On Maximal 3-planar Graphs

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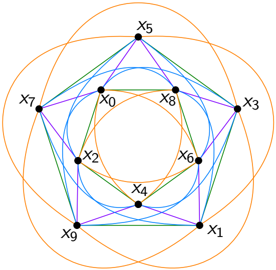

Planar Graphs

Definition (Planar Graphs)

A graph is planar if it admits a drawning in the plane without crossings.

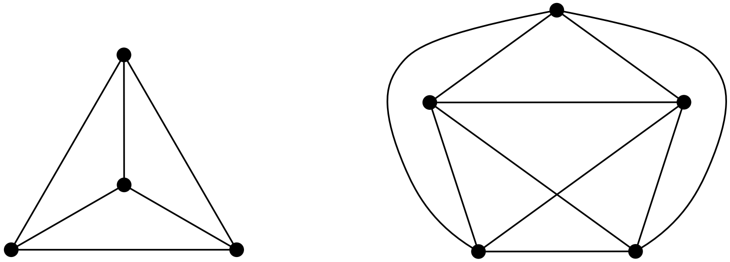

Figure: K_4 is a planar graph, while K_5 is not.

k-planar Graphs

Definition (k-planar Graphs)

A graph is k-planar if it admits a drawing in the plane such that every edge is crossed at most k times.

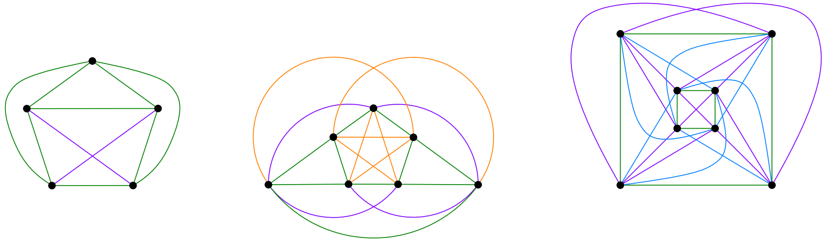

Figure: (Left)K_5 is a 1-planar. (Middle)K_7 is 2-planar. (Right)K_8 is 3-planar.

Edges are colored to indicate the number of crossings over them.
Planar edges — green; Singly crossed edges — purple;
Doubly crossed edges — orange; Triply crossed edges — blue.

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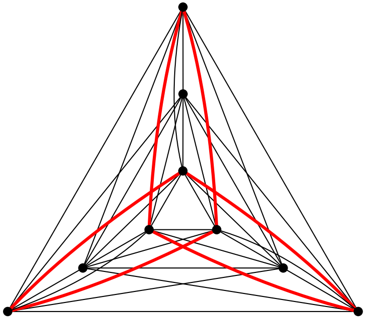

Figure: A drawing of K_9 where red edges are crossed more than 3 times.

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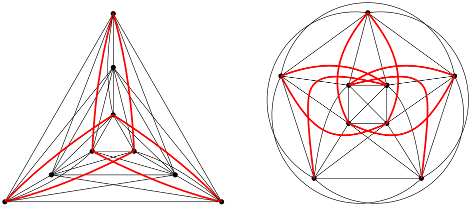

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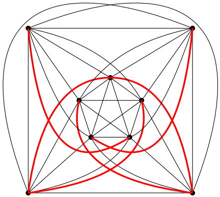
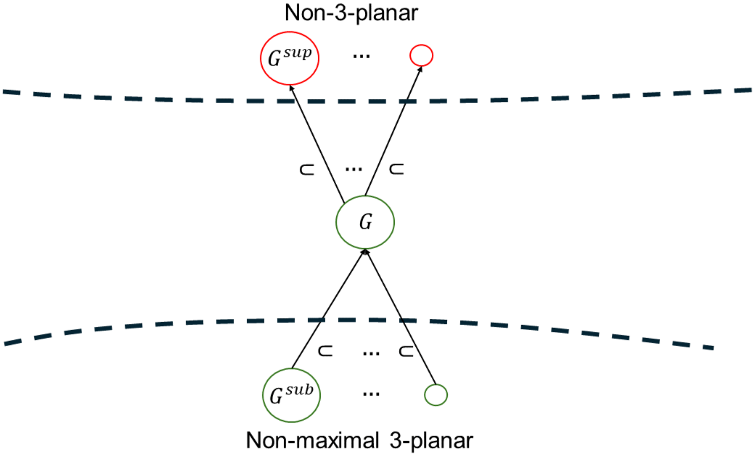


- There always exists an edge with at least four crossings.

Maximal k-planar Graphs

Definition (Maximal k-planar Graphs)

A k-planar graph is maximal if addition of any edge results in a graph that is not k-planar.



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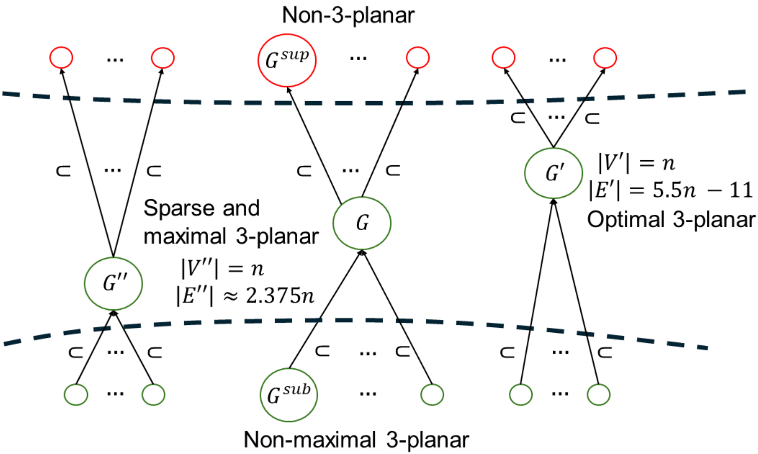

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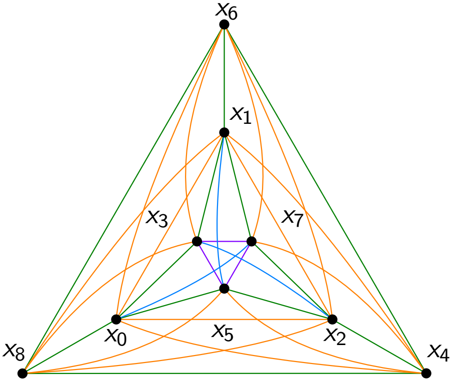

Figure: $K_9 \setminus \{\{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}\}$

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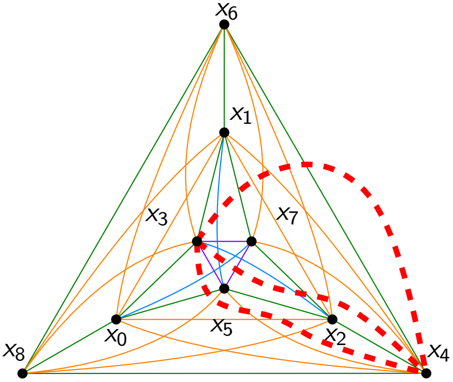

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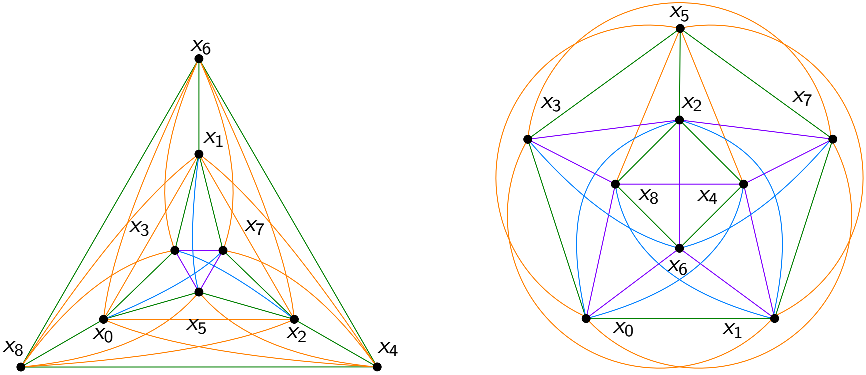

Figure: $K_9 \setminus \{\{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}\}$

Simple 3-plane Drawings

Definition (Simple Drawings)

A drawing is simple if every pair of edges shares at most one common point, including end points, and any edge does not cross itself.

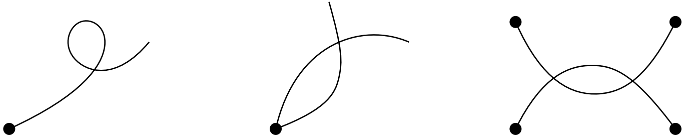

Figure: Forbidden structures in simple drawings.

Simple 3-plane Drawings

Theorem (Pach, Radoičić, Tardos, and Tóth, 2006)

Every 3-planar graph admits a simple 3-plane drawing.

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If a 3-planar graph G is not maximal 3-planar, then there exists a simple 3-plane drawing of G that is not maximal 3-plane.

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Proof.

There exists an edge e s.t. $G' = G \cup e$ is still 3-planar, and admits a simple 3-plane drawing. Remove e from the drawing, we get a simple 3-plane drawing for G. □

K_9 Based Graphs

Graphs	Simple 3-plane Drawing
K_9	$\times$
K_9/K_2	$\times$
$K_9/(K_2+K_2)$	$\times$
$K_9/(P_3)$	$\times$
$K_9/(K_2+K_2+K_2)$	$\checkmark$
$K_9/(P_3+K_2)$	$\checkmark$
$K_9/(P_4)$	$\checkmark$
$K_9/(K_3)$	$\checkmark$
$K_9/(\text{3-Star})$	$\checkmark$

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$K_9/(K_2 + K_2)$	$\times$
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$K_9/(K_2 + K_2 + K_2)$	$\checkmark$
$K_9/(P_3 + K_2)$	$\checkmark$
$K_9/(P_4)$	$\checkmark$
$K_9/(K_3)$	$\checkmark$
$K_9/$(3-Star)	$\checkmark$

Theorem

A graph on nine vertices is 3-planar if and only if it has at most 33 edges, and it is maximal 3-planar if and only if it has exactly 33 edges.

Unique Simple 3-plane Drawing

- $G_1 \cong K_{10} \setminus \{\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_6, x_7\}, \{x_8, x_9\}\}$

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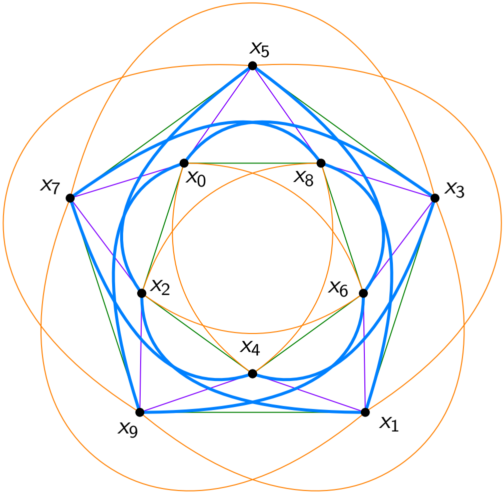

2-connectivity for Maximal Near-planar Graphs

Theorem (Michael Hoffmann, Meghana M. Reddy 2023)

For $k \leq 2$, every maximal k-planar graph on $n \geq 3$ vertices is 2-connected.

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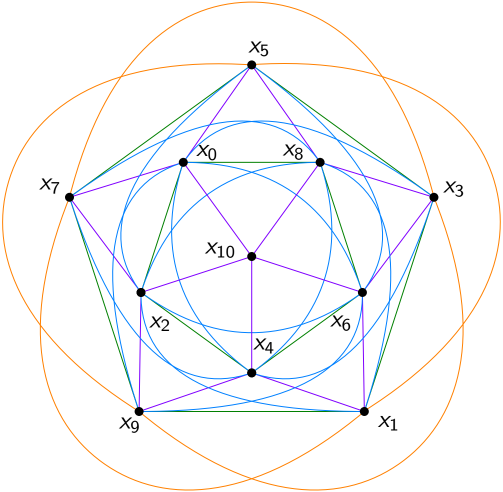
Theorem

There exist infinitely many maximal 3-planar graphs that are not 2-connected.

Cut Vertex in Maximal 3-planar Graphs



- $G_2 \cong G_1 \cup \{\{x_{10}, x_0\}, \{x_{10}, x_2\}, \{x_{10}, x_4\}, \{x_{10}, x_6\}, \{x_{10}, x_8\}\}$ (5-star).

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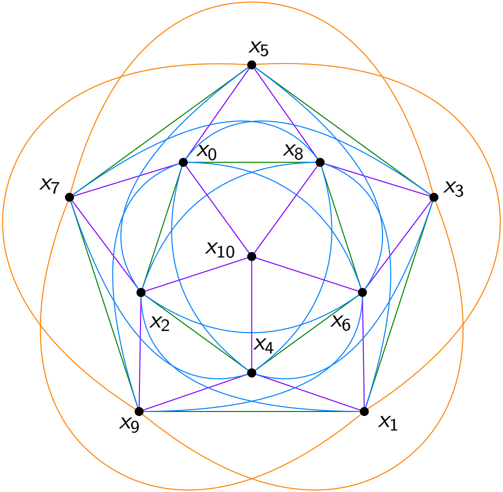

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- G_2 is also maximal 3-planar.

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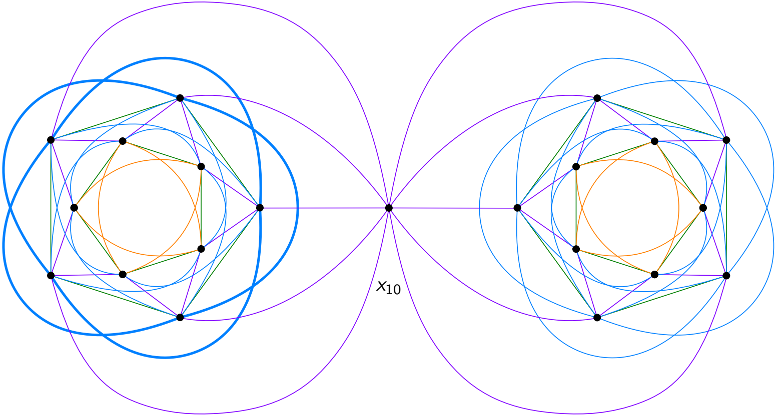

- $G_3 \cong$ Two copies of G_2 glued together at five-degree vertex.

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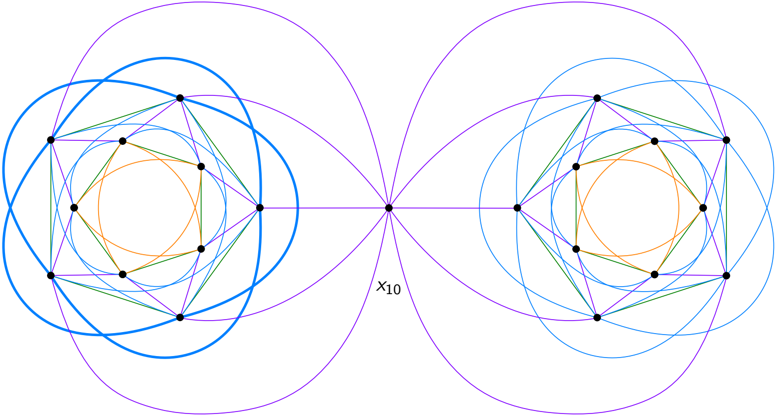

- $G_3 \cong$ Two copies of G_2 glued together at five-degree vertex.
- G_3 is also maximal 3-planar.

Sparse Maximal k-planar Graphs

- For connected graphs on n vertices.

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Graph Families	Minimal Edge Density		Optimal
	Lower Bound	Upper Bound	
Maximal 1-planar Graphs	$2.22n$	$2.647n$	$4n - 8$
Maximal 2-planar Graphs	$2n$	$2n + O(1)$	$5n - 10$

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Maximal 2-planar Graphs	$2n$	$2n + O(1)$	$5n - 10$
Maximal 3-planar Graphs	?	$2.375n + O(1)$	$5.5n - 11$

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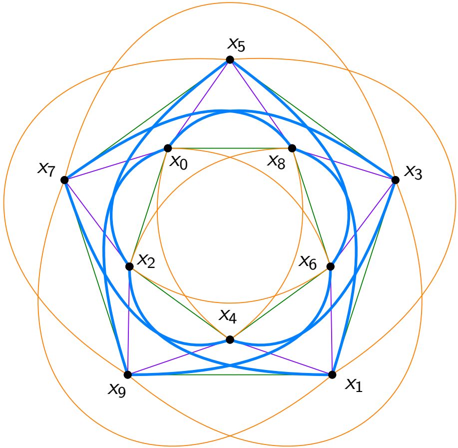

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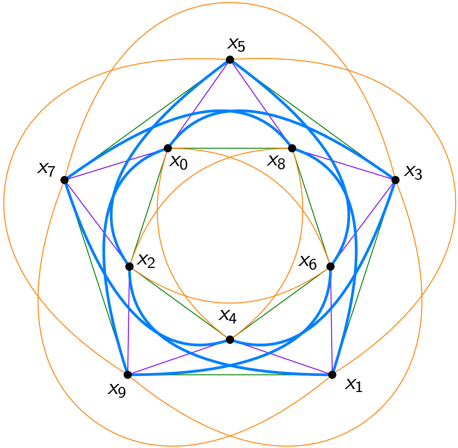

- Attach a two-degree vertex to each planar edge.

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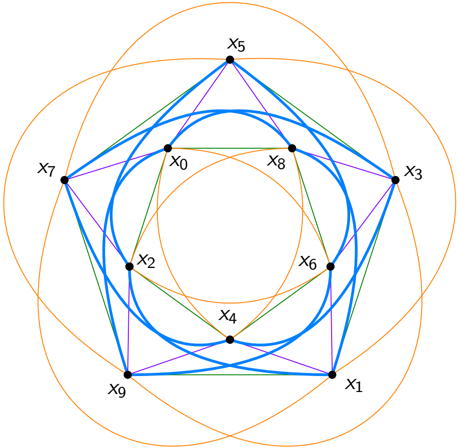
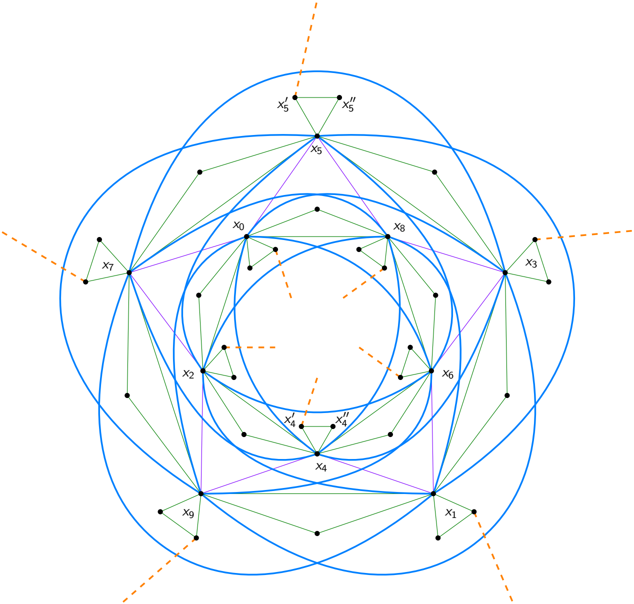


- Attach a two-degree vertex to each planar edge.
- Attach a triangle to each x_i.

Sparse Maximal 3-planar Graphs

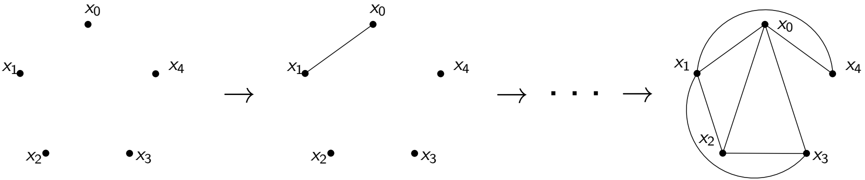


Open Problems

- Sparser maximal 3-planar graphs?
  - It is conjectured that there exist maximal 3-planar graphs on n vertices with $2n + O(1)$ edges.
  - More tools are needed for maximality proof.
- Lower bound of the edge density of maximal 3-planar graphs?
  - Looking into the local property of drawings.
  - Existence and density of low-degree vertices.
  - Often faced with exhaustive case analysis.
- $\cdots$

Enumerating Drawings

- Adding edges one by one.



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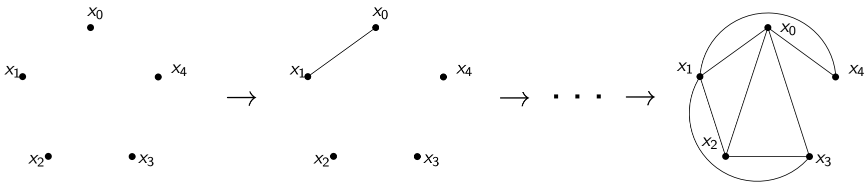
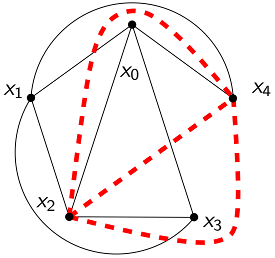


- Attemp to add edge $\{x_2, x_4\}$.



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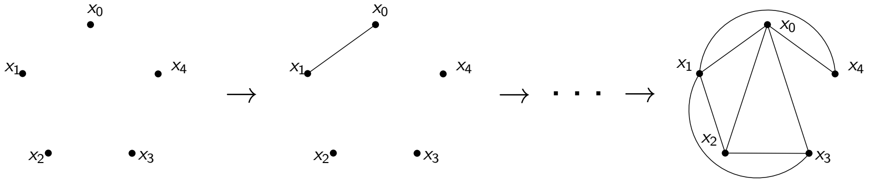

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