Monochromatic triangles in the max-norm plane

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based on joint works with
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Classical problem

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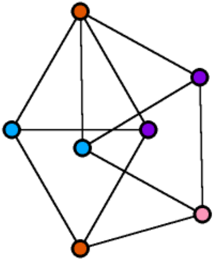
Known bounds: $5 \leq \chi \leq 7$.



Classical problem: Euclidean plane

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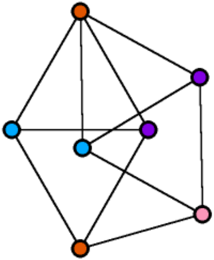


Variation: Max-norm plane

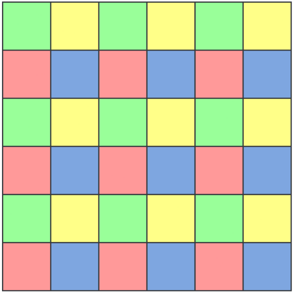
Let χ be the minimum number of colors needed to color the plane such that no pair of points at the ℓ_∞-distance* 1 is monochromatic.

*The ℓ_∞-distance between (x_1, y_1) and (x_2, y_2) is defined by $\max(|x_1 - x_2|, |y_1 - y_2|)$.

Variation: Max-norm plane

Let χ be the minimum number of colors needed to color the plane such that no pair of points at the ℓ_∞-distance* 1 is monochromatic.

Tight result: $\chi = 4$.



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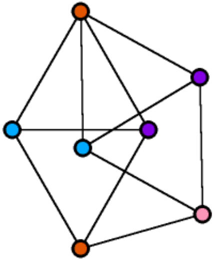

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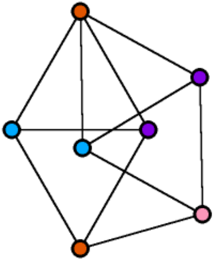

Variation: triangle on the Euclidean plane

Let $\chi(T)$ be the minimum number of colors needed to color the plane such that no Euclidean isometric copy of T is monochromatic.

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Observation: $2 \leq \chi(\triangle) \leq 7$.

Example: $\chi(\triangle) = 2$.

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Conjecture (Graham)

If $T \neq \triangle$, then $\chi(T) = 3$.

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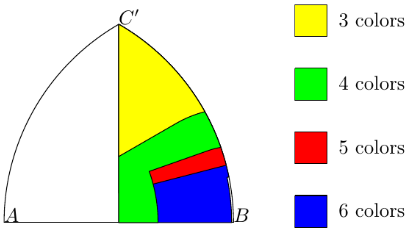

Variation: triangle on the max-norm plane

Let $\chi(T)$ be the minimum number of colors needed to color the plane such that no ℓ_∞-isometric copy of T is monochromatic.

Observation: $2 \leq \chi(\triangle) \leq 4$.

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Let $\chi(T)$ be the minimum number of colors needed to color the plane such that no ℓ_∞-isometric copy of T is monochromatic.

Observation: $2 \le \chi(\triangle) \le 4$.

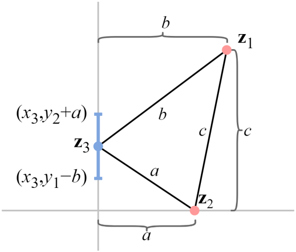
Theorem (Natalchenko, S.)

Let the side length $a \le b \le c$ of T be positive integers such that $c < a + b$ and $\gcd(a, b, c) = 1$. If

- $a + b + c$ is odd, or
- a and b are odd, $c \ge a + b - \gcd(a, b)$,

then $\chi(T) = 2$. Otherwise, $\chi(T) = 3$.

Proof illustrations



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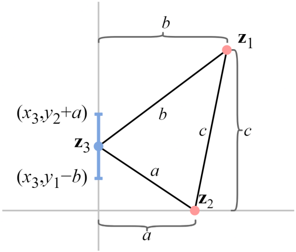
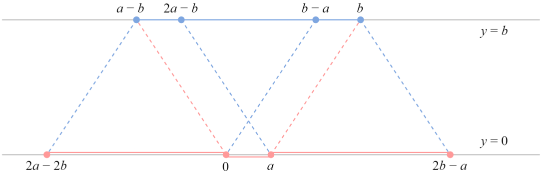
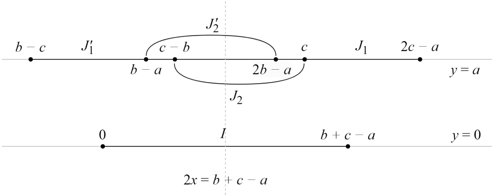
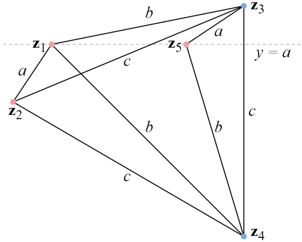








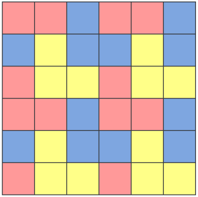
Open problem: degenerate triangle on the max-norm plane

Let $\chi(T)$ be the minimum number of colors needed to color the plane such that no ℓ_∞-isometric copy of T is monochromatic.

Observation: $2 \leq \chi(\triangle) \leq 4$.

Conjecture (Natalchenko, S.)

If the side length $a \leq b < a + b$ of T be positive integers such that $a \not\equiv b \pmod 3$, then $\chi(T) = 4$.



Thank You!

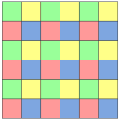
Monochromatic triangles in the max-norm plane

Let χ (resp. $\chi(T)$) be the minimum number of colors needed to color all the points of $\mathbb{R}^2$ such that no pair of points at distance 1 (resp. no copy of a triangle T) is monochromatic.

Euclidean norm

- $5 \leq \chi \leq 7$
- $\chi(\triangle) = 2$
- Conjecture (Graham): $T \neq \triangle \Rightarrow \chi(T) = 3$

Max-norm



- $\chi = 4$

Side lengths $a \leq b \leq c$ are integers such that $c < a + b$. Theorem:

- If (1) $a + b + c$ is odd, or
  (2) a and b are odd, $c \geq a + b - \gcd(a, b)$,
  then $\chi(T) = 2$
- Otherwise $\chi(T) = 3$