Representing Hypergraphs by Point-Line Incidences

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Hypergraph Visualizations

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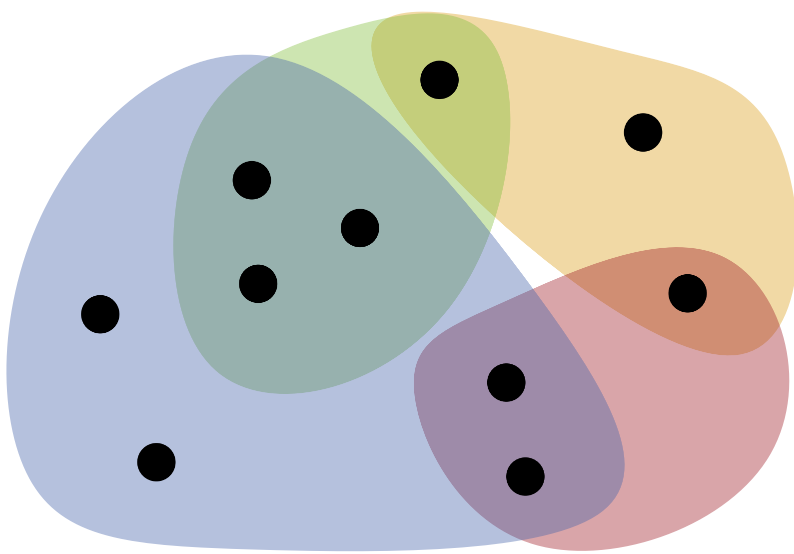
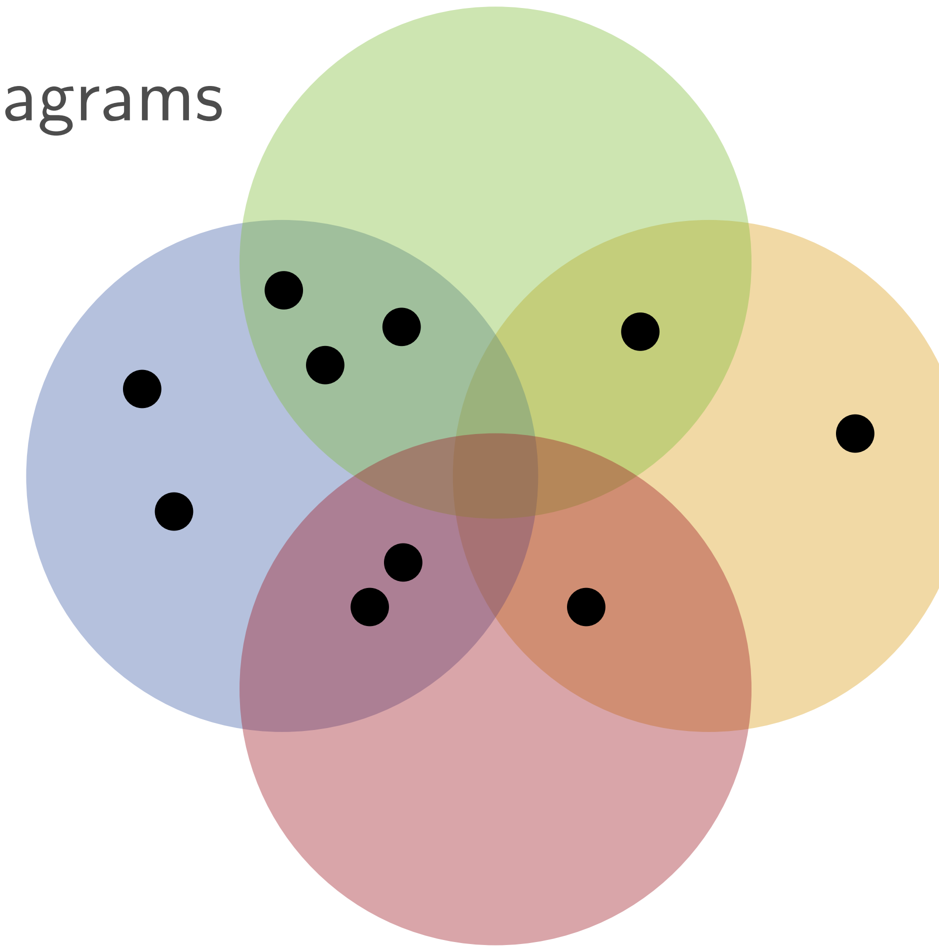
Hypergraph $H = (V, E)$
V . . . elements/vertices
E . . . sets/hyperedges

Euler Diagrams



Venn Diagrams



MetroSets

Hypergraph $H = (V, E)$
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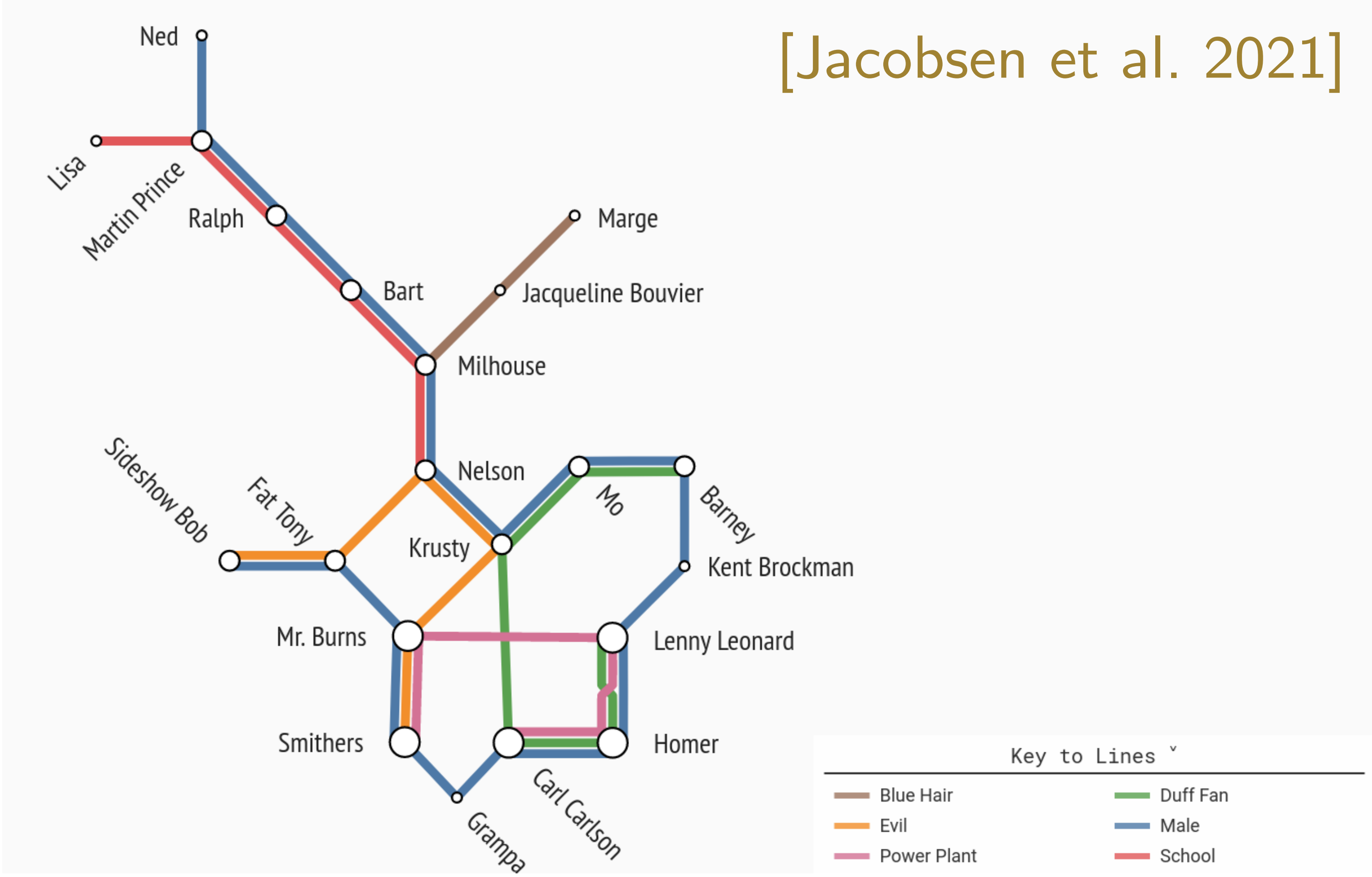
[Jacobsen et al. 2021]



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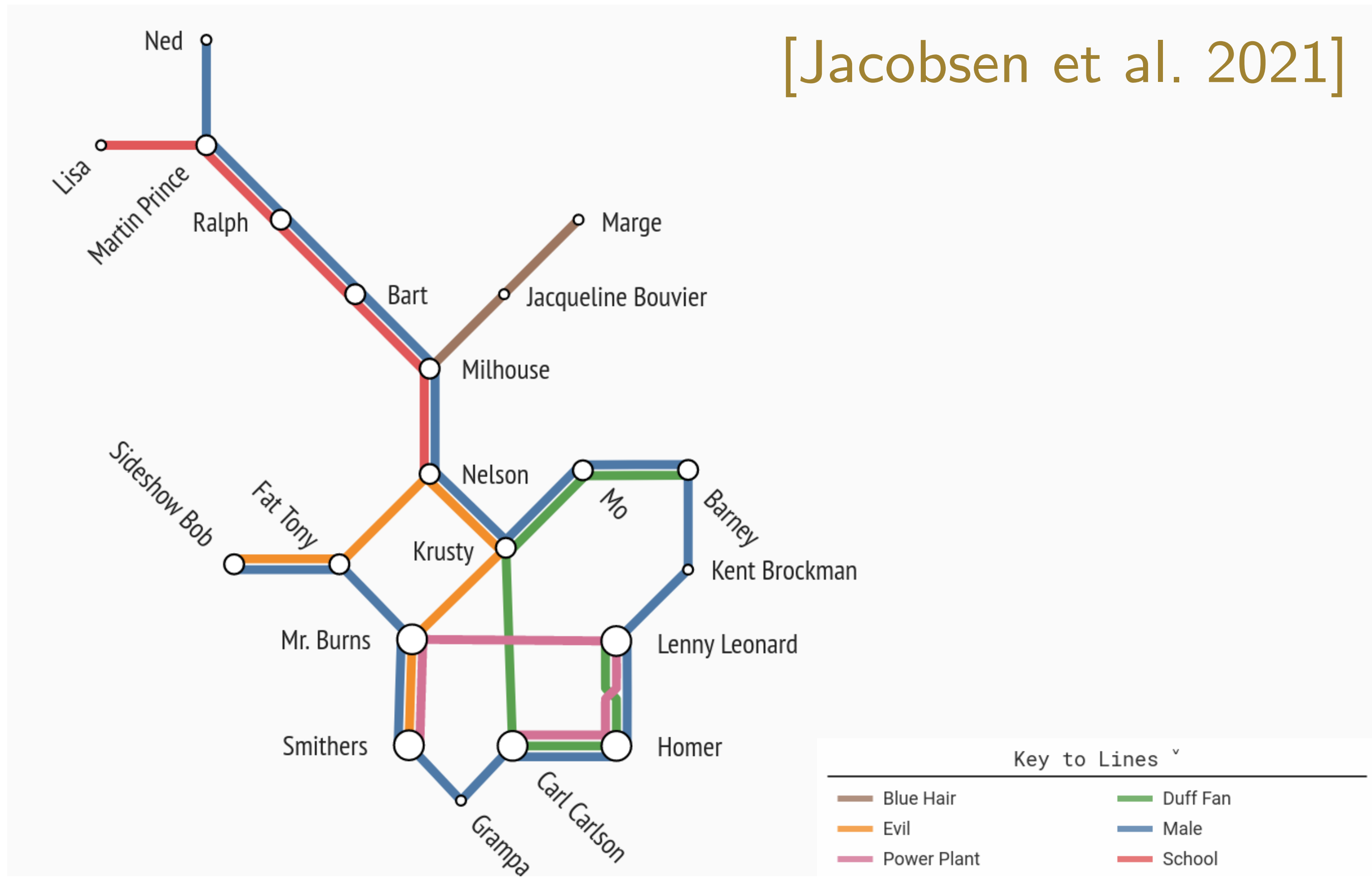

Many bends can happen
Can we find representations without bends/with few bends?

Hypergraphs Visualizations as Point-Line Incidences

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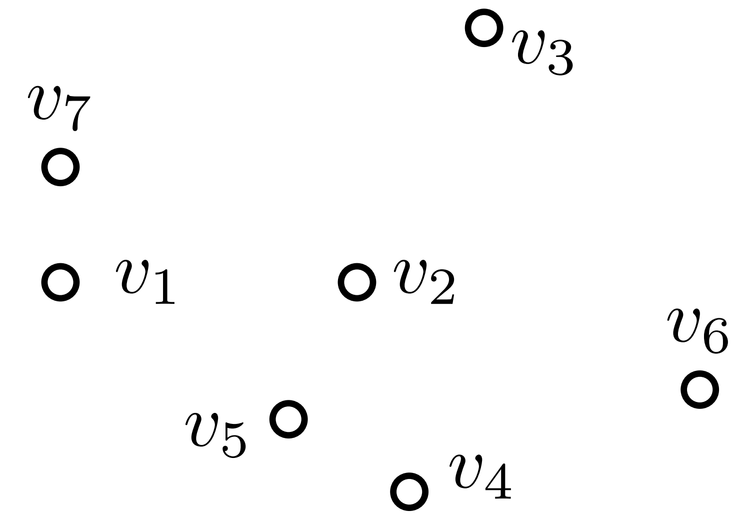

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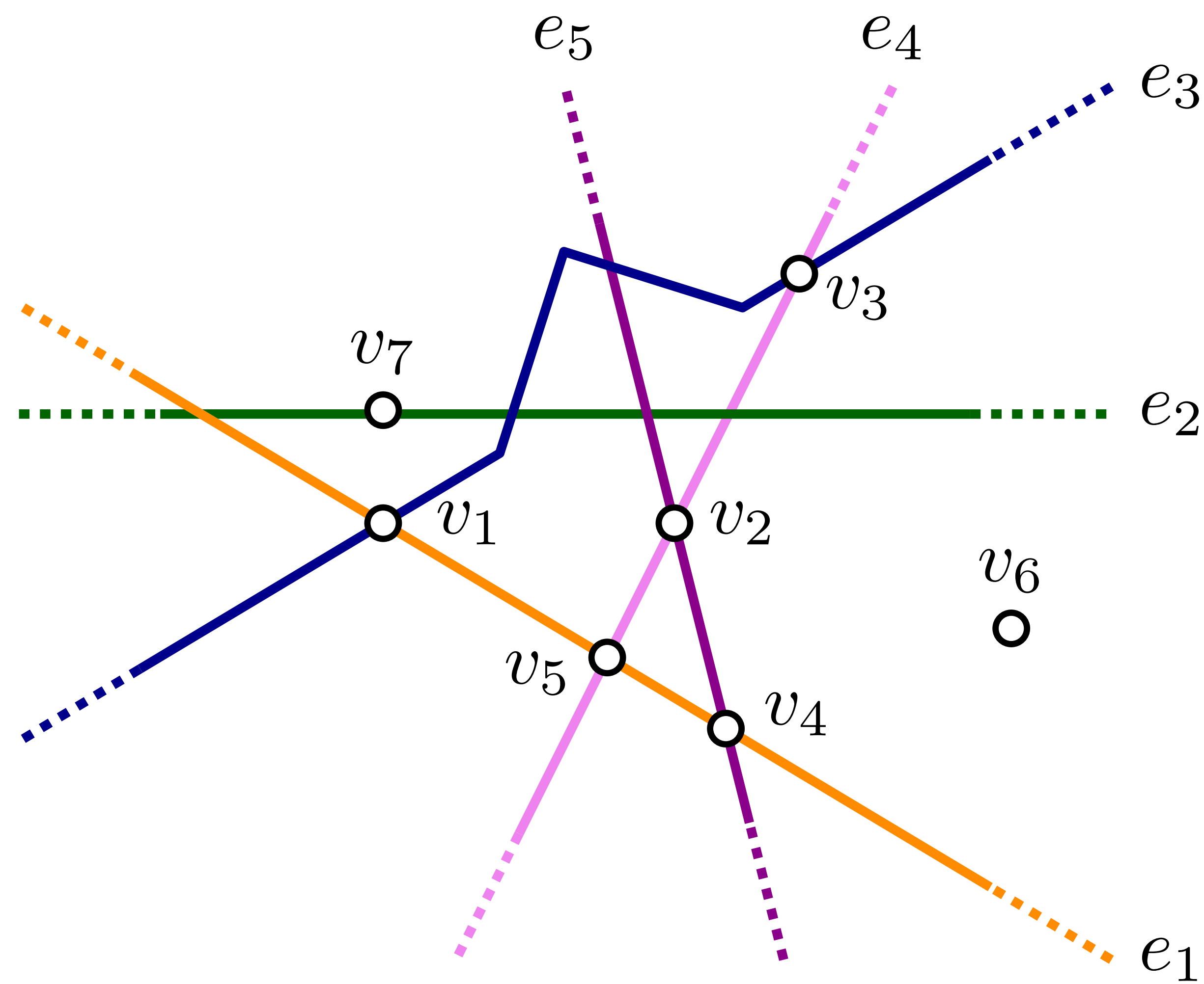

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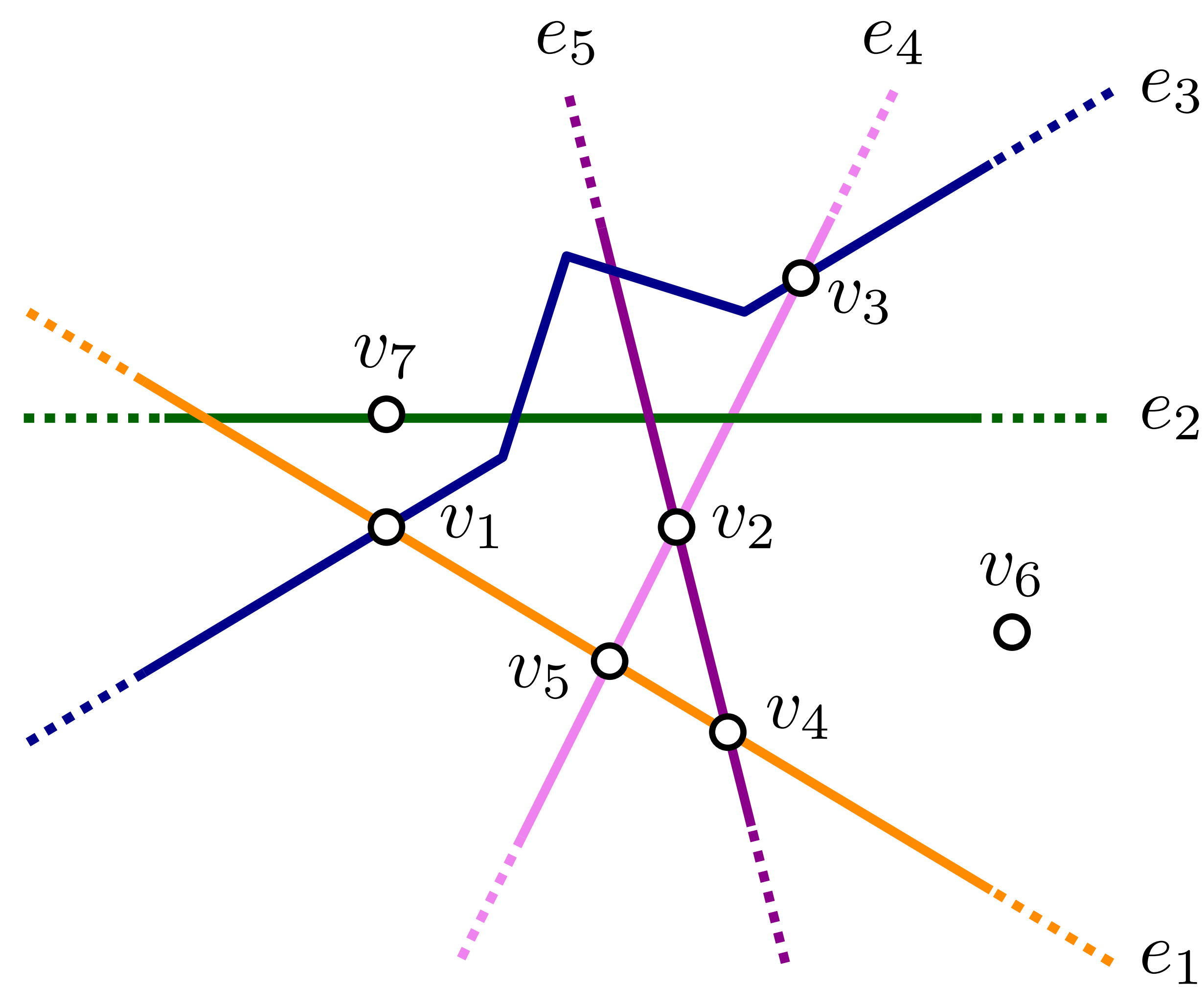

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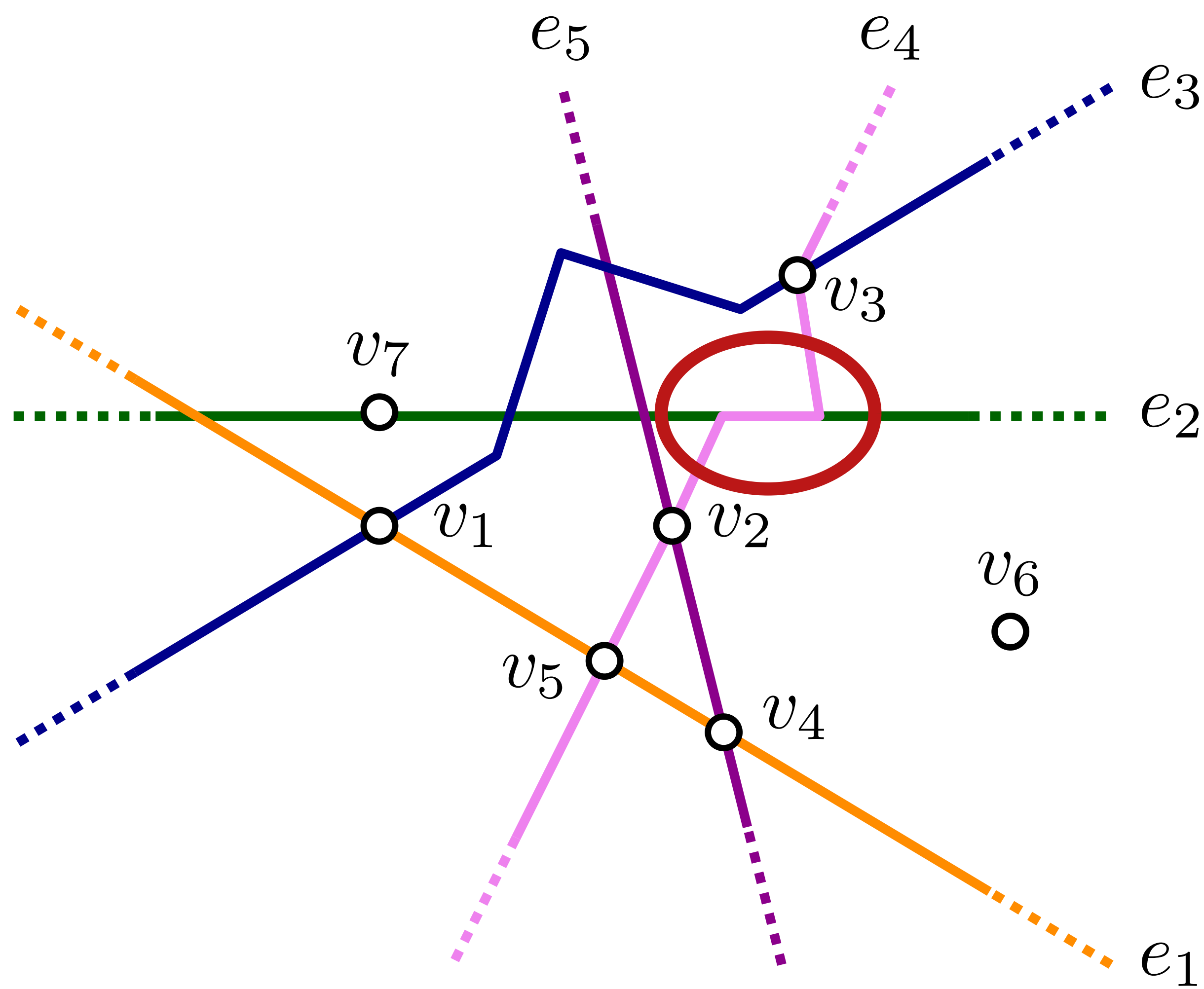

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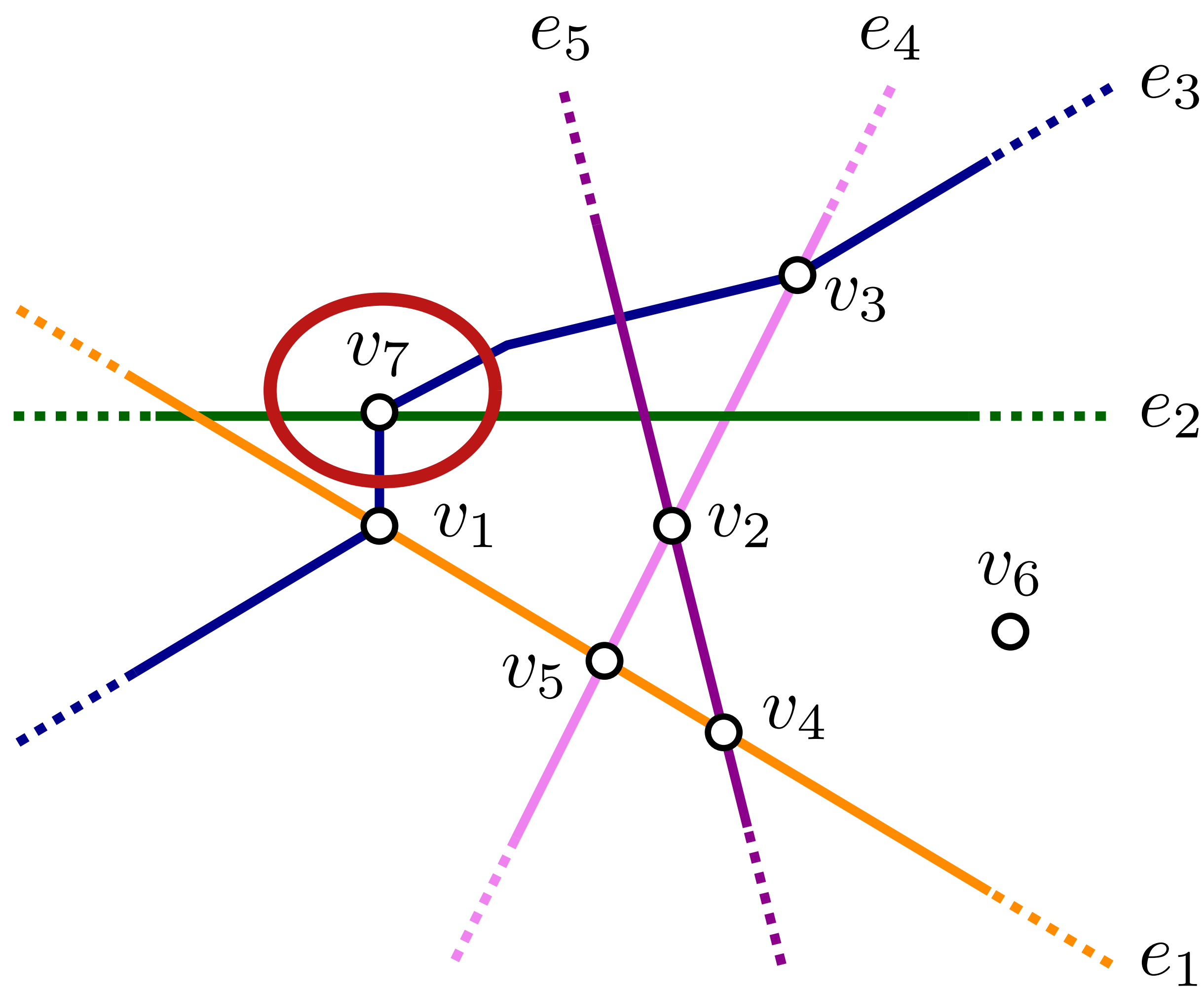

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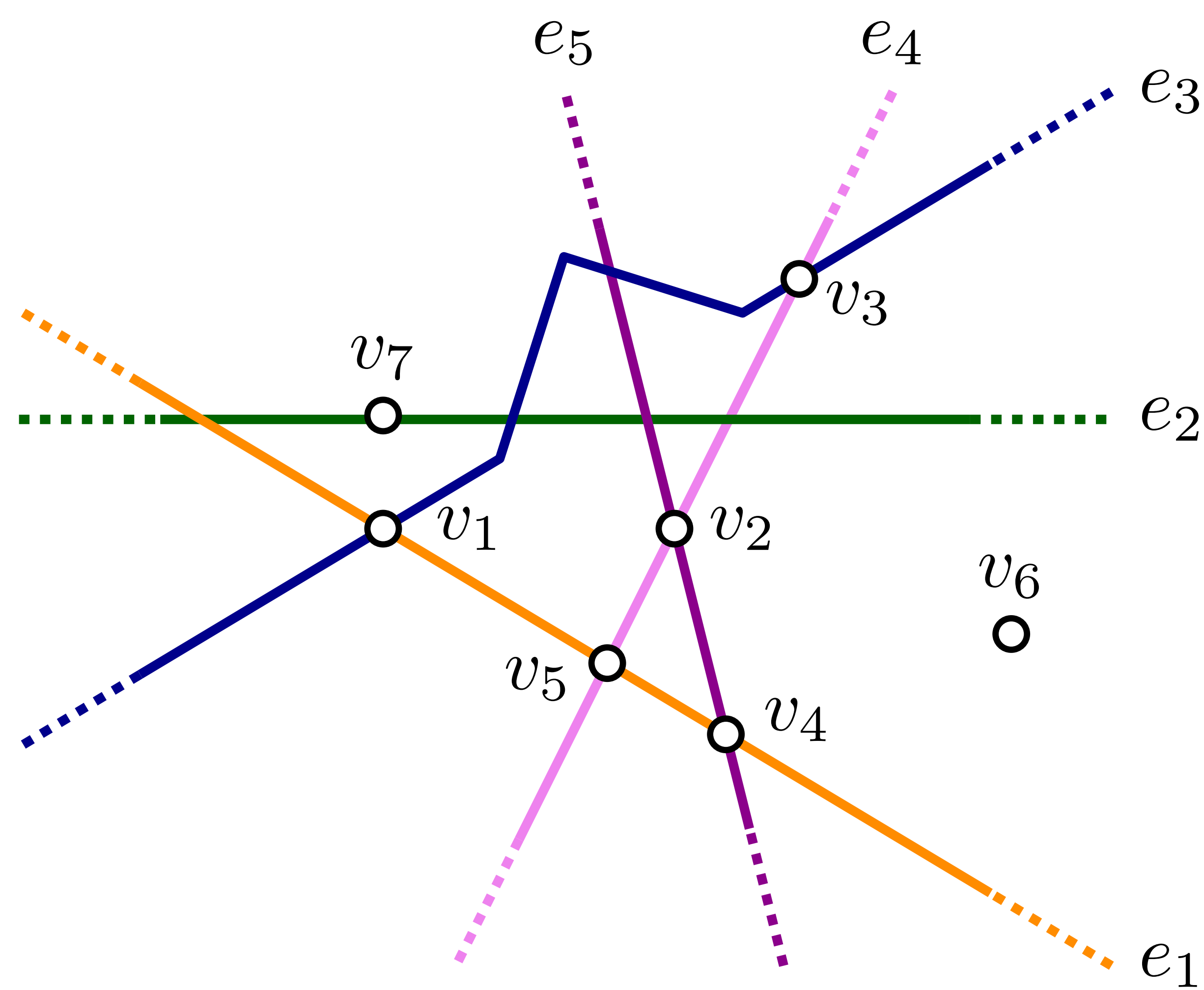

Why infinite lines? - because we can use point-line incidence theory of **Pappus, Möbius, Kantor, Steinitz, Grünbaum, ...**

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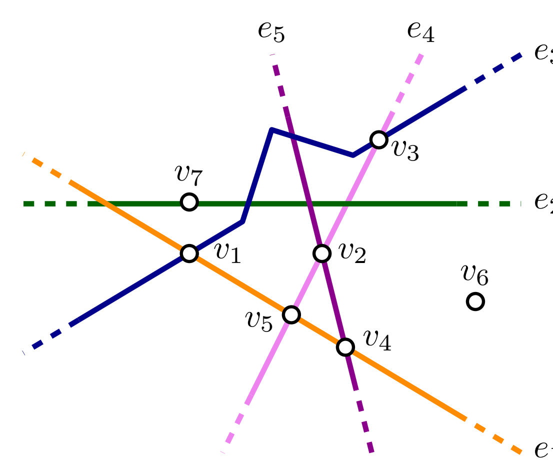
Questions:

- Which hypergraphs can be realized without bends?
- Upper/lower bounds on required bends?
- Complexity of deciding if zero bends possible?



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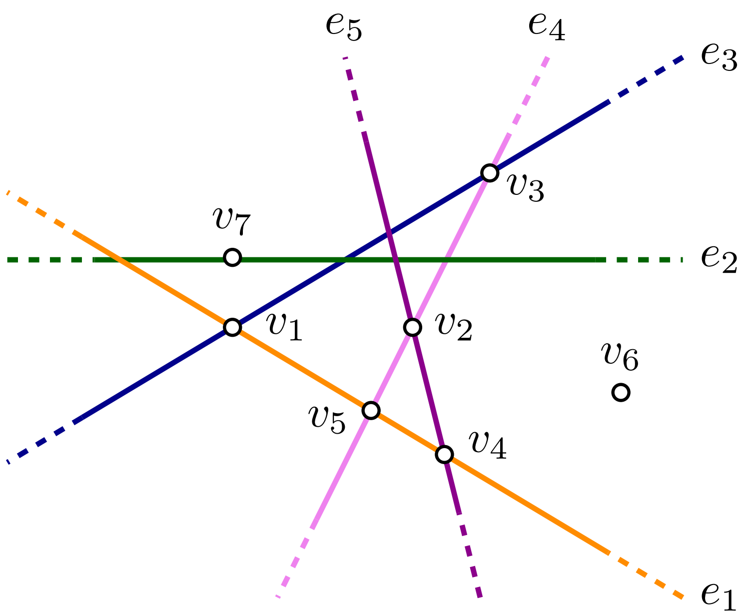

- $|E|$ lines in general position
- place degree-2 vertices at unique intersection of incident lines/hyperedges

An Old Result by Steinitz

Theorem. [Steinitz 1894] Every connected **3-uniform 3-regular linear** hypergraph can be realized with zero bends, except for one line with one bend.

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3-regular: $\forall v \in V$: degree$(v) = 3$
3-uniform: $\forall e \in E : |e| = 3$

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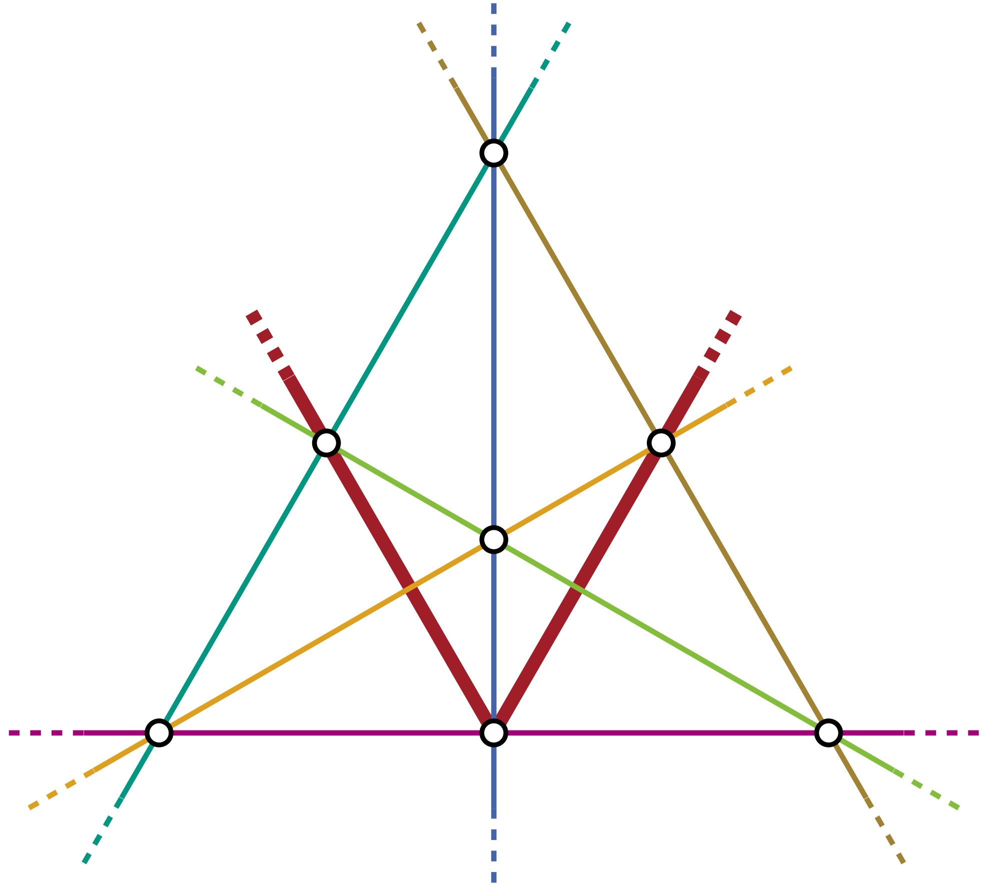
Example: Fano-plane (Fano-configuration, Fano hypergraph)



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Result is wrong! Pointed out by [Grünbaum 2009]

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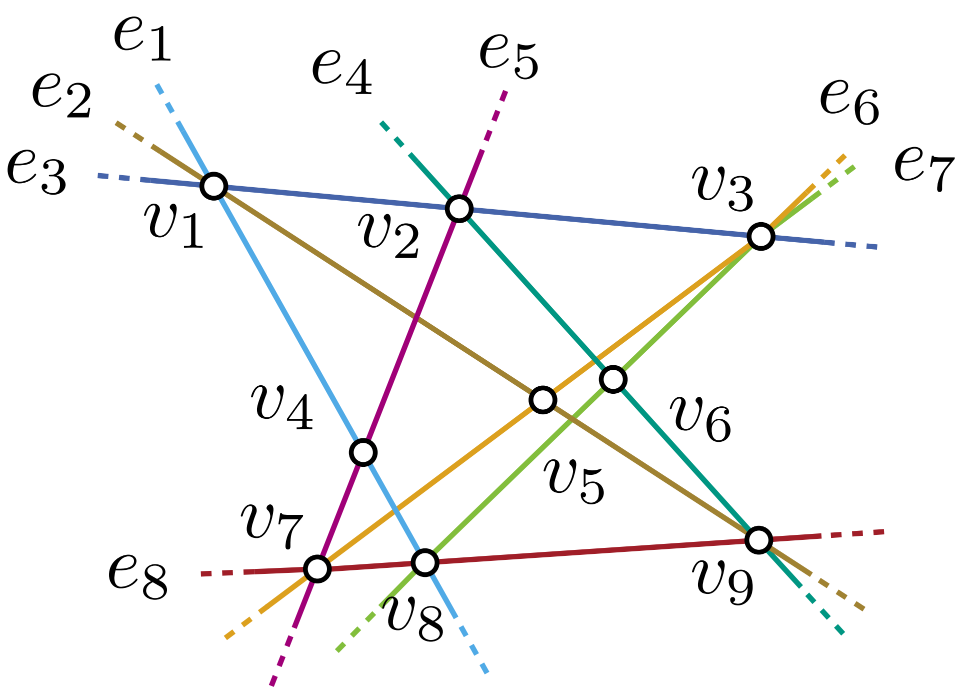
A Counterexample

Realization of hypergraph P:



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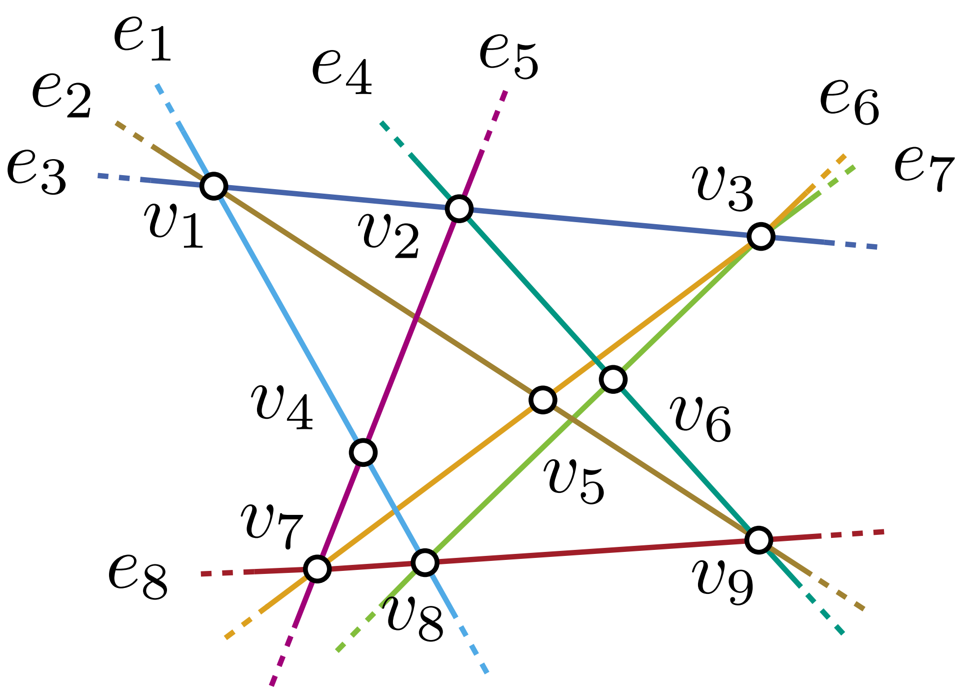

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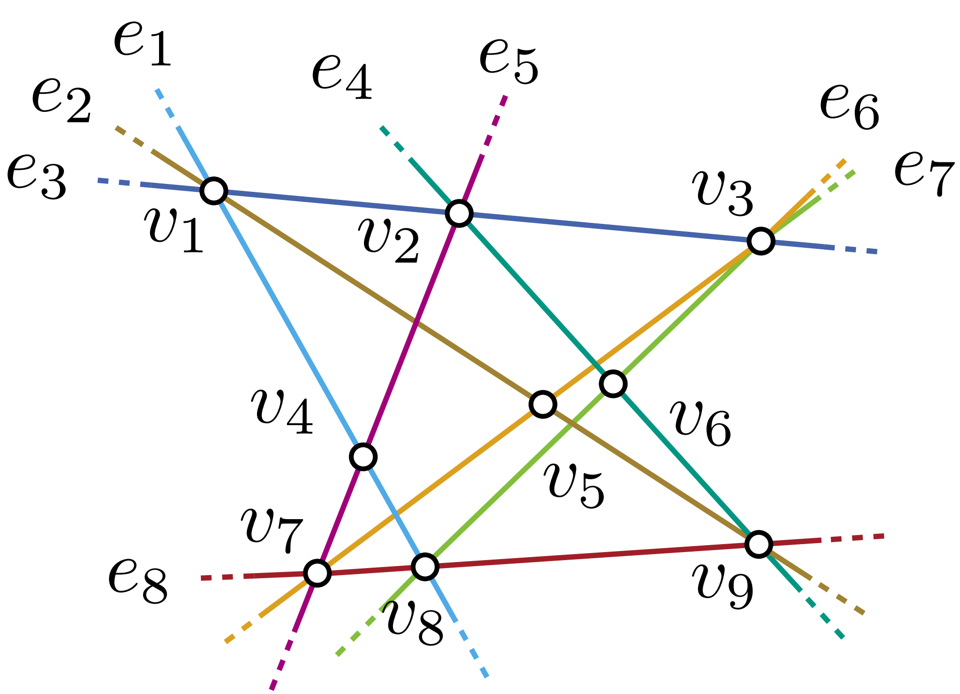

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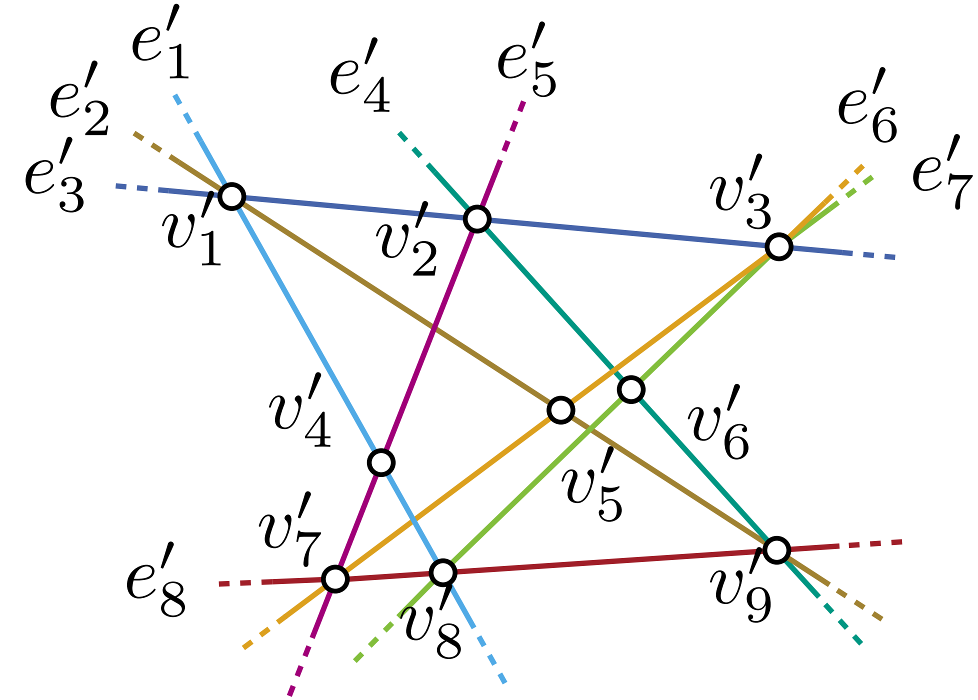

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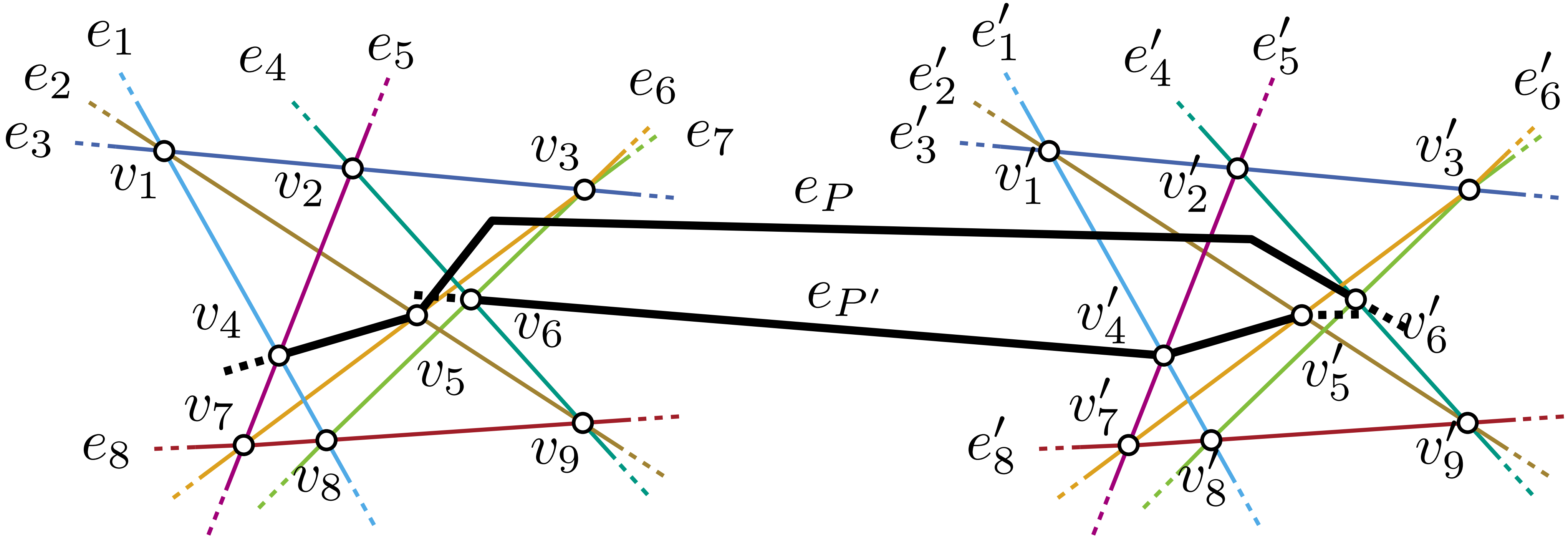

Hypergraph $H = P \cup P' \cup$ hyperedges $e_P = \{v_4, v_5, v'_6\}$ and $e_{P'} = \{v'_4, v'_5, v_6\}$

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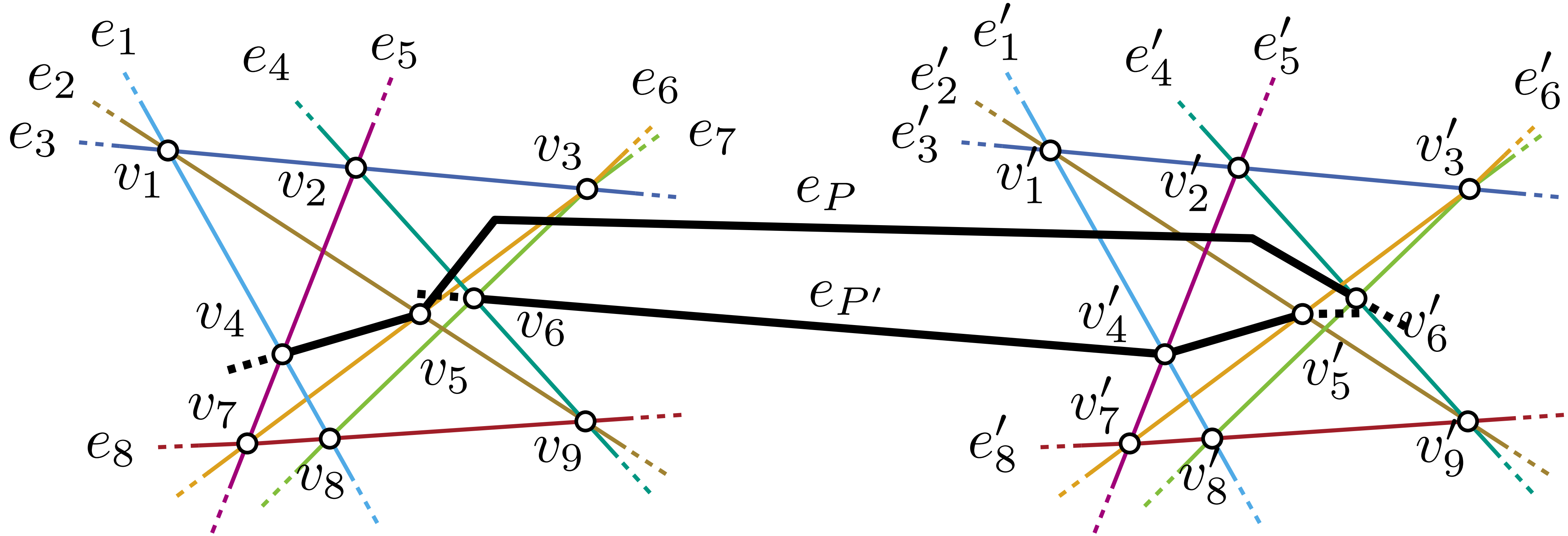

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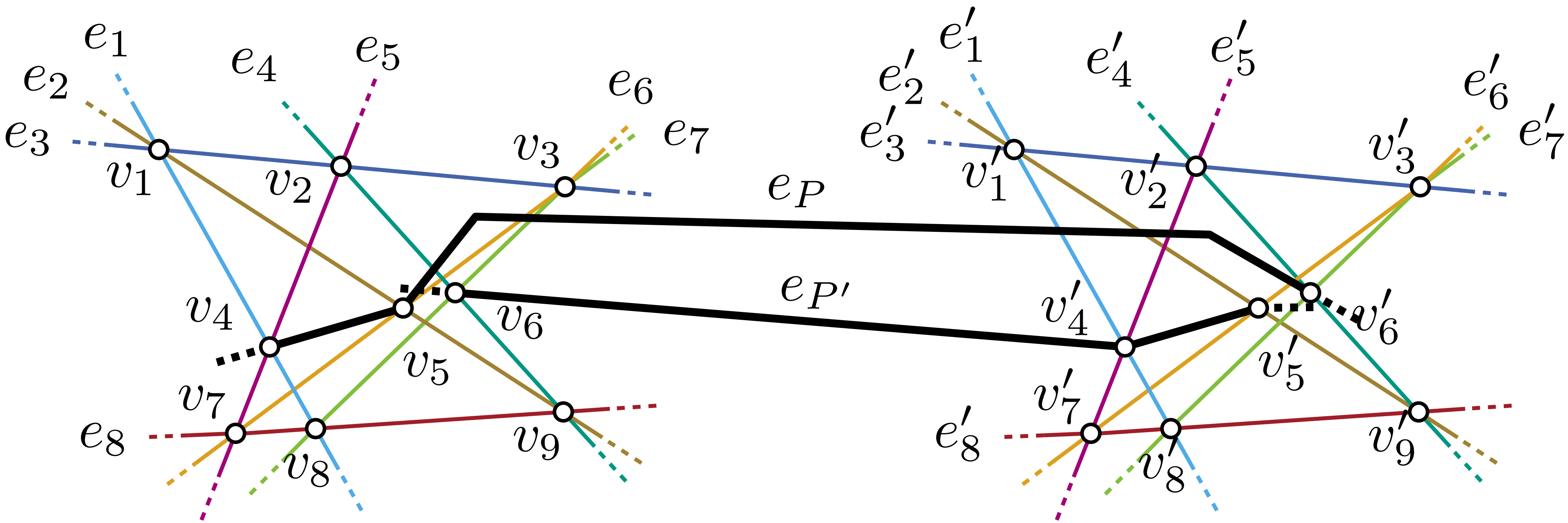

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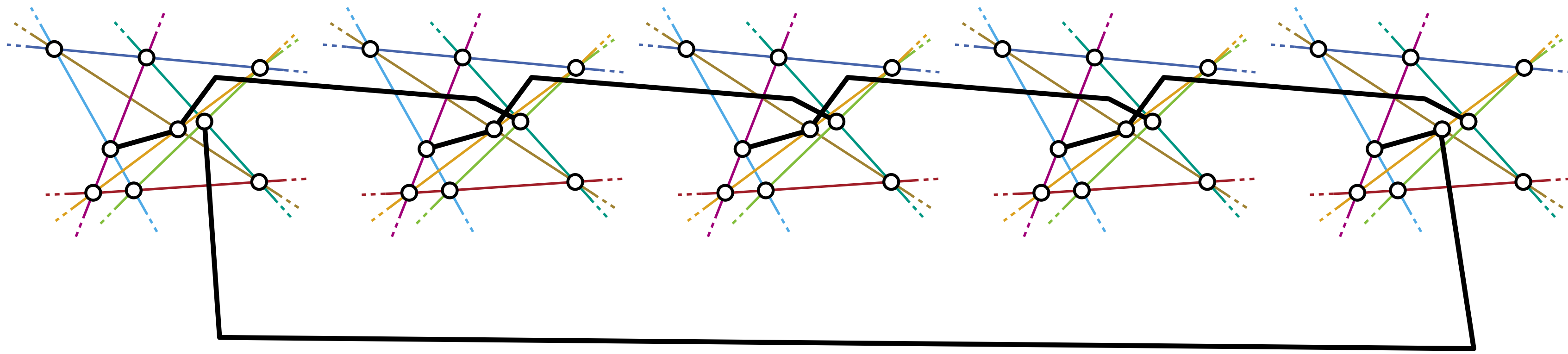
Proof.
- P realized without bends $\Rightarrow e_P$ requires bend
- P' realized without bends $\Rightarrow e_{P'}$ requires bend

A General Construction

Hypergraph H_5



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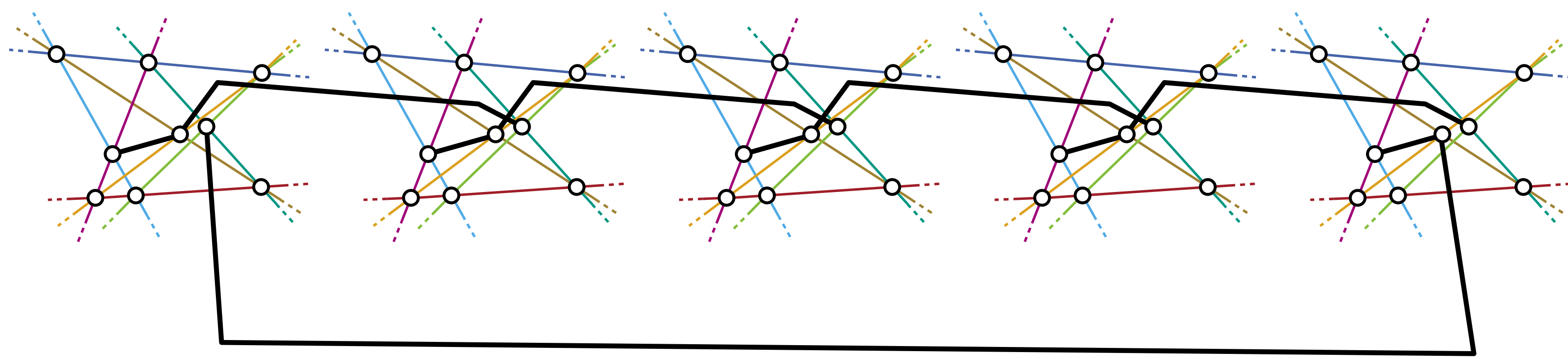

Theorem. For each $k \in \mathbb{N}$ there exists a linear 3-regular 3-uniform hypergraph H_k such that at least k hyperedges need to be realized with bends.

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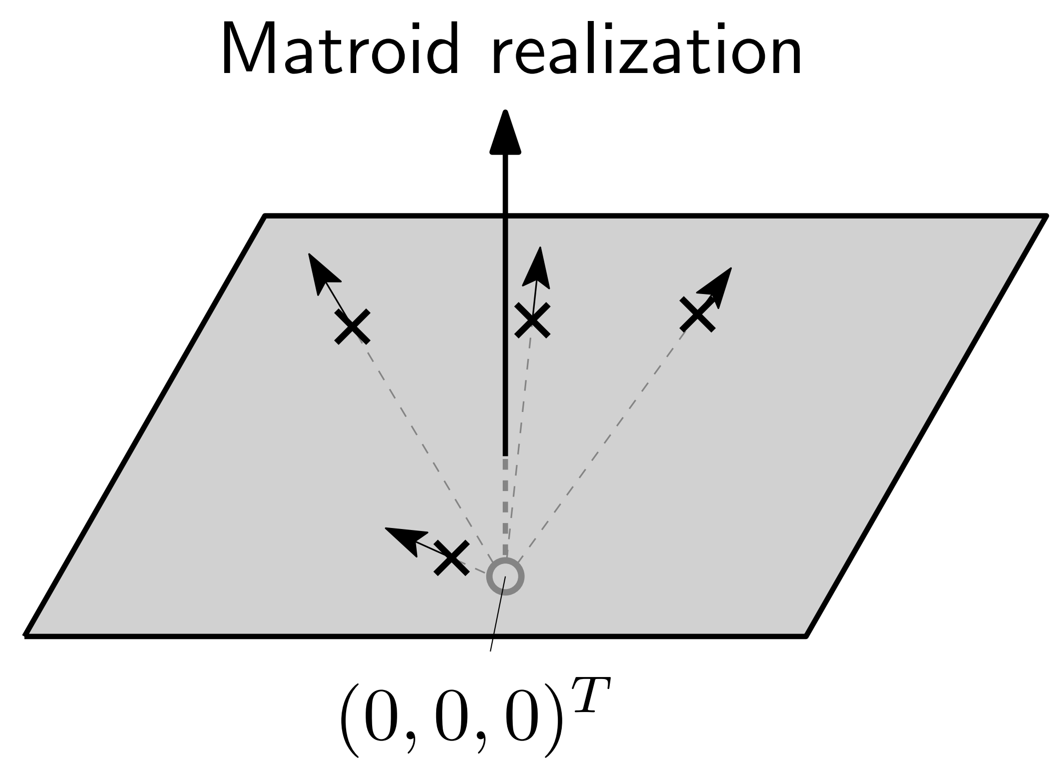

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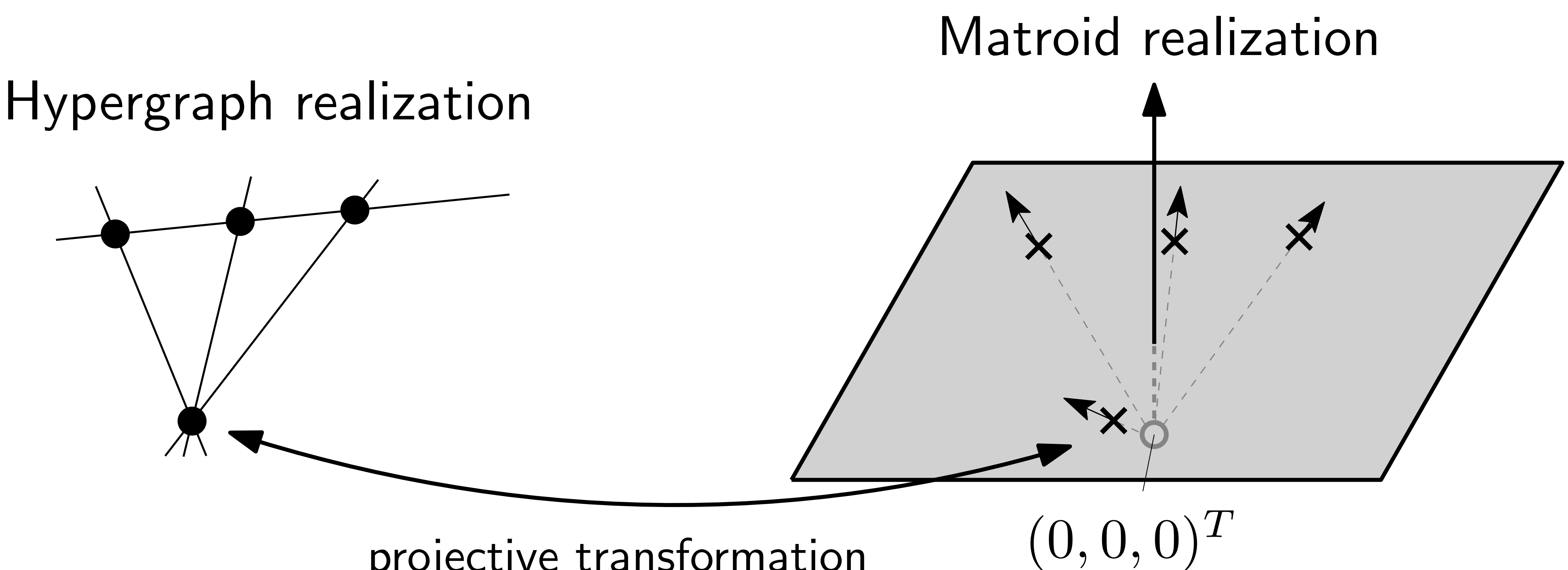

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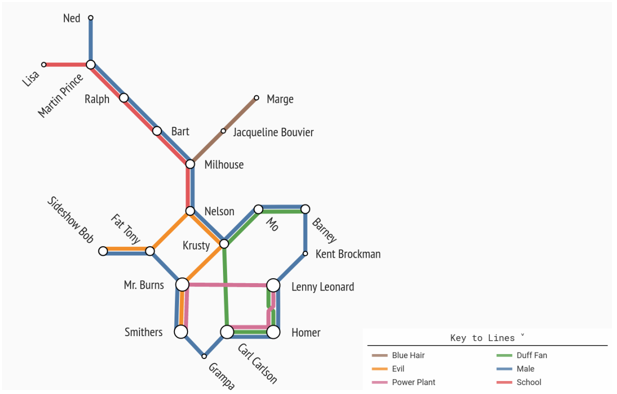

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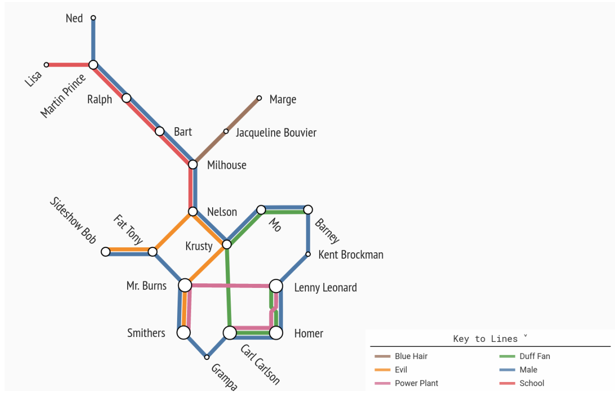

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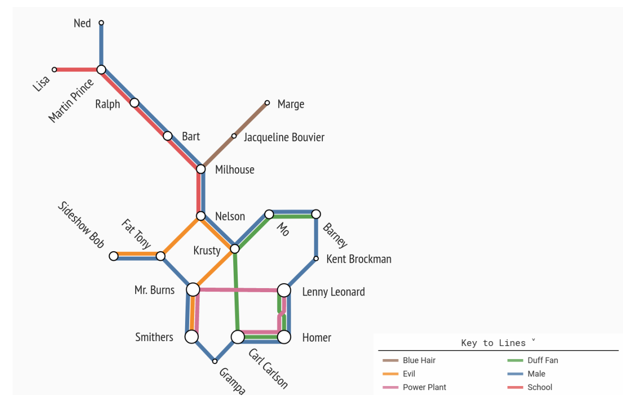

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Questions