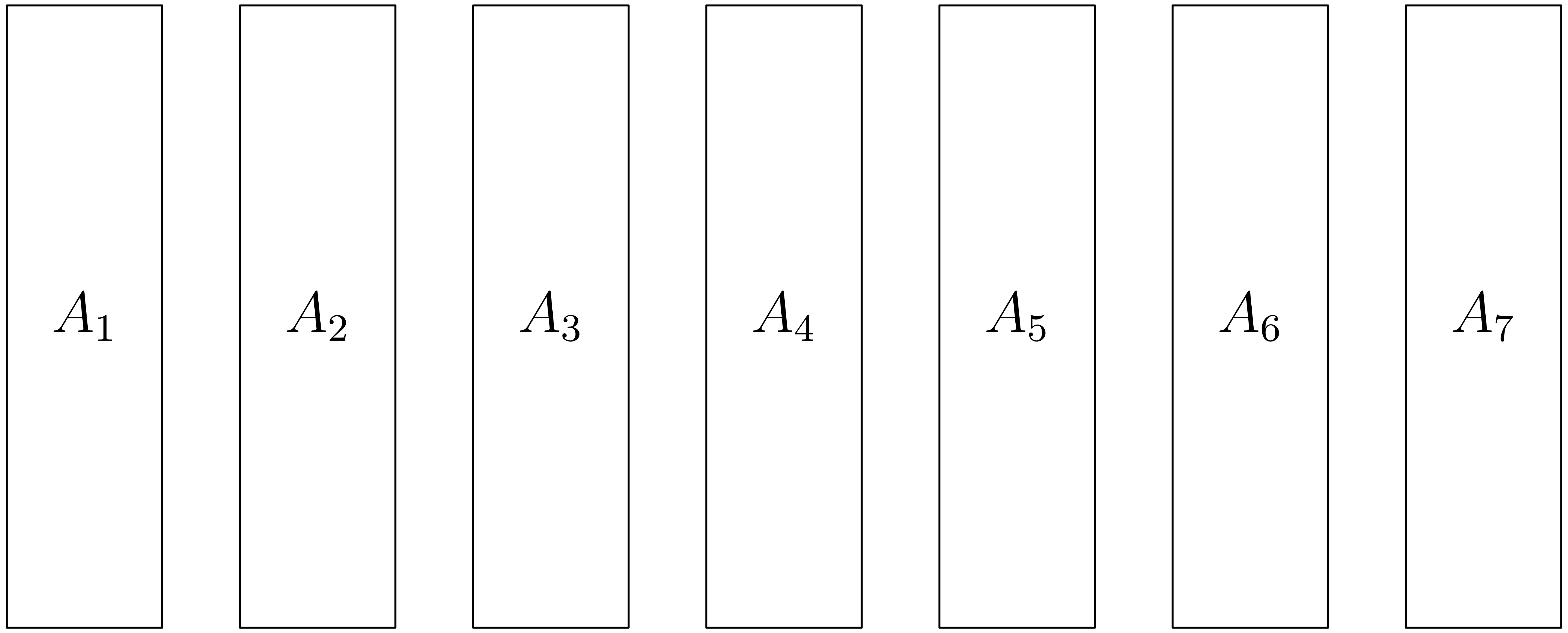
Unit Interval Graphs & Maximum c-Independent Sets Maximizing the Number of Isolated Vertices

Linda Kleist and Kai Kobbe

Technische Universität Braunschweig

Machine conflicts and unit interval graphs



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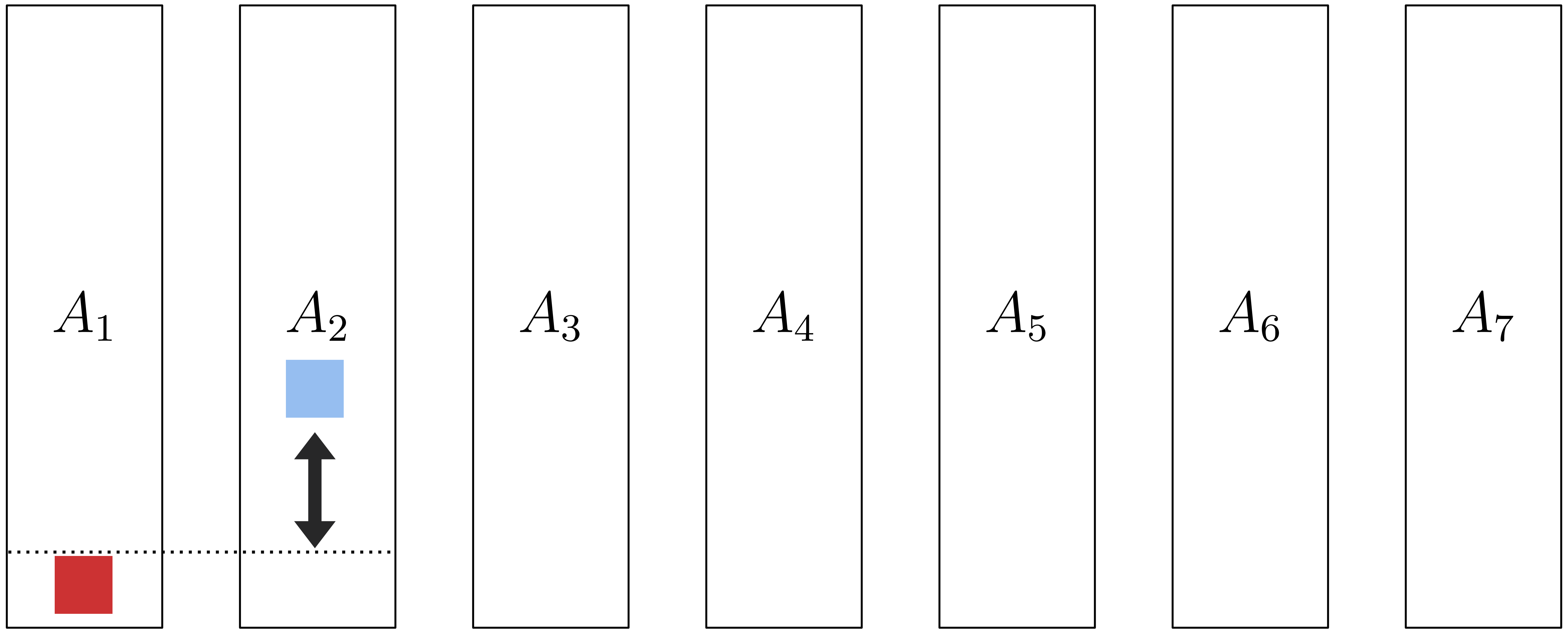

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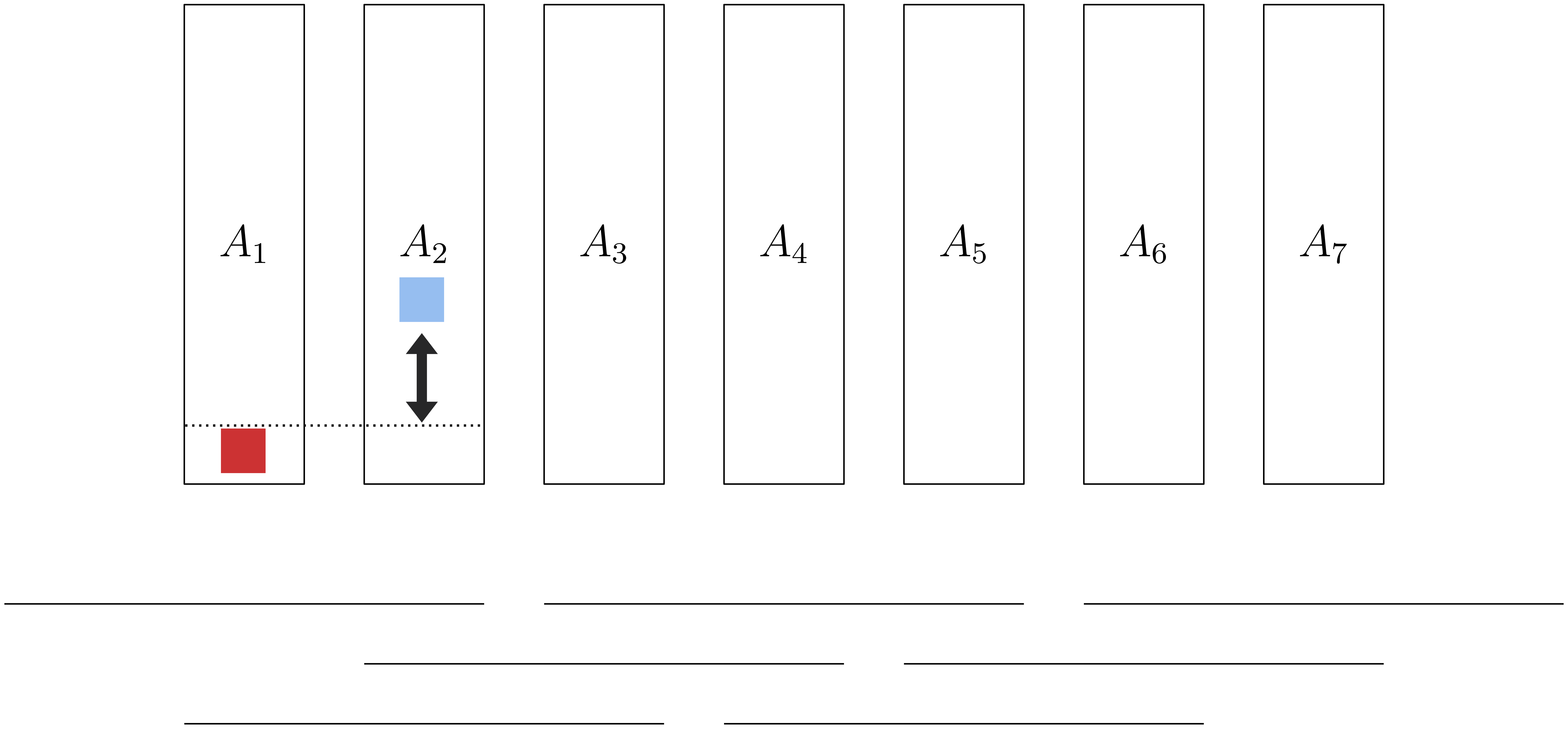

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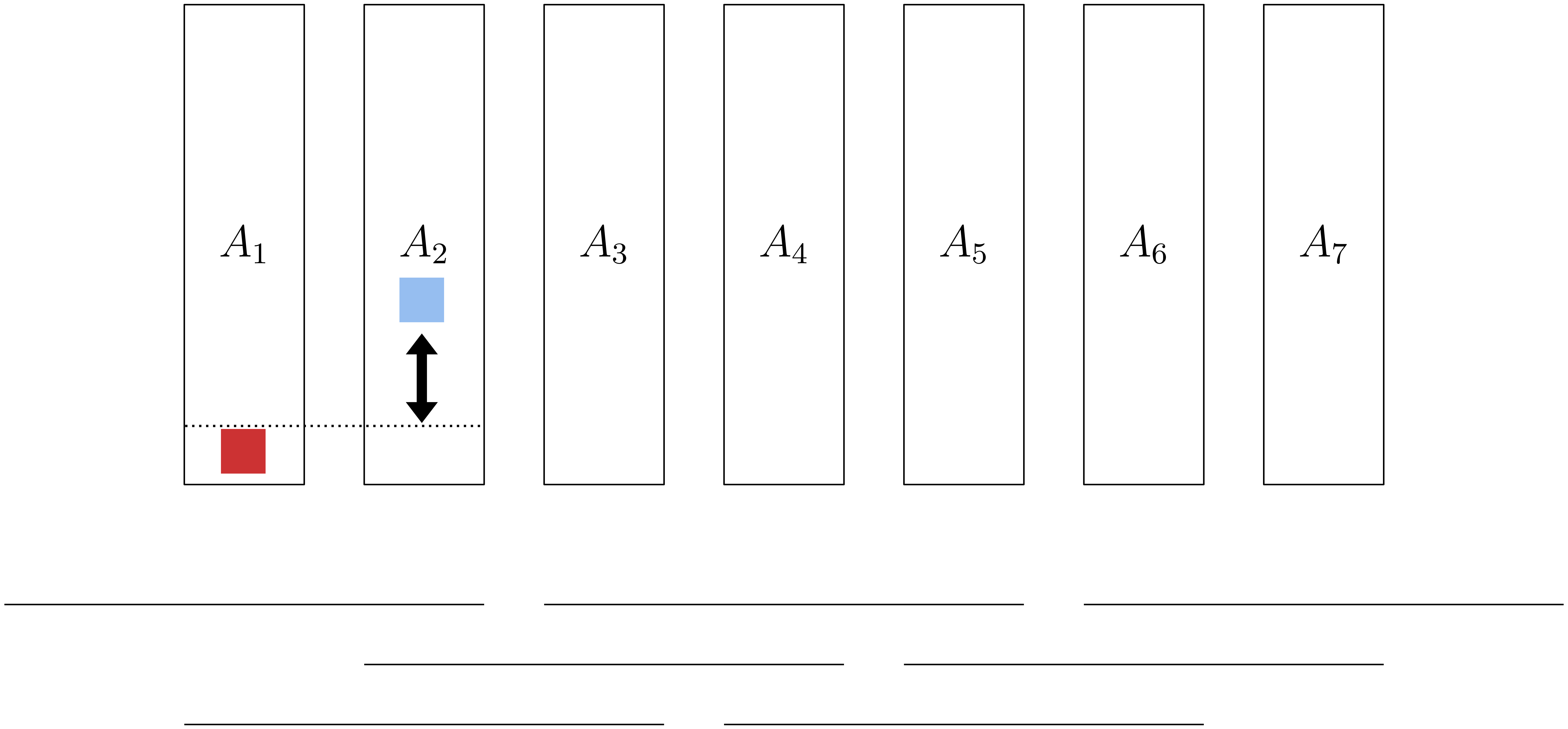

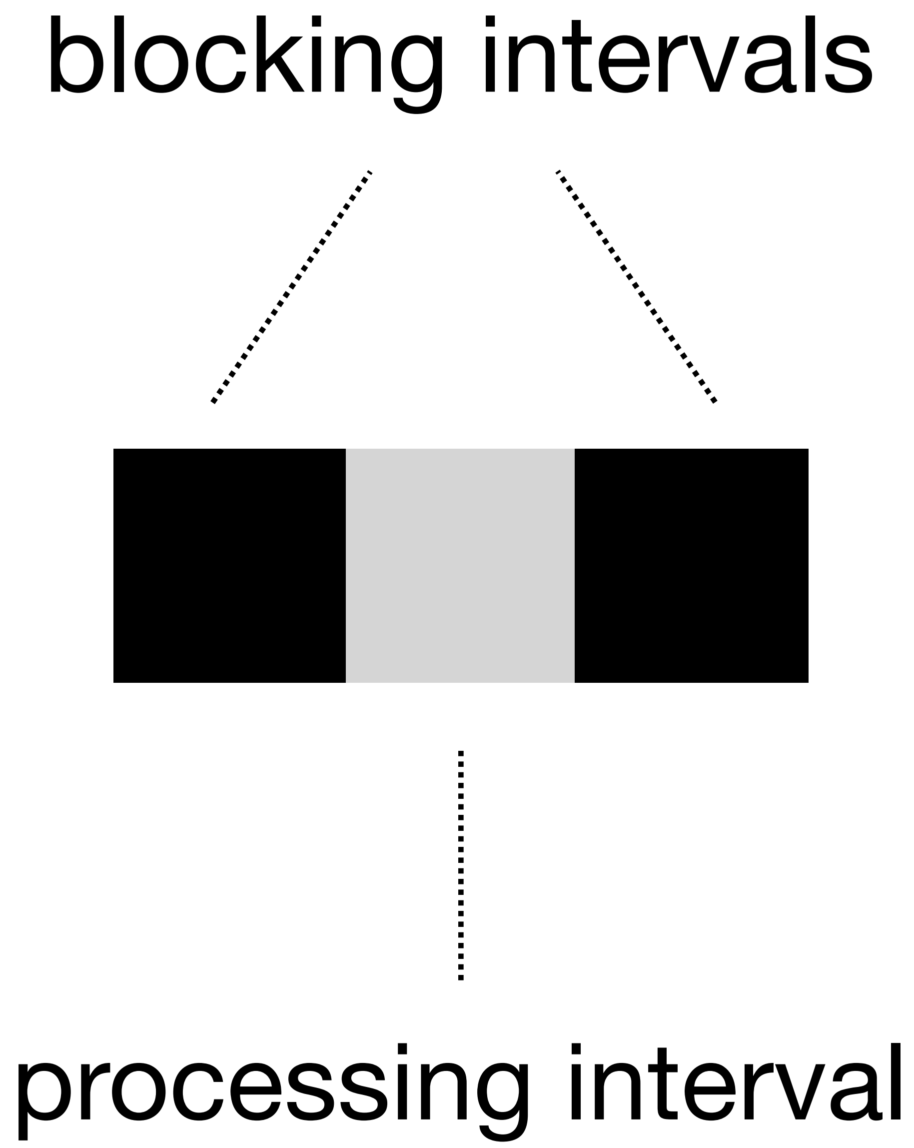

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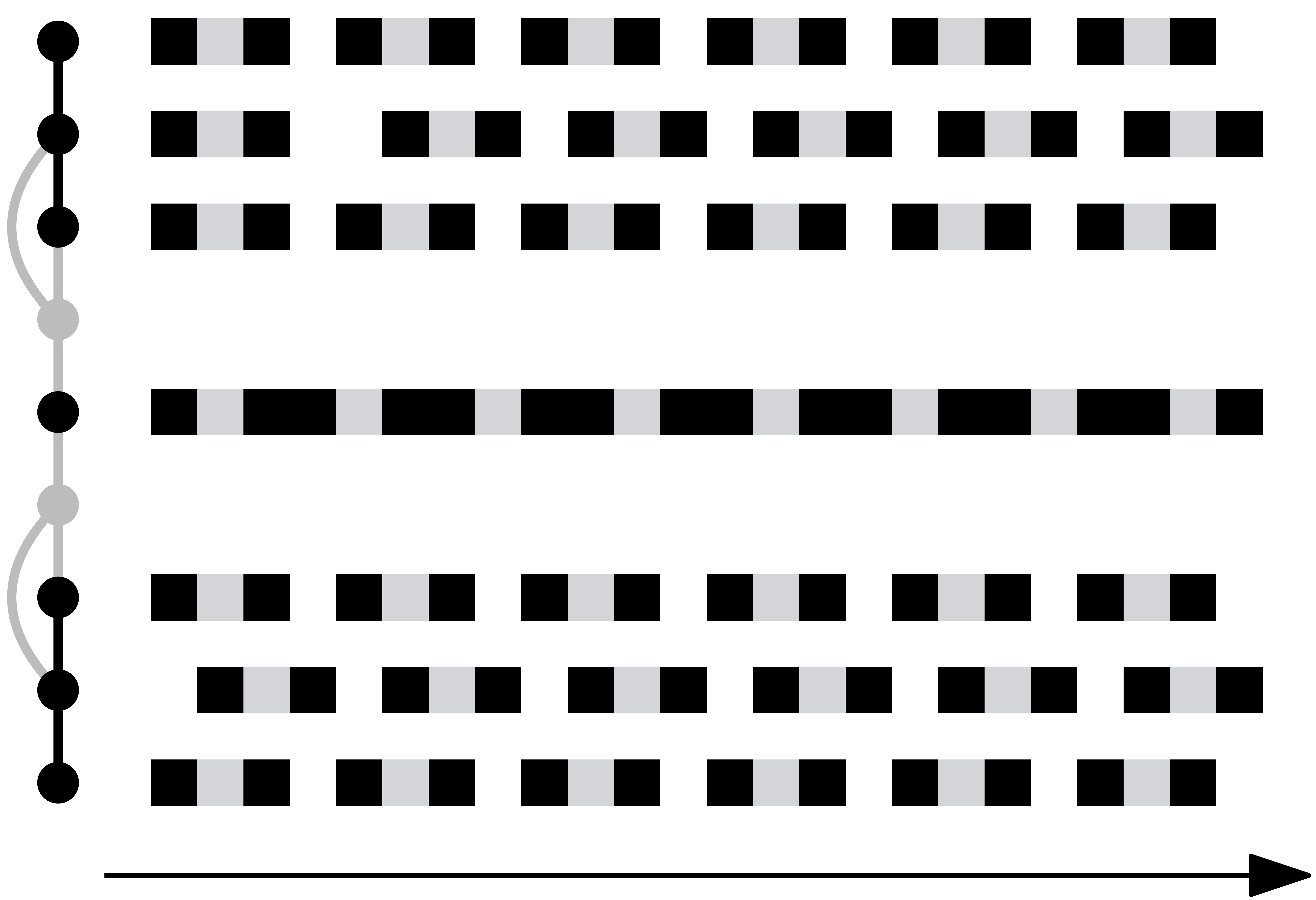

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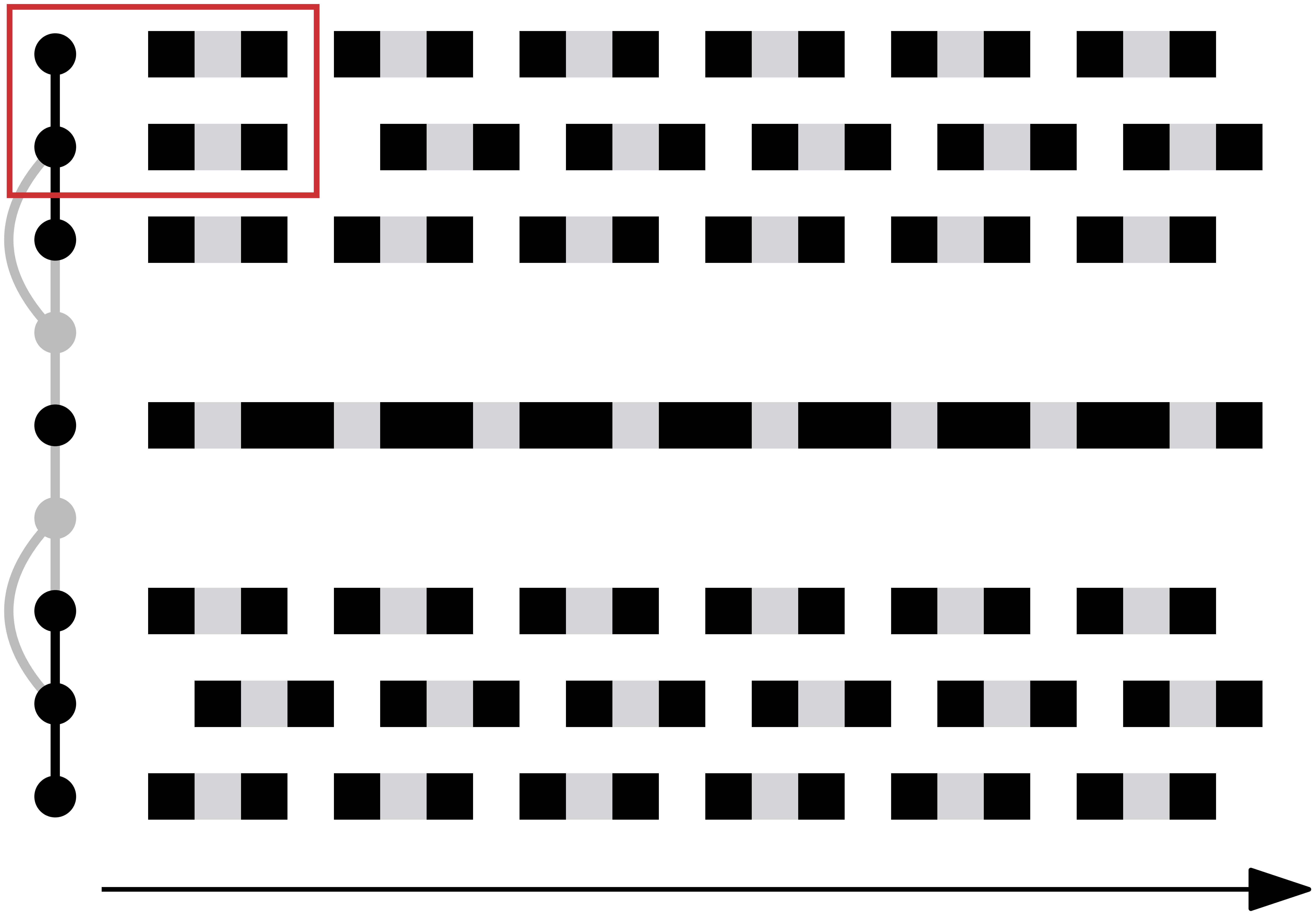

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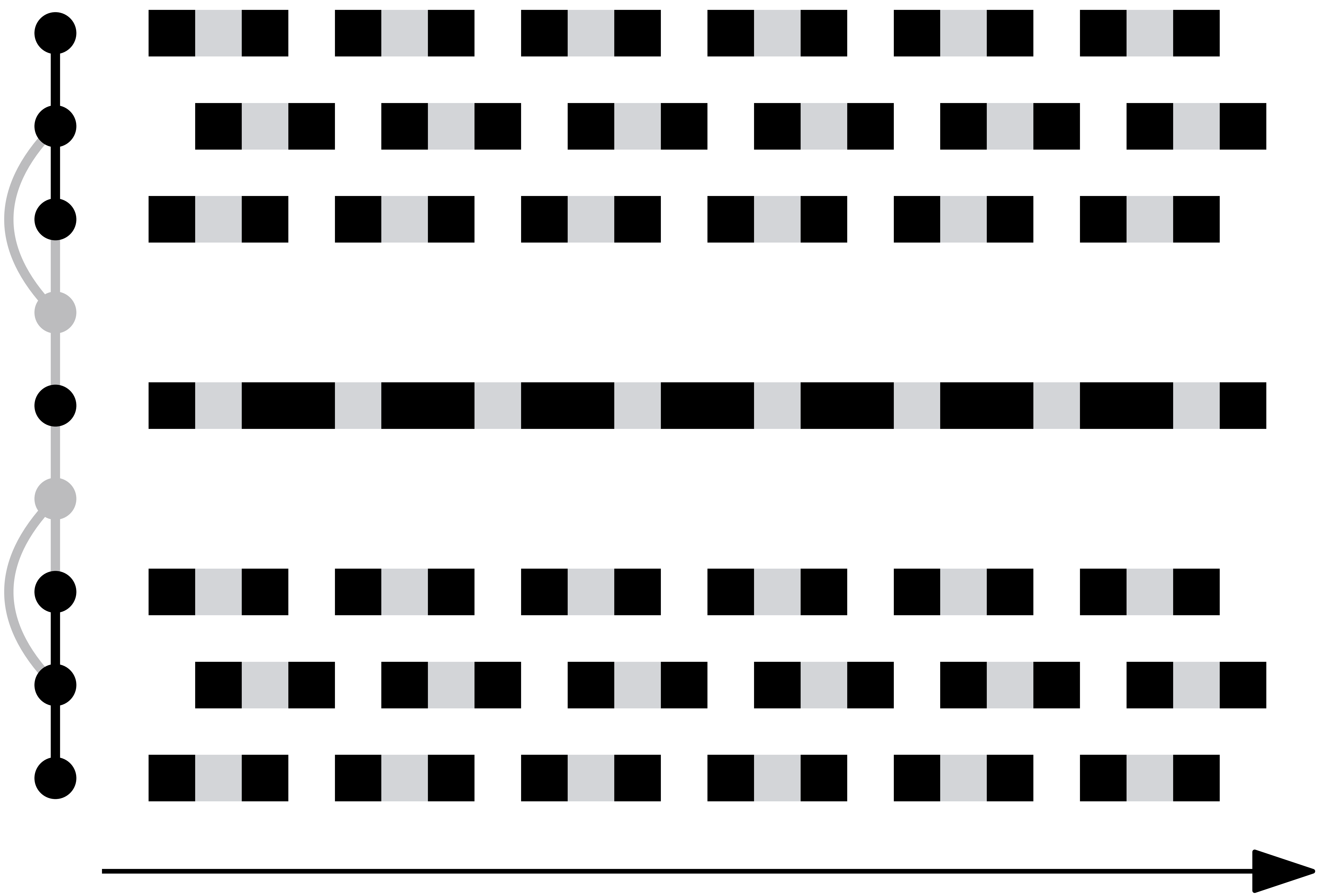

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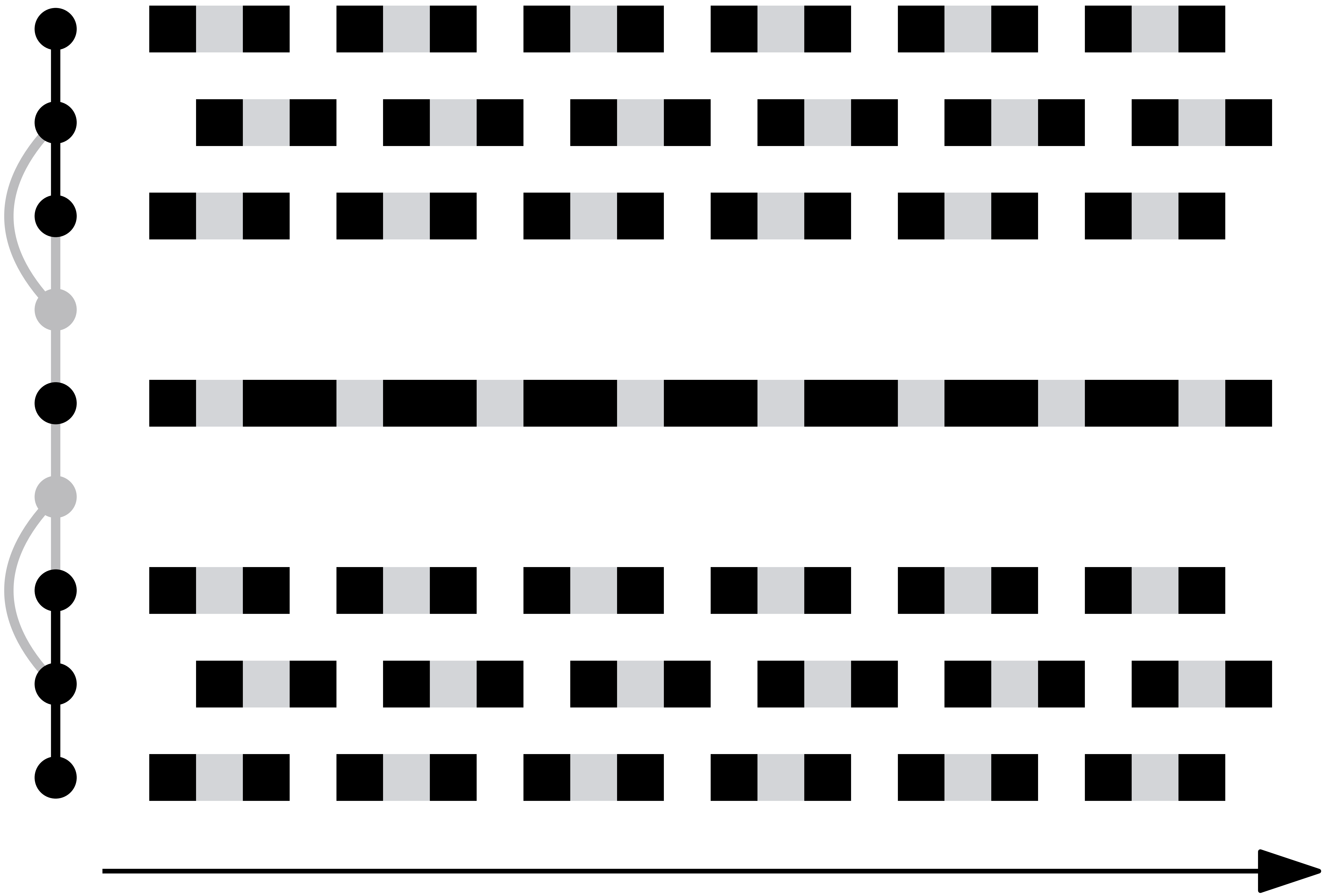

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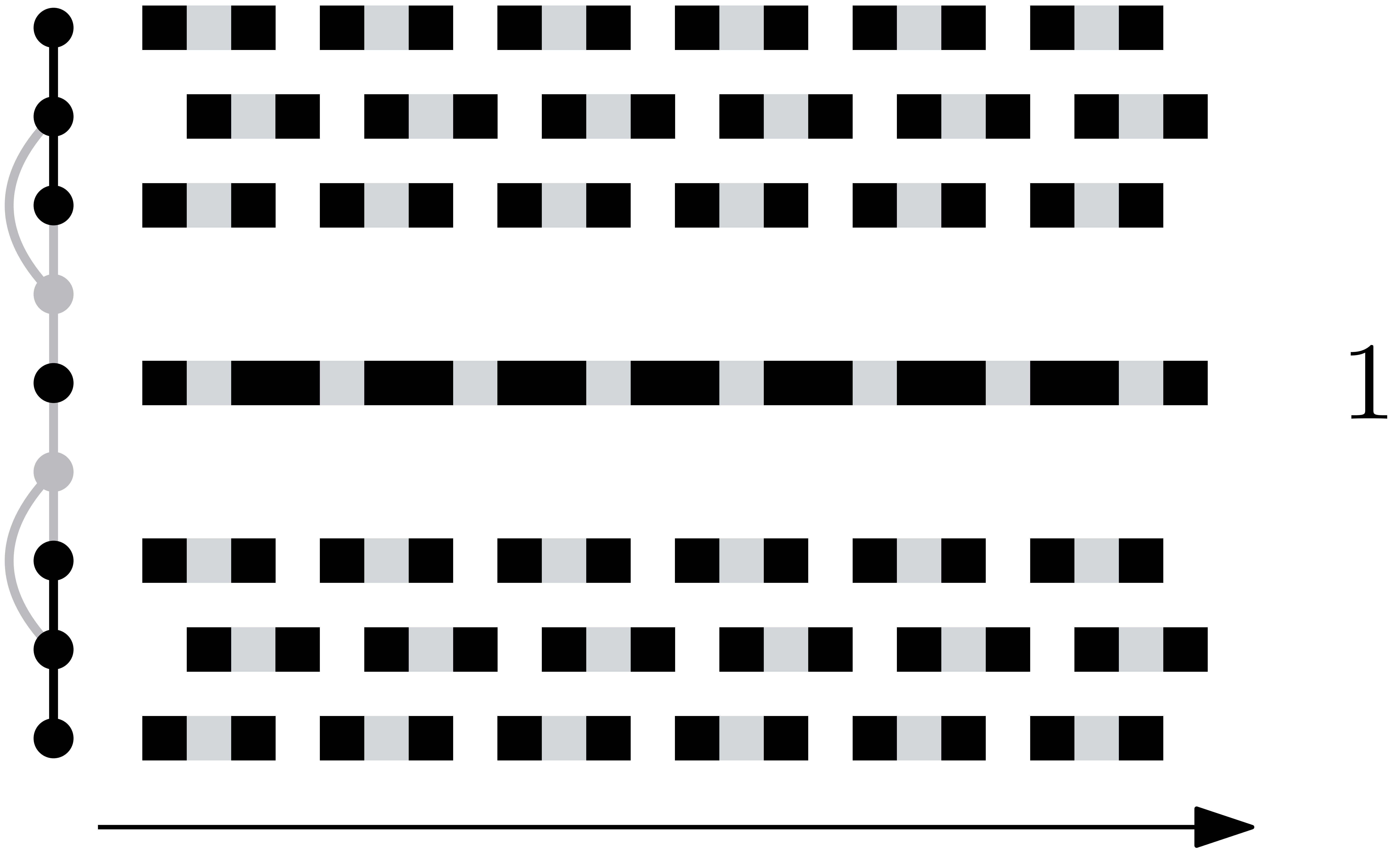

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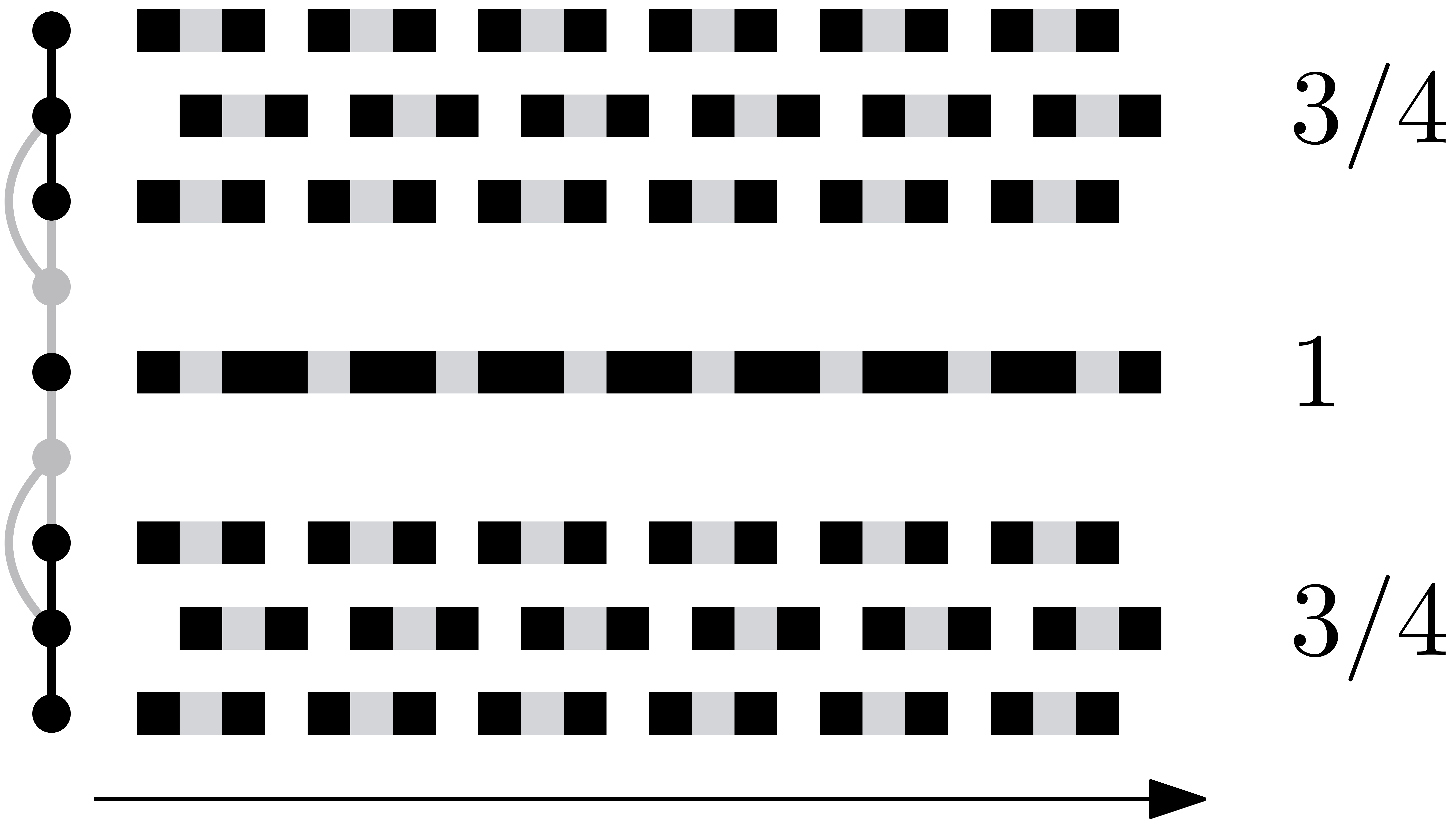

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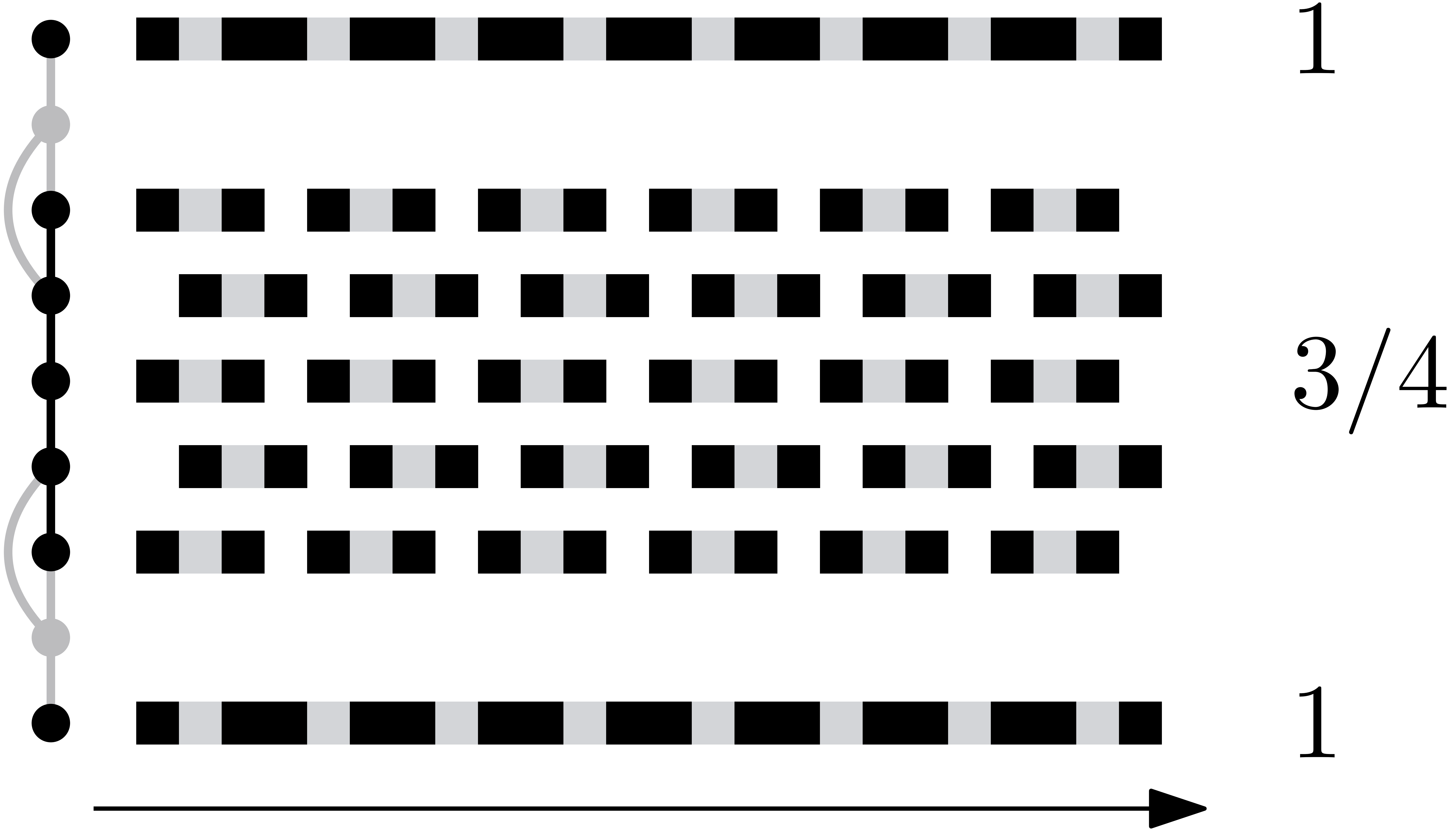

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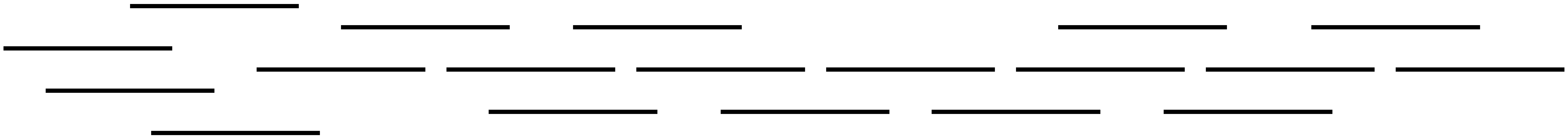
▶ **Theorem 1.1.** *There exists an algorithm that computes a max-iso c-IS for every unit interval graph on n vertices with a running time in $O(n)$, even if c is part of the input.*

Recap: Greedy for maximum c-IS in unit interval graphs



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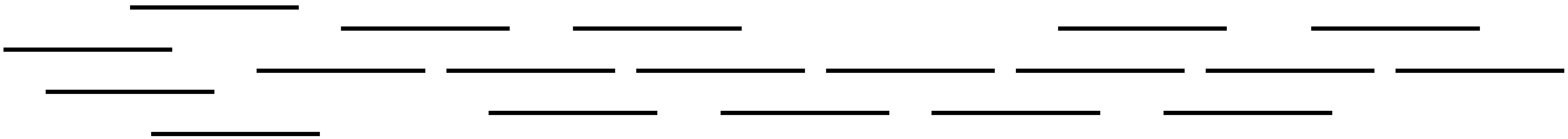

1. Start with $I = \emptyset$.
2. Consider the vertices in **left-right-order** and add them respectively to I if this maintains a c-IS.

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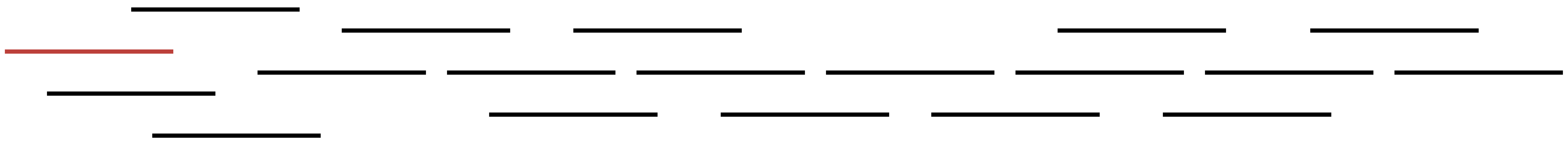

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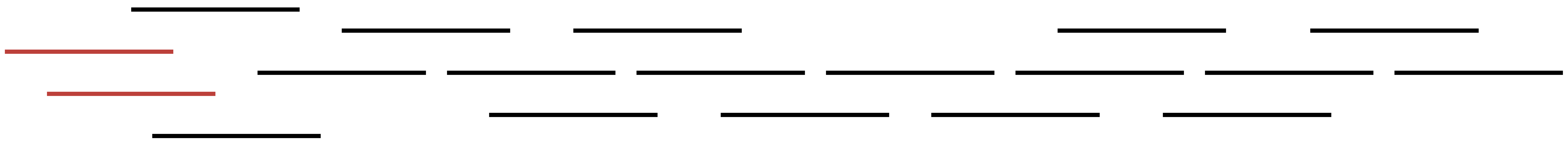

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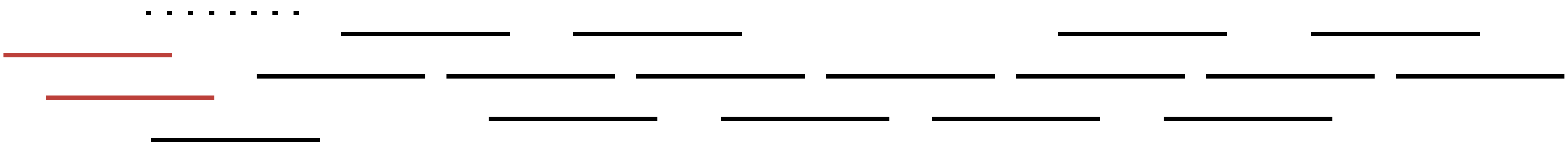

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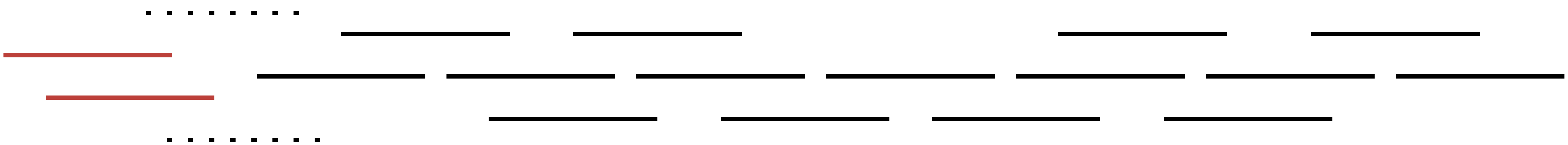

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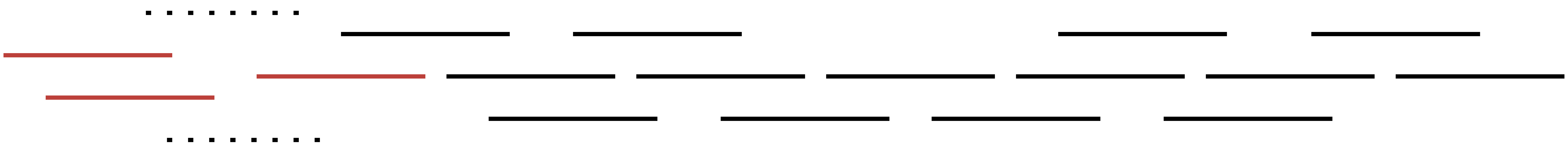

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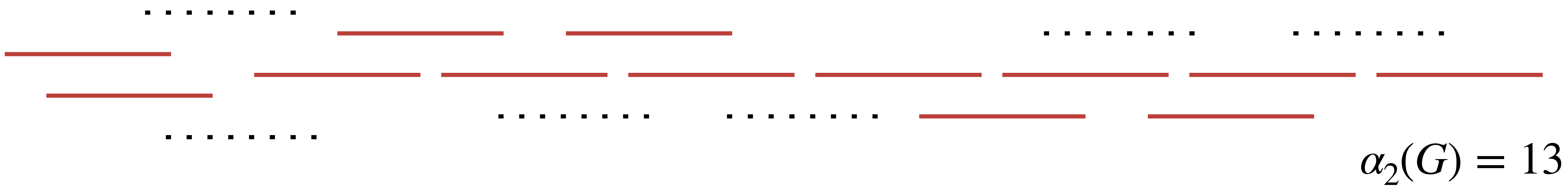

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Identifying candidates

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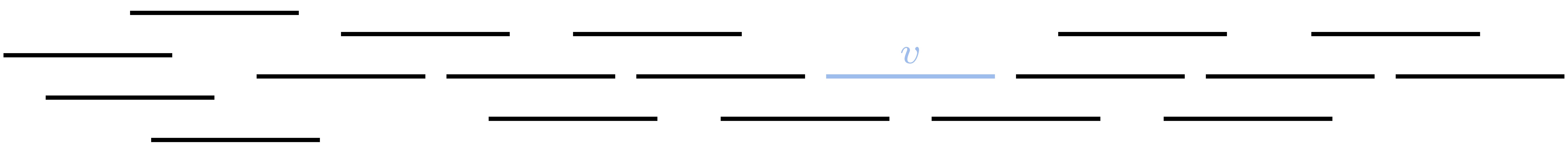

For every valid candidate, it holds that:

$$\alpha_c(G - N(v)) = \alpha_c(G)$$

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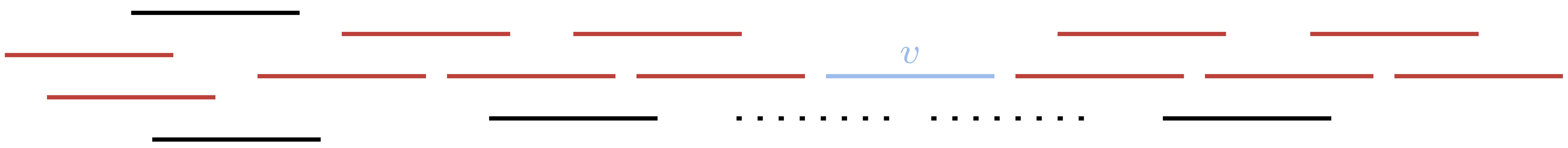

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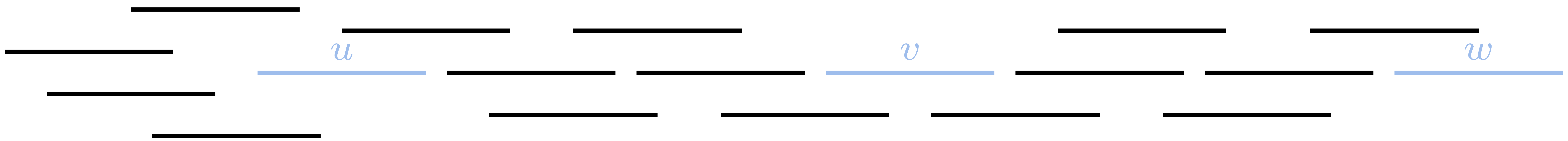

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We should always isolate the leftmost candidate!

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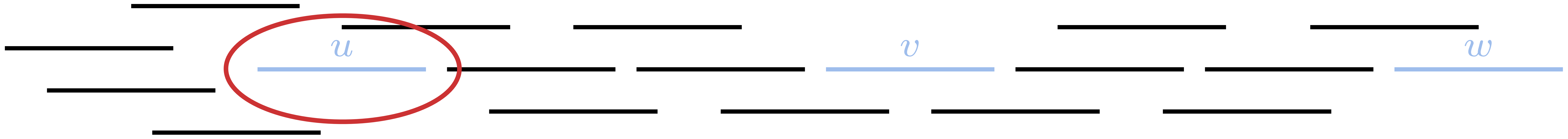

Crucial Lemma:

For the leftmost candidate v, $G - N(v)$ contains a max-iso c-independent set in G.

Quadratic algorithm

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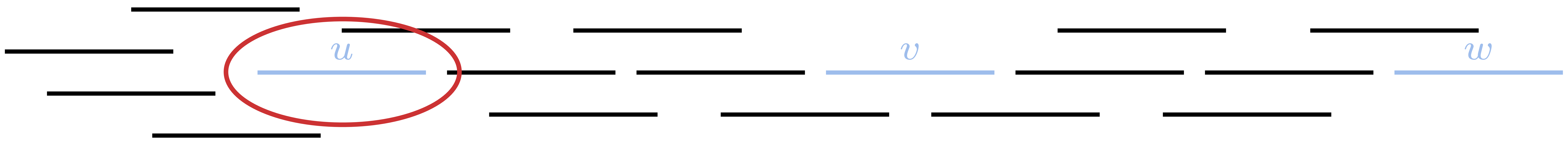

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2. Delete $N(v)$ from G and repeat 1. with the next vertex.

3. Return a maximum c-IS in the modified graph.

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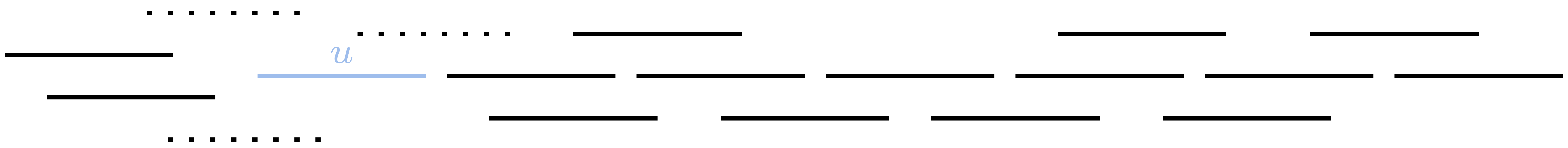

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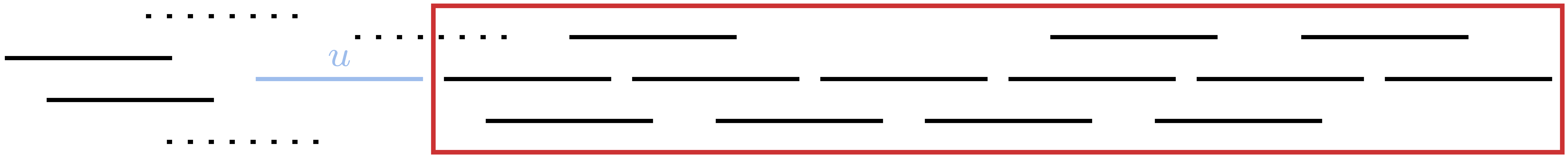

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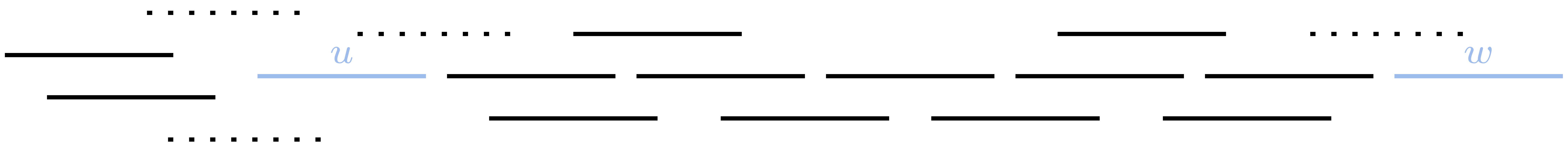

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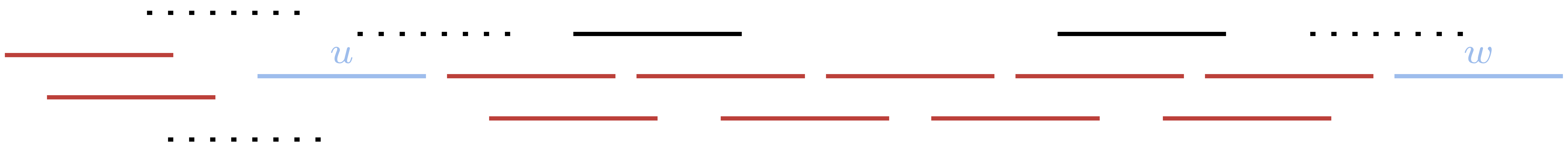

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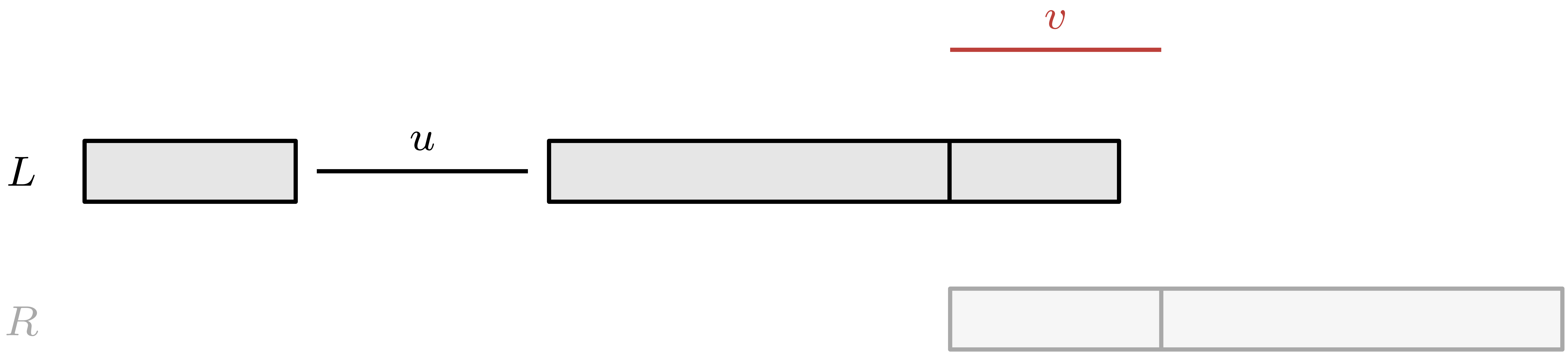
Linear-time implementation: Greedy from both sides!



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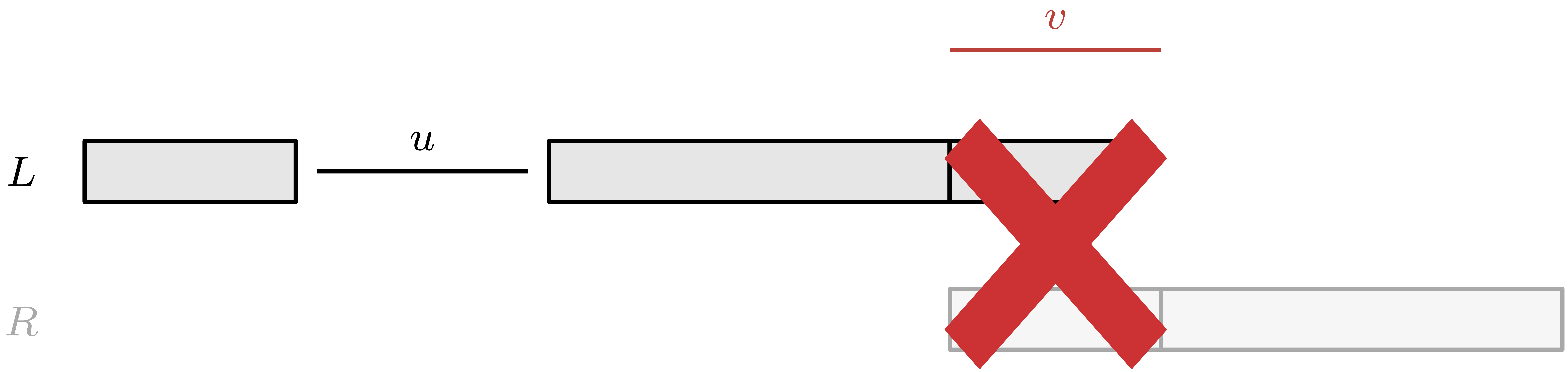

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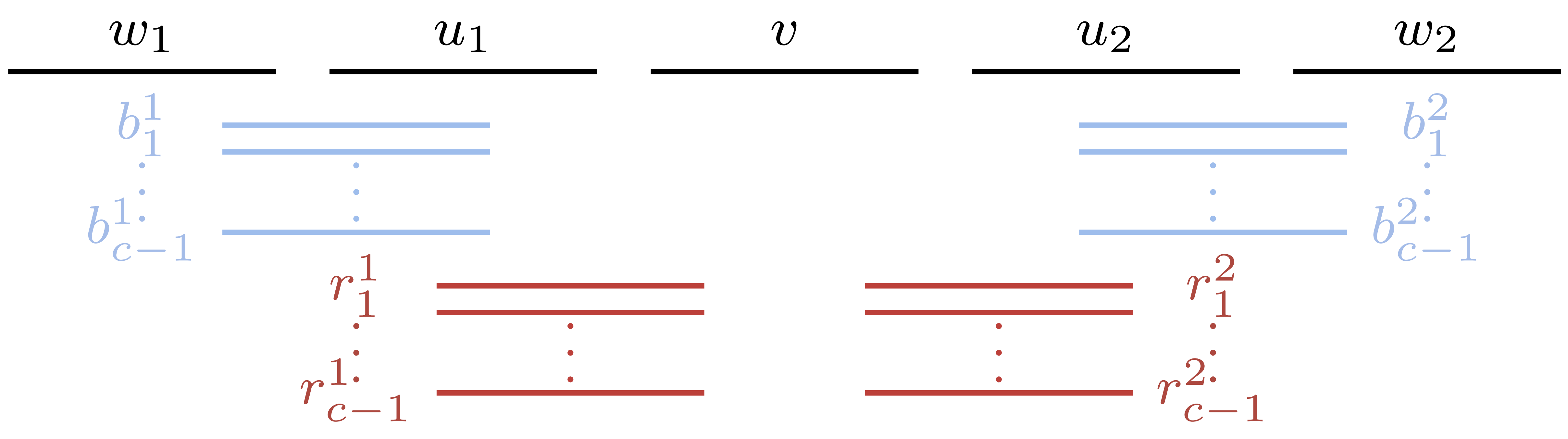

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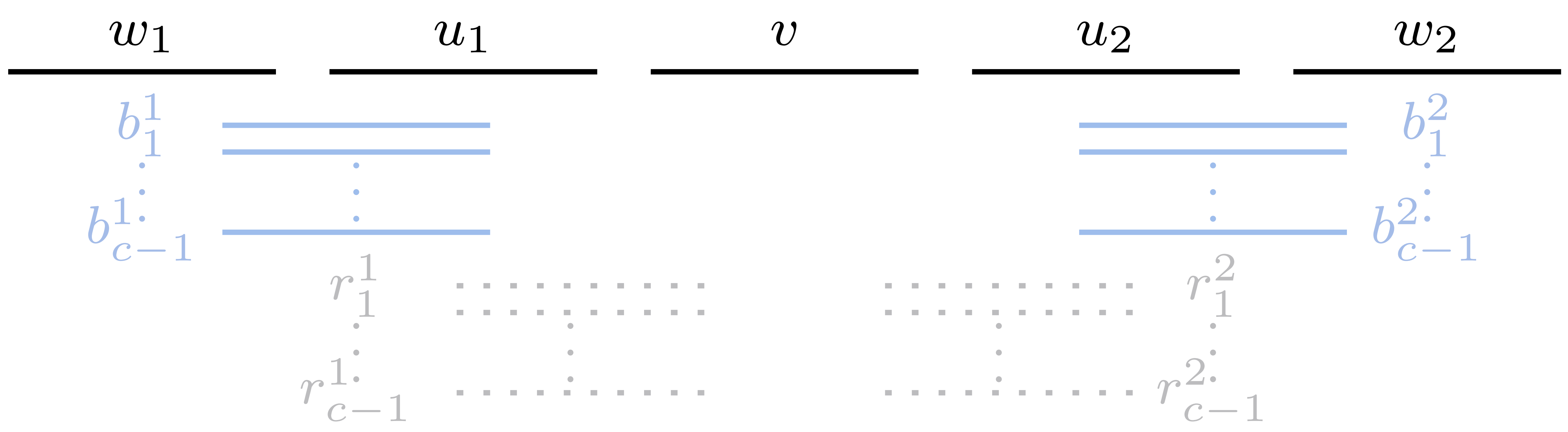

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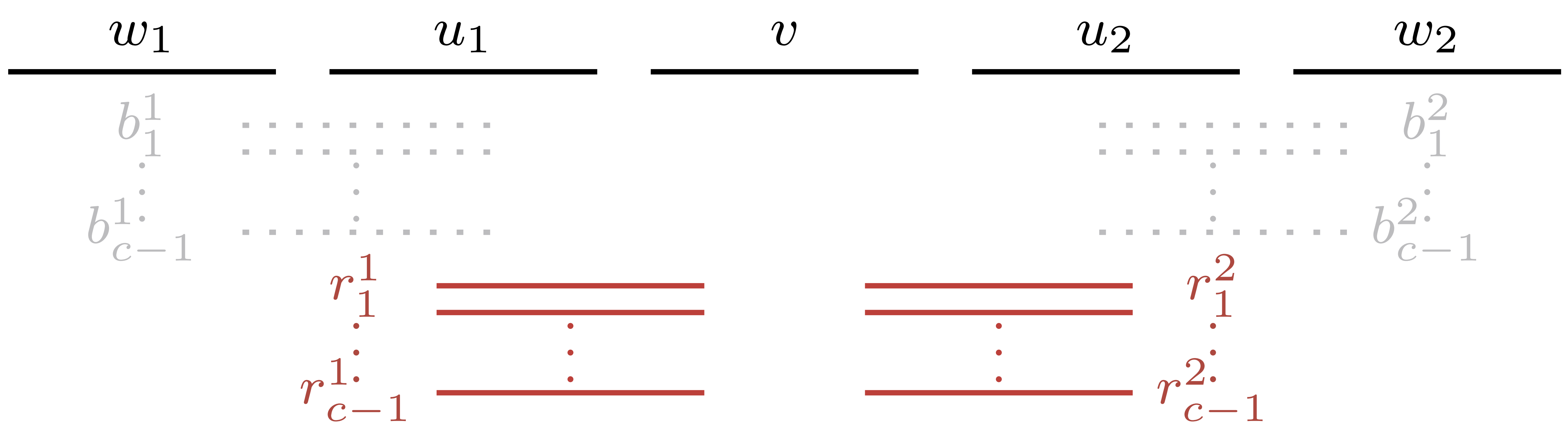

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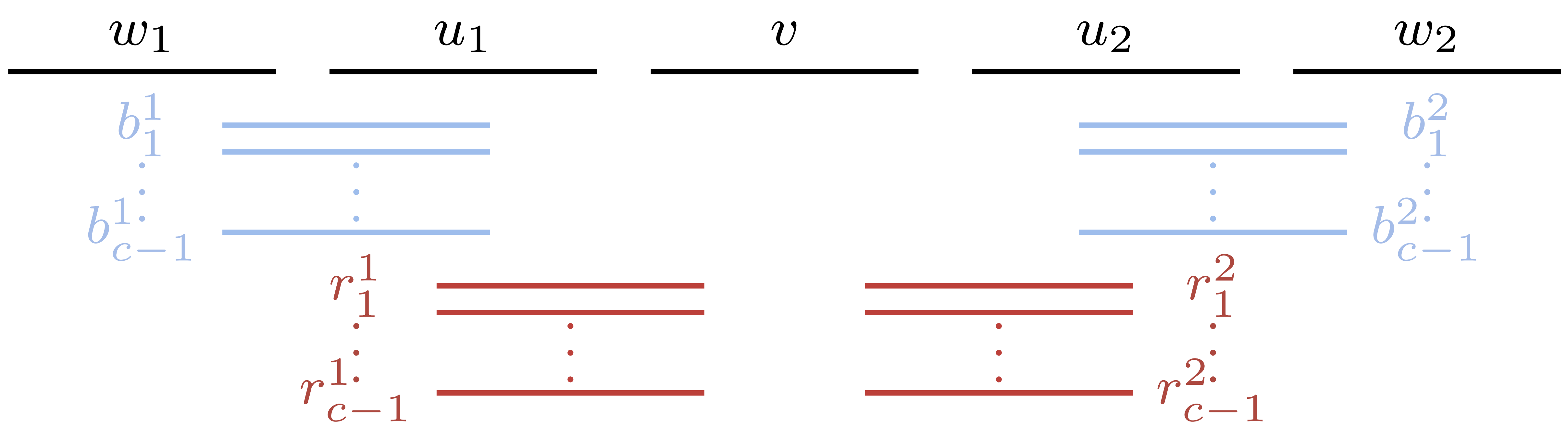

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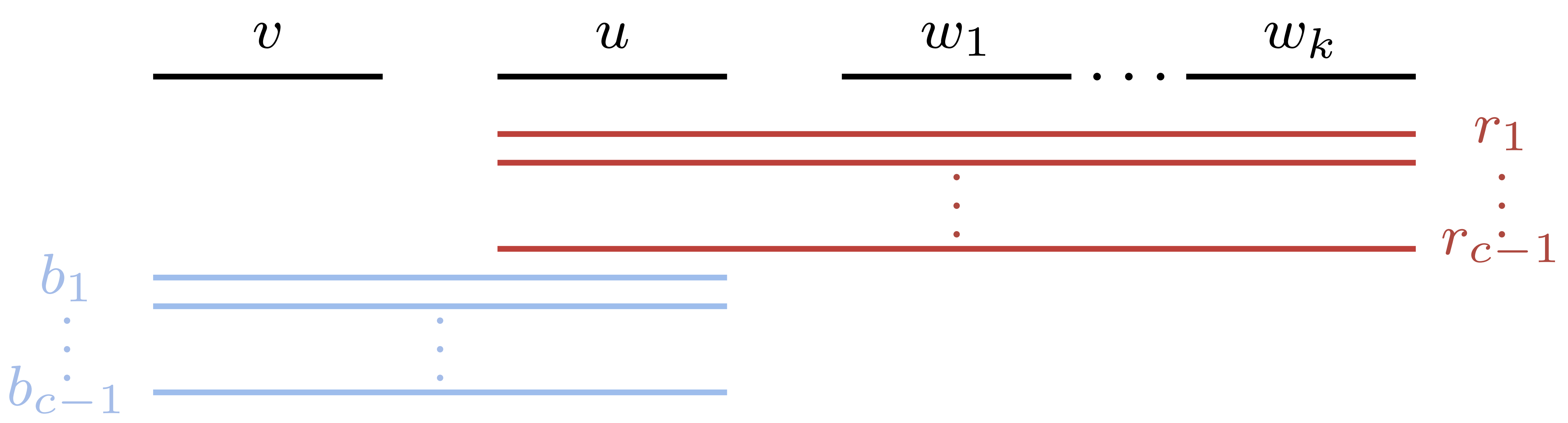

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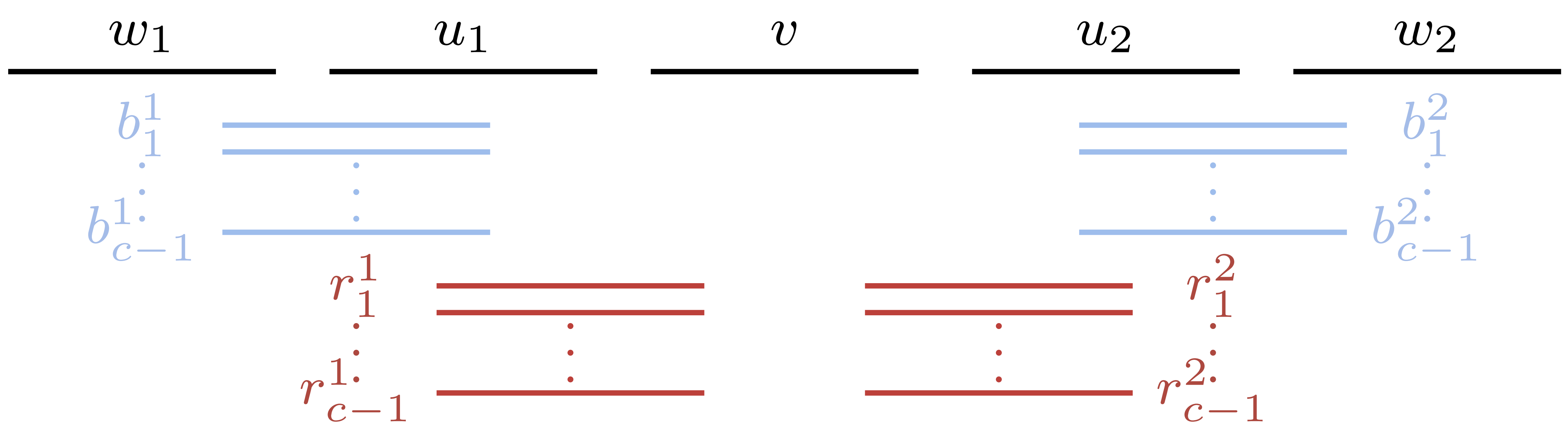

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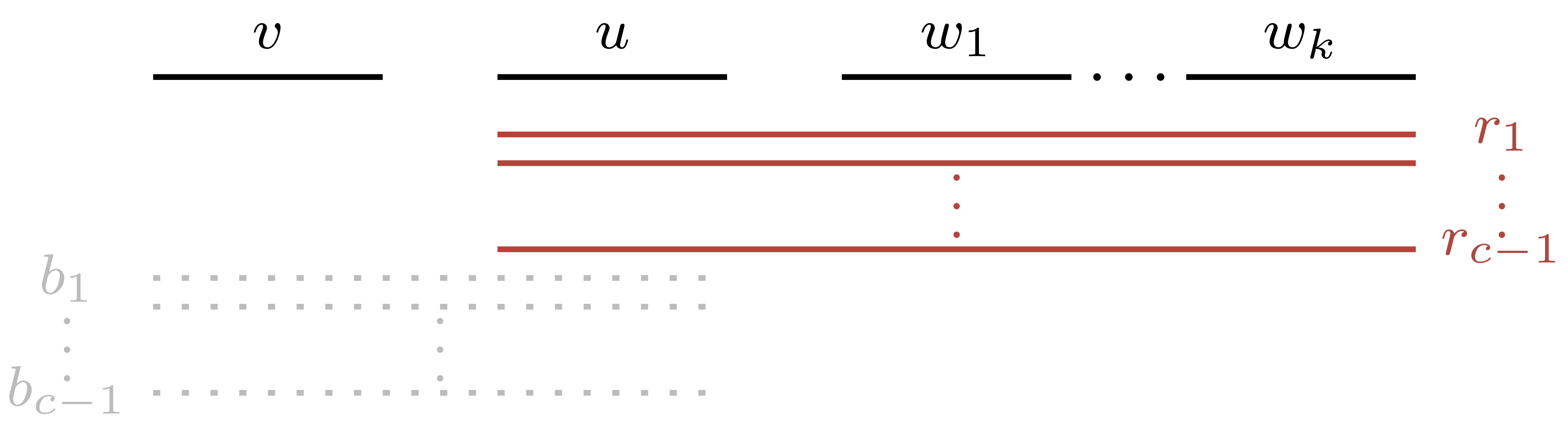

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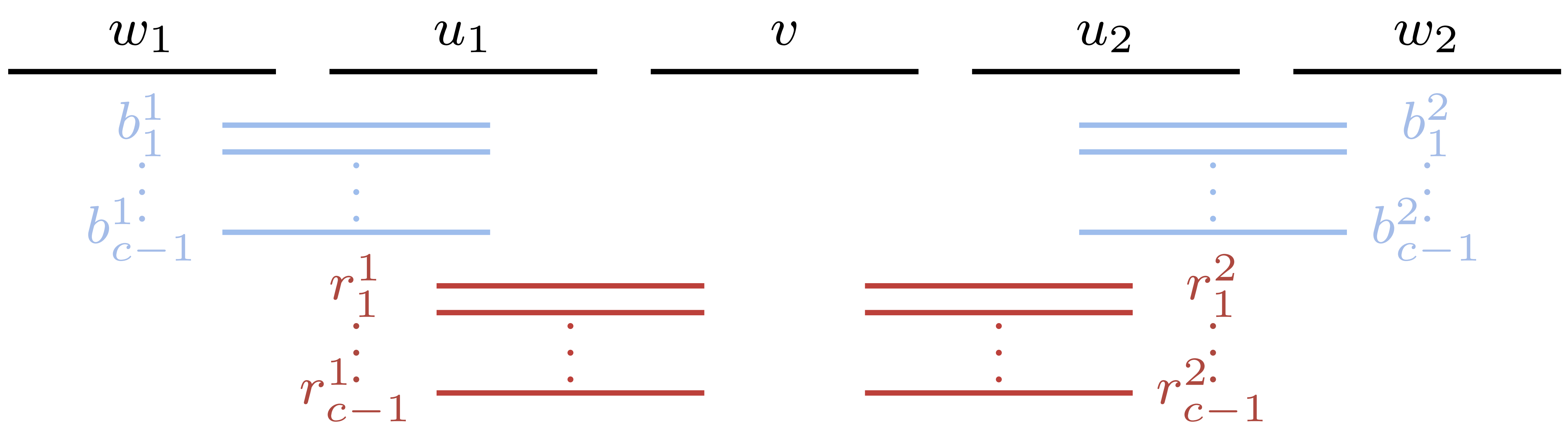

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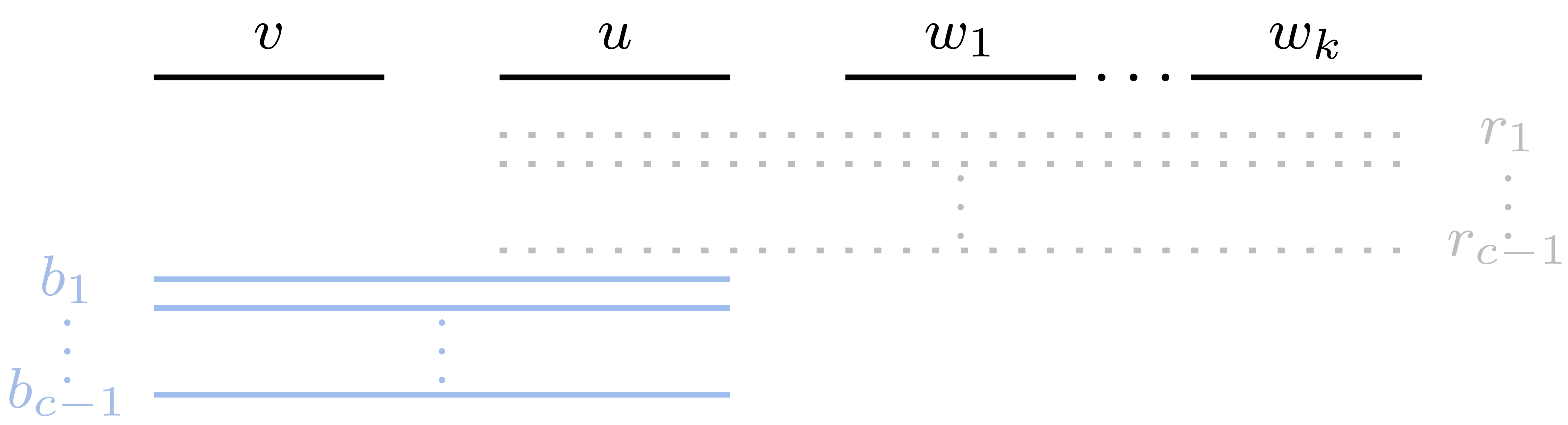

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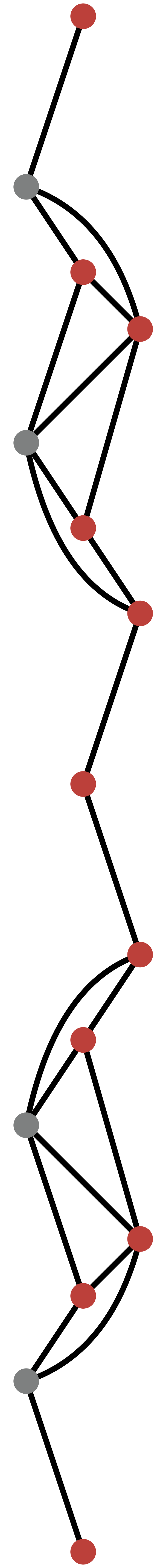
- Surprisingly, the algorithm does not even yield an approximation for general interval graphs (open question!).



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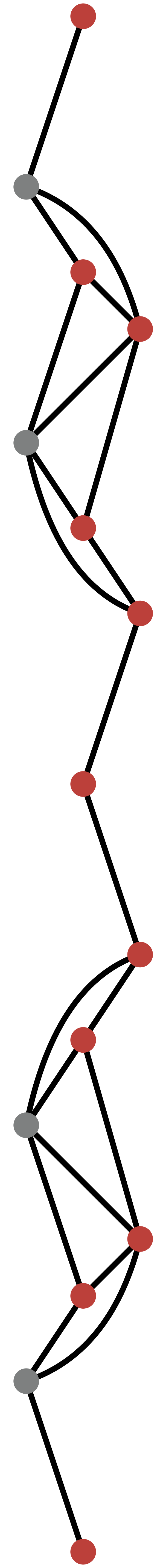

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