Revisiting the Fréchet distance between piecewise smooth curves

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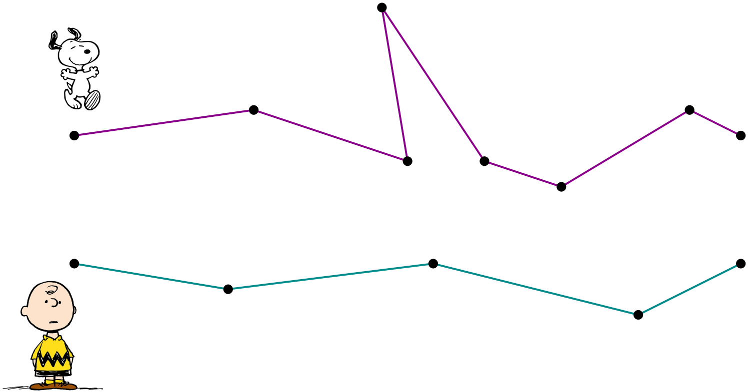
The Fréchet distance



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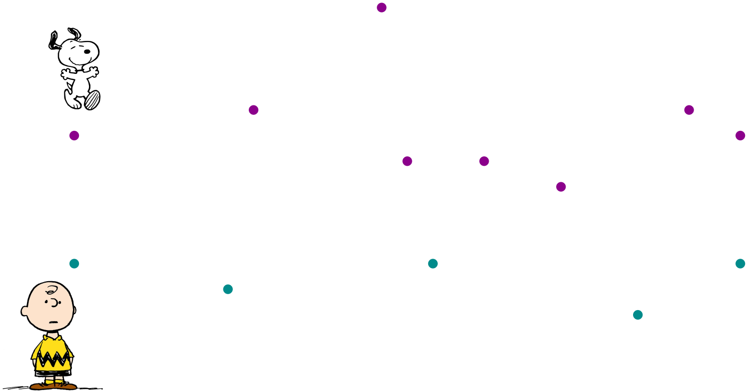

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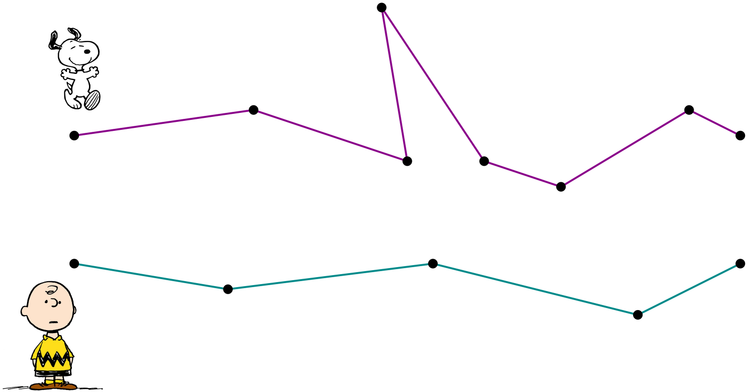

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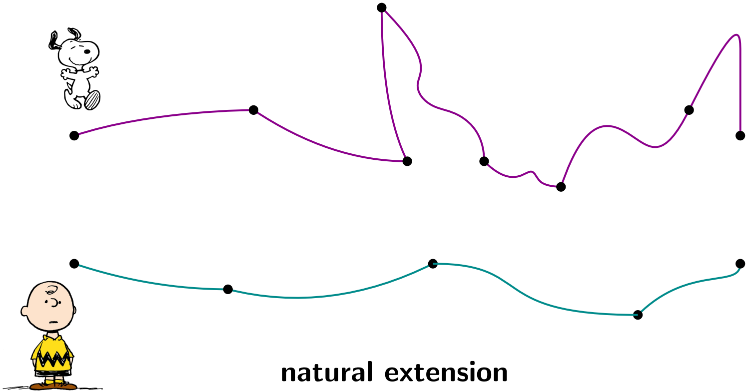
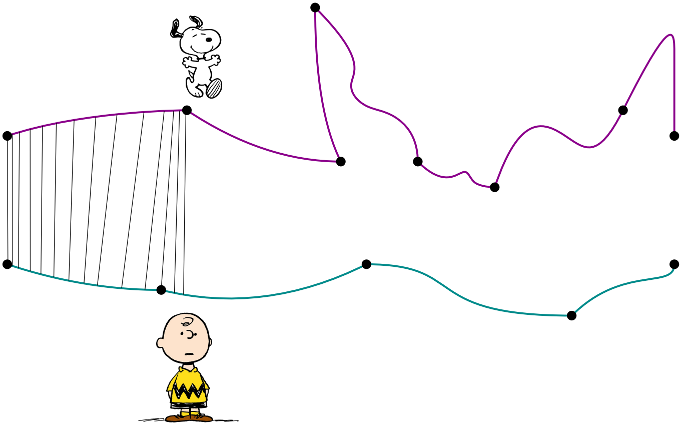


natural extension

The Fréchet distance



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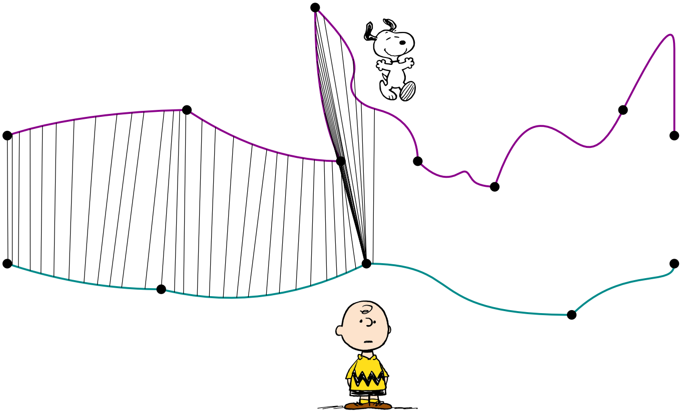
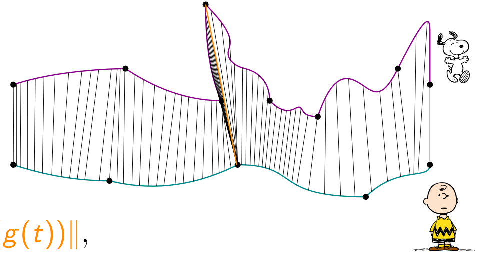


The Fréchet distance

Definition (Fréchet distance)

The **Fréchet distance** $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2)$ for curves γ_1, γ_2 in $\mathbb{R}^d$ *is*



$$\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) = \inf_{f,g:[0,1]\to[0,1]} \max_{t\in[0,1]} \|\gamma_1(f(t)) - \gamma_2(g(t))\|,$$

where f and g are continuous, non-decreasing and surjective.

How does one usually compute the Fréchet distance?

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Definition (Decision problem)

Given $\delta > 0$, decide if

$$\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) \leq \delta.$$

- corner stone of algorithms computing the Fréchet distance
- Standard techniques (parametric search) lead to algorithm to compute $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2)$ (incurring an additional log runtime factor)

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State-of-the-art for polygonal curves

- For polygonal curves γ_1, γ_2 consisting of m and n pieces:
  - An $O(mn)$ algorithm for the decision problem (more or less) (Alt and Godau 1995)
  - A resulting $O(mn \log(mn))$ algorithm to compute $d_{\mathcal{F}}(\gamma_1, \gamma_2)$.
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Free space diagram

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Definition (Free space diagram)

Given $\delta > 0$ and γ_1, γ_2, the FSD_δ is a decomposition of $[0,1]^2$:

1. Decomposition of $[0,1]^2$ into cells, one for each pair of pieces of γ_1 and γ_2
2. $(t_1, t_2) \in [0,1]^2$ is associated to a pair of points $(\gamma_1(t_1), \gamma_2(t_2))$;

   (t_1, t_2) is in the **free space** $\mathcal{D}_\delta$ if $\gamma_1(t_1)$ and $\gamma_2(t_2)$ are at most δ apart.

   - **free space** $\mathcal{D}_\delta = \{(t_1, t_2) \in [0,1]^2 | \|\gamma_1(t_1) - \gamma_2(t_2)\| \leq \delta\}$
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Monotone paths in FSD_δ

Lemma

There is a path, monotone in both coordinates, from $(0,0)$ *to* $(1,1)$ *inside* $\mathcal{D}_\delta$ *iff*

$$\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2) \leq \delta.$$

Observation

For polygonal curves, $\mathcal{D}_\delta$ is **convex** in each cell, so checking for monotone paths is straightforward.

What curves are we interested in?

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Definition (Piecewise smooth curve of complexity n)

A piecewise smooth curve $\gamma : [0,1] \to \mathbb{R}^d$ consisting of m pieces that are each C^2

Definition (Algebraically bounded curves)

Set $\mathcal{S}$ of piecewise algebraic curves with maximal degree of each piece bounded by a constant

State-of-the-art for piecewise smooth curves

- Known: **Planar** algebraically bounded curves are just like polygonal curves:
  - An $O(nm)$ algorithm for the decision problem (Rote 2007).
  - An $O(mn\log(mn))$ algorithm to compute $\mathrm{d}_{\mathcal{F}}(\gamma_1, \gamma_2)$.
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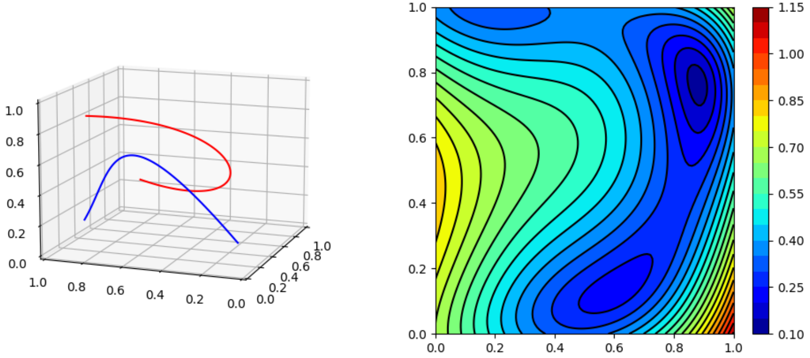
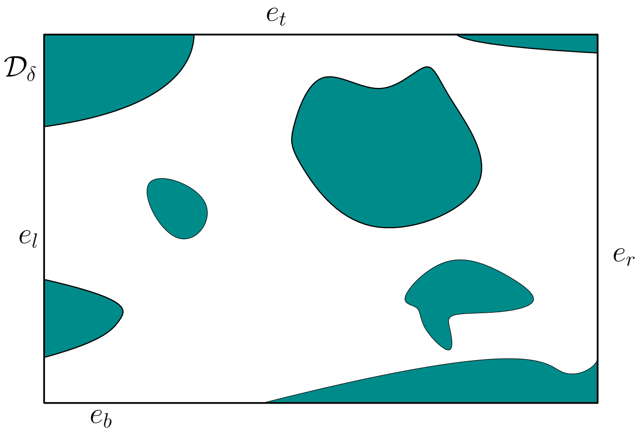
Contour plot of the free space diagram



How could we make each cell easier to analyze? Refine FSD_δ



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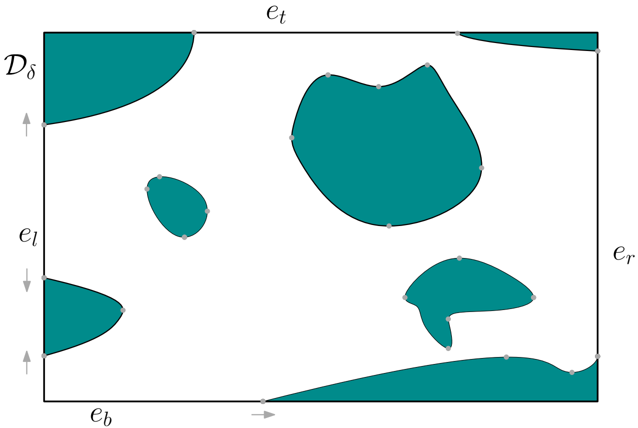

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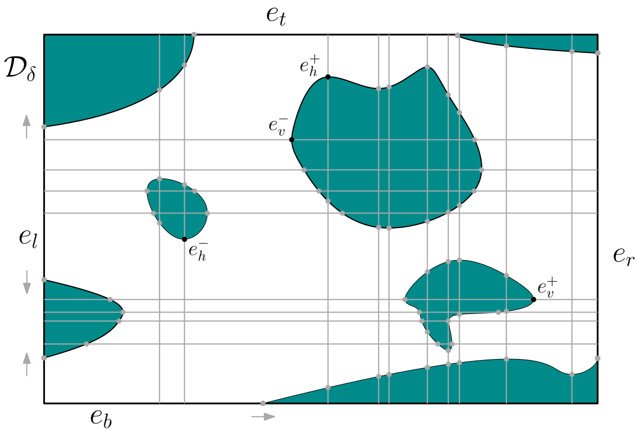
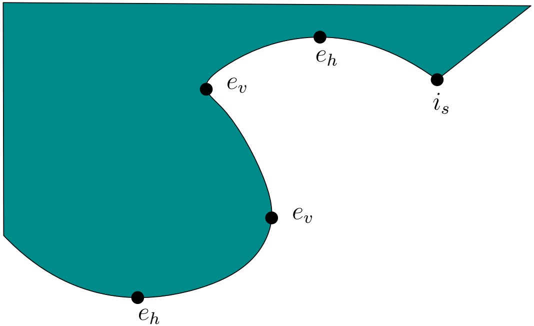


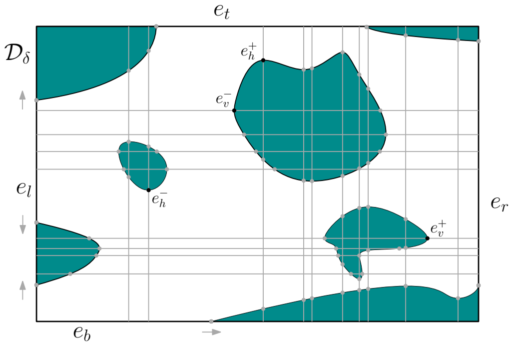
What does this do?



Observation

Each piece of boundary between points (cell walls) is monotone.

Reconstructing FSD_δ

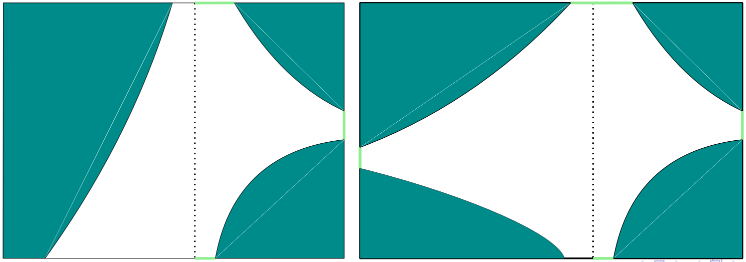


Lemma

The marked points and the shown arrows allow the reconstruction of FSD_δ.

Solving the decision problem

- Starting from the bottom-left, trace the intervals on cell walls that are reachable by a monotone path (BFS on cells)
- coordinates of intervals present in coordinates of marked points on cell walls



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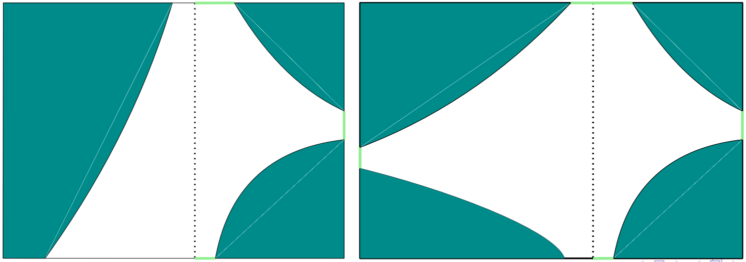


Solving the decision problem

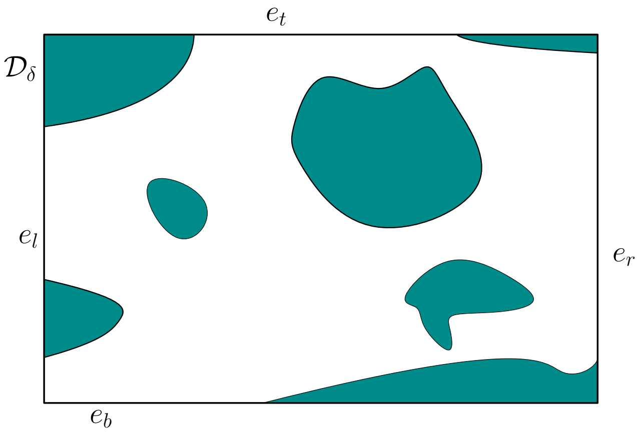
Proposition

The refinement of FSD_δ of algebraically bounded curves consisting of m resp. n pieces consists of $O(mn)$ subcells, i.e., each cell splits into $O(1)$ subcells.

Theorem

For such curves, decision problem can be solved in $O(mn)$ time.

What can go wrong in the above approach? How to avoid critical values?



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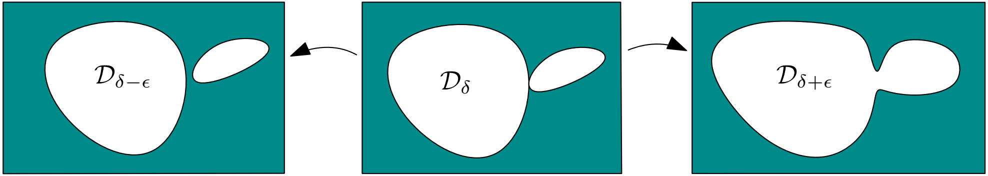

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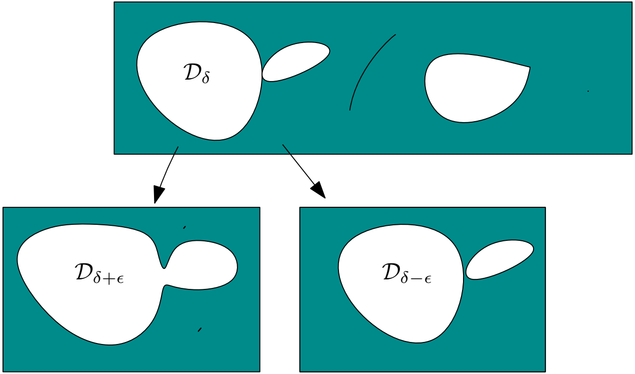

Critical values

Critical values

Theorem

For algebraically bounded curves, the number of values of δ where the decision problem is subject to change is finite.

- In particular, there are only finitely many values of δ where the boundary of the free space is not smooth.
- The theorem holds for all ℓ_p norms with $p \neq 1, \infty$ and with some restrictions for all p (in the definition of $\mathrm{d}_\mathcal{F}(\gamma_1, \gamma_2)$)

Take-home message

- Instead of analyzing the curves γ_1, γ_2 directly, look at FSD_δ for nice values of δ.
- The maxima and minima in x and y direction yield a decomposition.
- The decision problem can be solved in the same time as for polygonal curves.
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Thank you for your attention!

Questions?