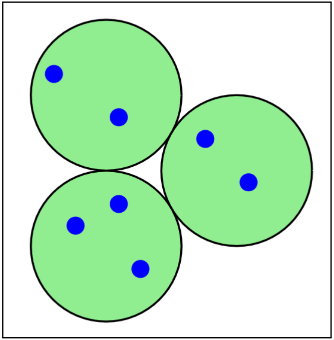
Exact covering with unit disks

Ji Hoon Chun, **Christian Kipp**, Sandro Roch

EuroCG 2024

Inaba's puzzle

- *Show that any set of $n = 10$ points in $\mathbb{R}^2$ can be covered by* **disjoint** *unit disks.*



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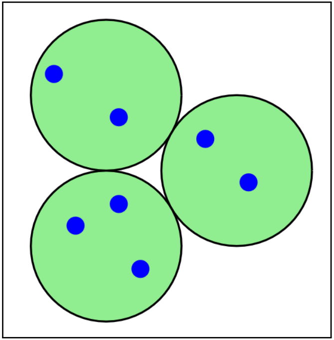

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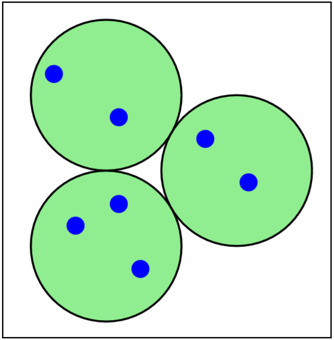

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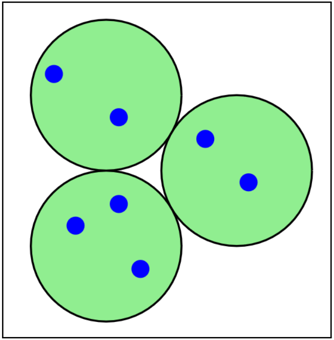

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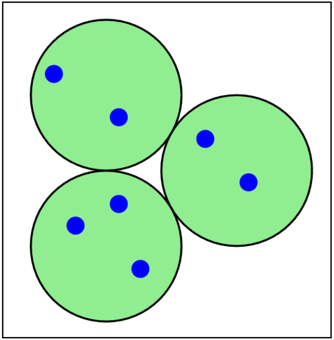
  - lower bound: 12,



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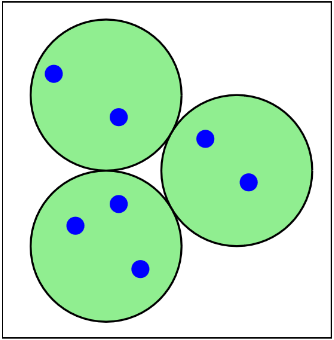

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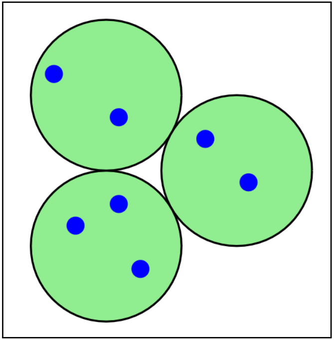
- We consider a relaxation.



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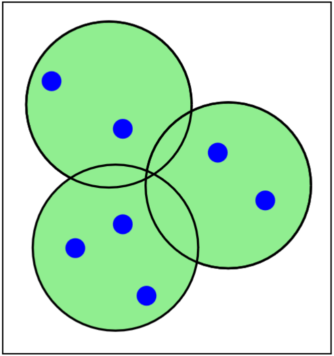

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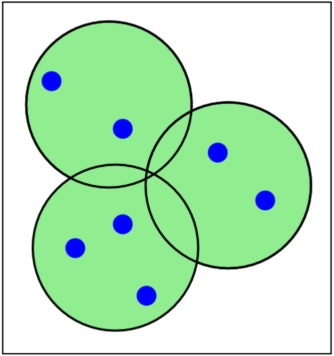

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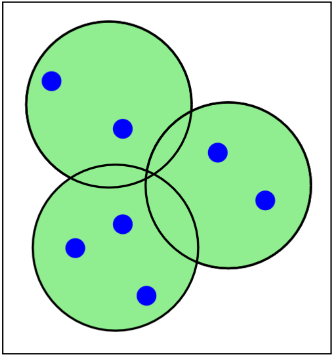

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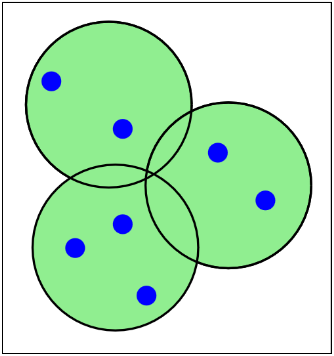

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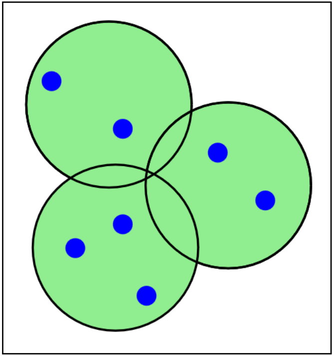

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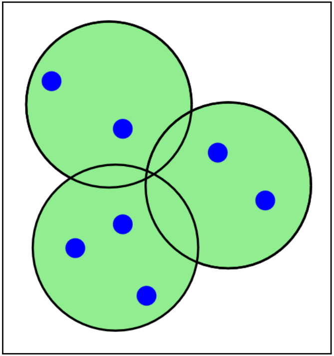
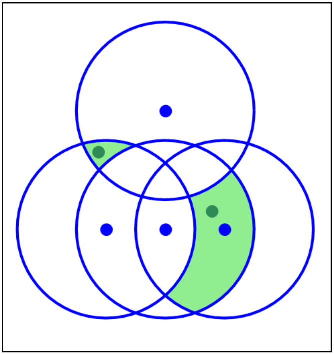


The dual formulation

- Given: $X \subset \mathbb{R}^2$ finite.



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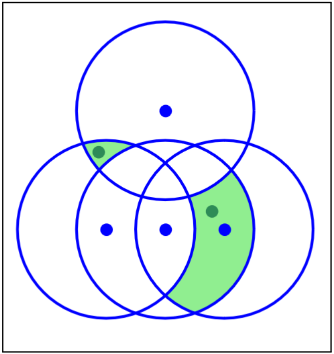

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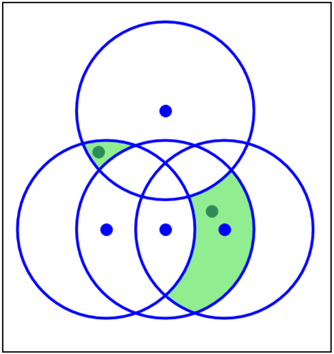

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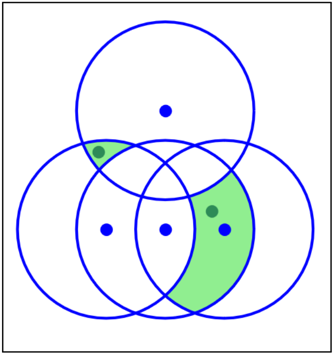

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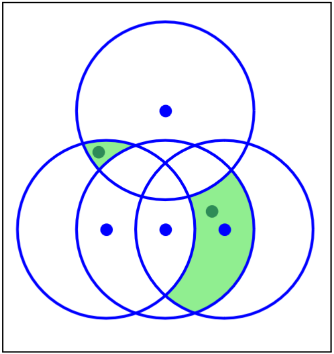

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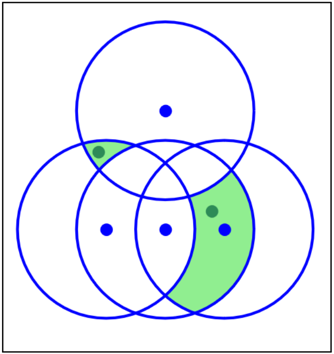
- Special case of `EXACT COVER`.
  - Can be solved with Algorithm X, SAT solvers or integer programming.



The lower bound

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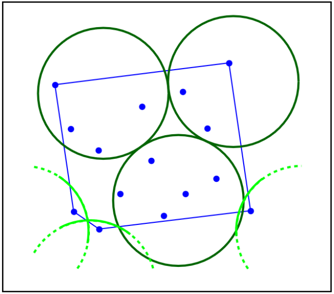

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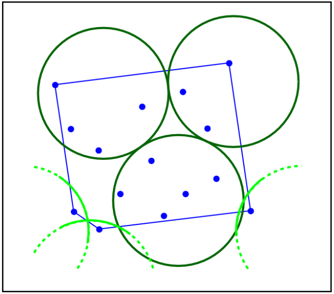
- Extension argument: find disjoint cover of the points in the interior of $\operatorname{conv} X$, extend to an exact cover of X.



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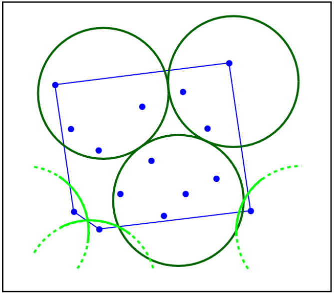

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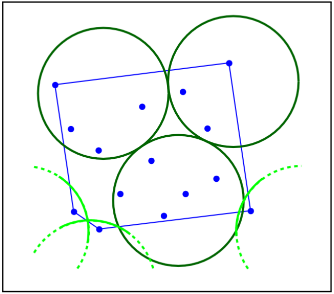

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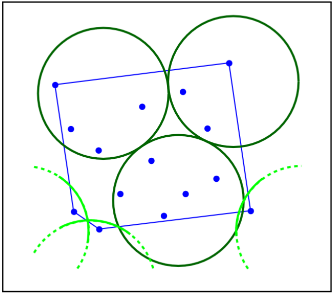

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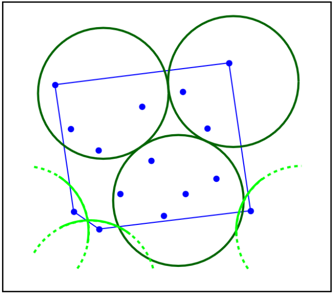
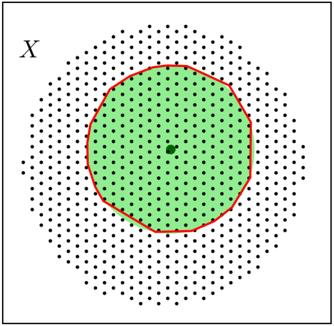
- Careful analysis of case distinctions improves the lower bound to 17.



Intuitive idea behind the upper bound

- Let $X \subset \mathbb{R}^2$ be finite and $Y \subset \mathbb{R}^2$ be a the set of disk centers of an exact cover.



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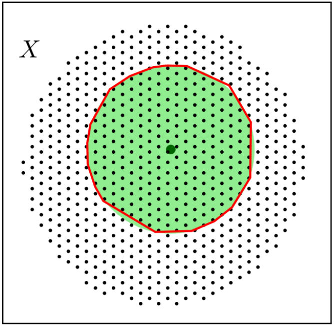
- Consider the Voronoi diagram of Y.



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- Let $X \subset \mathbb{R}^2$ be finite and $Y \subset \mathbb{R}^2$ be a the set of disk centers of an exact cover.
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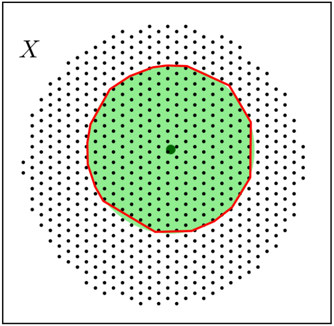
- If X is "sufficiently dense", then the Voronoi regions of Y must be almost spherical.



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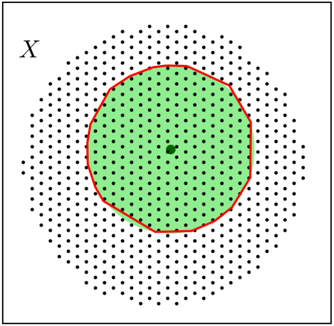
- But three almost spherical polytopes cannot meet at a common vertex.



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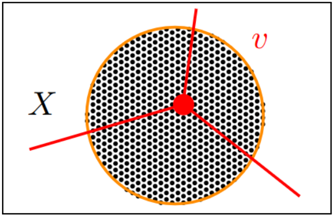
- Let $M \subset \mathbb{R}^2$ and $\varepsilon > 0$.



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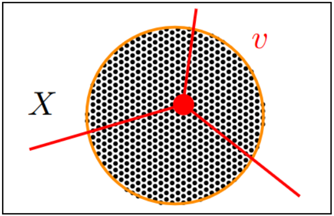
- Let $M \subset \mathbb{R}^2$ and $\varepsilon > 0$.
- X is an ε-*net* of M if $M \subset X + \varepsilon D$, where D is the unit disk centered at 0.



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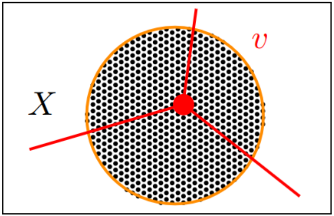
- M is an *ε-blocker* if no ε-net of M has an exact cover.



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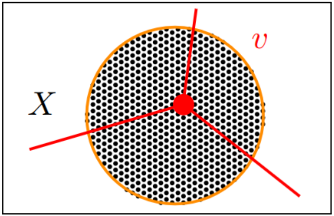

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- **Thm.** Every disk of radius $R > 1$ is an ε-blocker for some $\varepsilon > 0$.
- Upper bound of 656 is obtained by constructing an ε-net of $\frac{3+3\varepsilon}{2}D$, where $\varepsilon \approx 0.07$.