## Revisiting the Fréchet distance between piecewise smooth curves

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EuroCG 2024, Ioannina, Greece





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The Fréchet distance			



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#### The Fréchet distance



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Definition (Fréchet distance)
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The **Fréchet distance**  $d_{\mathcal{F}}(\gamma_1, \gamma_2)$  for curves  $\gamma_1, \gamma_2$  in  $\mathbb{R}^d$  is



$$d_{\mathcal{F}}(\gamma_1, \gamma_2) = \inf_{f,g:[0,1]\to[0,1]} \max_{t\in[0,1]} \|\gamma_1(f(t)) - \gamma_2(g(t))\|,$$

where f and g are continuous, non-decreasing and surjective.

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Preliminaries

Computing the Fréchet distance - in a nutshell

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## How does one usually compute the Fréchet distance?

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## How does one usually compute the Fréchet distance?

Preliminaries

#### Definition (Decision problem)

Given  $\delta > 0$ , decide if

 $\mathbf{d}_{\mathcal{F}}(\gamma_1,\gamma_2) \leq \delta.$ 

corner stone of algorithms computing the Fréchet distance

 Standard techniques (parametric search) lead to algorithm to compute d<sub>F</sub>(γ<sub>1</sub>, γ<sub>2</sub>) (incurring an additional log runtime factor)

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# State-of-the-art for polygonal curves

Preliminaries

- For polygonal curves  $\gamma_1, \gamma_2$  consisting of *m* and *n* pieces:
  - An O(mn) algorithm for the decision problem (more or less) (Alt and Godau 1995)
  - A resulting  $O(mn \log(mn))$  algorithm to compute  $d_{\mathcal{F}}(\gamma_1, \gamma_2)$ .
  - No strongly subquadratic algorithm unless SETH fails (Bringmann 2014)

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#### Free space diagram

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## Free space diagram

#### Definition (Free space diagram)

#### Given $\delta > 0$ and $\gamma_1, \gamma_2$ , the $FSD_{\delta}$ is a decomposition of $[0, 1]^2$ :

- 1. Decomposition of  $[0,1]^2$  into cells, one for each pair of pieces of  $\gamma_1$  and  $\gamma_2$
- 2.  $(t_1, t_2) \in [0, 1]^2$  is associated to a pair of points  $(\gamma_1(t_1), \gamma_2(t_2));$ 
  - $(t_1,t_2)$  is in the **free space**  $\mathcal{D}_{\delta}$  if  $\gamma_1(t_1)$  and  $\gamma_2(t_2)$  are at most  $\delta$  apart.
    - ▶ free space  $\mathcal{D}_{\delta} = \{(t_1, t_2) \in [0, 1]^2 | \|\gamma_1(t_1) \gamma_2(t_2)\| \leq \delta\}$
    - forbidden region  $[0,1]^2 \setminus \mathcal{D}_{\delta}$

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    - forbidden region  $[0,1]^2 \setminus \mathcal{D}_{\delta}$

## Monotone paths in $\mathrm{FSD}_\delta$

Lemma

There is a path, monotone in both coordinates, from (0,0) to (1,1) inside  $\mathcal{D}_{\delta}$  iff

 $\mathbf{d}_{\mathcal{F}}(\gamma_1,\gamma_2) \leq \delta.$ 

Observation For polygonal curves,  $\mathcal{D}_{\delta}$  is **convex** in each cell, so checking for monotone paths is straightforward.

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What are we dealing with?			

#### What curves are we interested in?

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#### What curves are we interested in?

#### Definition (**Piecewise smooth curve of complexity** *n*)

A piecewise smooth curve  $\gamma: [0,1] o \mathbb{R}^d$  consisting of m pieces that are each  $C^2$ 

#### Definition (Algebraically bounded curves)

Set  $\mathcal S$  of piecewise algebraic curves with maximal degree of each piece bounded by a constant

#### State-of-the-art for piecewise smooth curves

► Known: **Planar** algebraically bounded curves are just like polygonal curves:

- An O(nm) algorithm for the decision problem (Rote 2007).
- An  $O(mn \log(mn))$  algorithm to compute  $d_{\mathcal{F}}(\gamma_1, \gamma_2)$ .
- **b** Depends on decomposing  $FSD_{\delta}$  using planar curvature and turning angle of curves
- Unclear how to generalize this approach to higher dimensions

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## State-of-the-art for piecewise smooth curves

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## Contour plot of the free space diagram





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Combinatorial decomposition of  $\mathrm{FSD}_\delta$   $\bullet \mathrm{OOOOO}$ 

#### How could we make each cell easier to analyze? Refine $FSD_{\delta}$



Combinatorial decomposition of  $\mathrm{FSD}_\delta$   $\bullet \mathrm{OOOOO}$ 

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Combinatorial decomposition of  $\mathrm{FSD}_\delta$   $\bullet \circ \circ \circ \circ$ 

#### How could we make each cell easier to analyze? Refine $FSD_{\delta}$



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#### What does this do?

Observation

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Each piece of boundary between points (cell walls) is monotone.

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# Reconstructing $FSD_{\delta}$



#### Lemma

The marked points and the shown arrows

allow the reconstruction of  $FSD_{\delta}$ .

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## Solving the decision problem

 Starting from the bottom-left, trace the intervals on cell walls that are reachable by a monotone path (BFS on cells)

coordinates of intervals present in coordinates of marked points on cell walls



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## Solving the decision problem

 Starting from the bottom-left, trace the intervals on cell walls that are reachable by a monotone path (BFS on cells)

coordinates of intervals present in coordinates of marked points on cell walls



## Solving the decision problem

#### Proposition

The refinement of  $FSD_{\delta}$  of algebraically bounded curves consisting of m resp. n pieces consists of O(mn) subcells, i.e., each cell splits into O(1) subcells.

#### Theorem

For such curves, decision problem can be solved in O(mn) time.

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#### What can go wrong in the above approach? How to avoid critical values?



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## What can go wrong in the above approach? How to avoid critical values?



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## What can go wrong in the above approach? How to avoid critical values?



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#### Critical values

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## Critical values

#### Theorem

For algebraically bounded curves, the number of values of  $\delta$  where the decision problem is subject to change is finite.

- In particular, there are only finitely many values of δ where the boundary of the free space is not smooth.
- The theorem holds for all ℓ<sub>p</sub> norms with p ≠ 1,∞ and with some restrictions for all p (in the definition of d<sub>F</sub>(γ<sub>1</sub>, γ<sub>2</sub>))

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## Take-home message

- lnstead of analyzing the curves  $\gamma_1, \gamma_2$  directly, look at  $FSD_{\delta}$  for nice values of  $\delta$ .
- $\triangleright$  The maxima and minima in x and y direction yield a decomposition.
- The decision problem can be solved in the same time as for polygonal curves.

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## Take-home message

- lnstead of analyzing the curves  $\gamma_1, \gamma_2$  directly, look at  $FSD_{\delta}$  for nice values of  $\delta$ .
- ▶ The maxima and minima in x and y direction yield a decomposition.
- ▶ The decision problem can be solved in the same time as for polygonal curves.
- leads to a computation of d<sub>F</sub>(γ<sub>1</sub>, γ<sub>2</sub>) and generalizes existing framework for polygonal case.

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## Thank you for your attention!

## Questions?

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