Exact solutions to the Weighted Region Problem

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Joint work with:

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Utrecht U.

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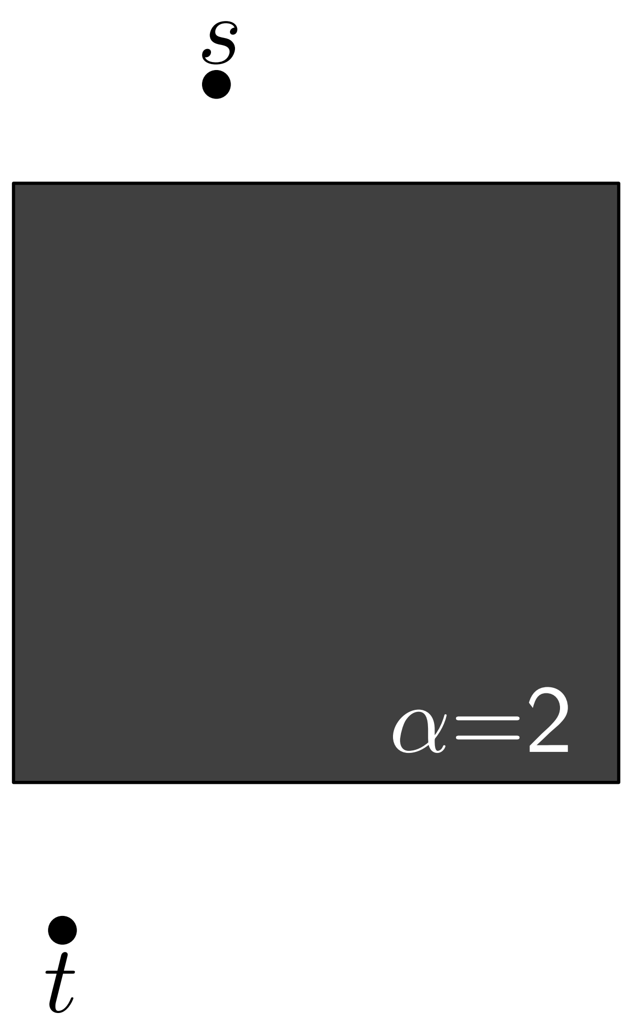
Utrecht U.

Guillermo Esteban
U. de Alcalá & Carleton U.

Shortest paths amid one weighted square

Given: two points s, t, one square with weight $\alpha \geq 0$

Output: shortest *weighted* path from s to t



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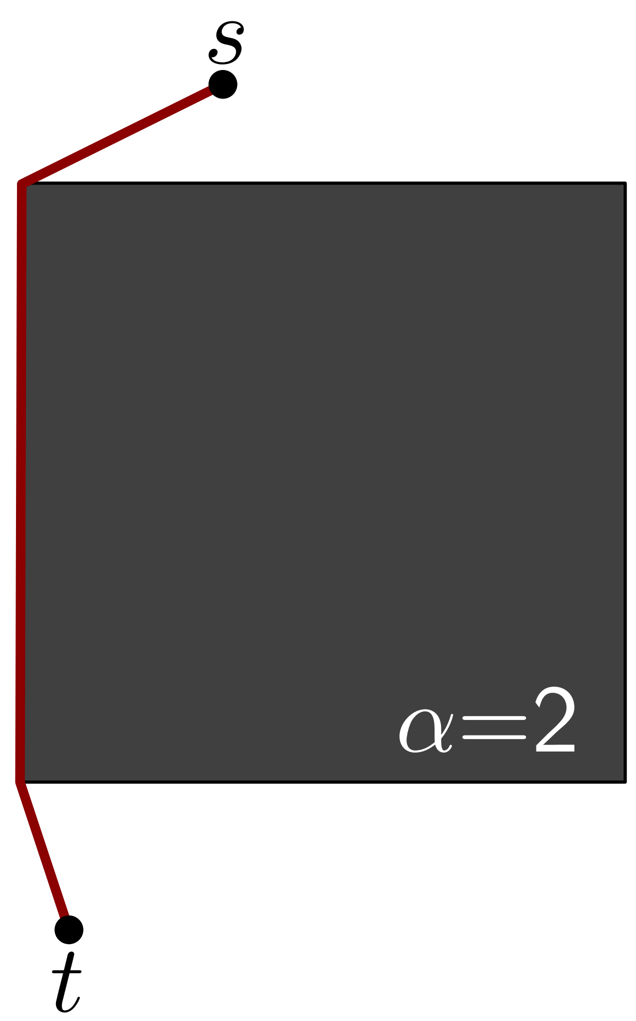

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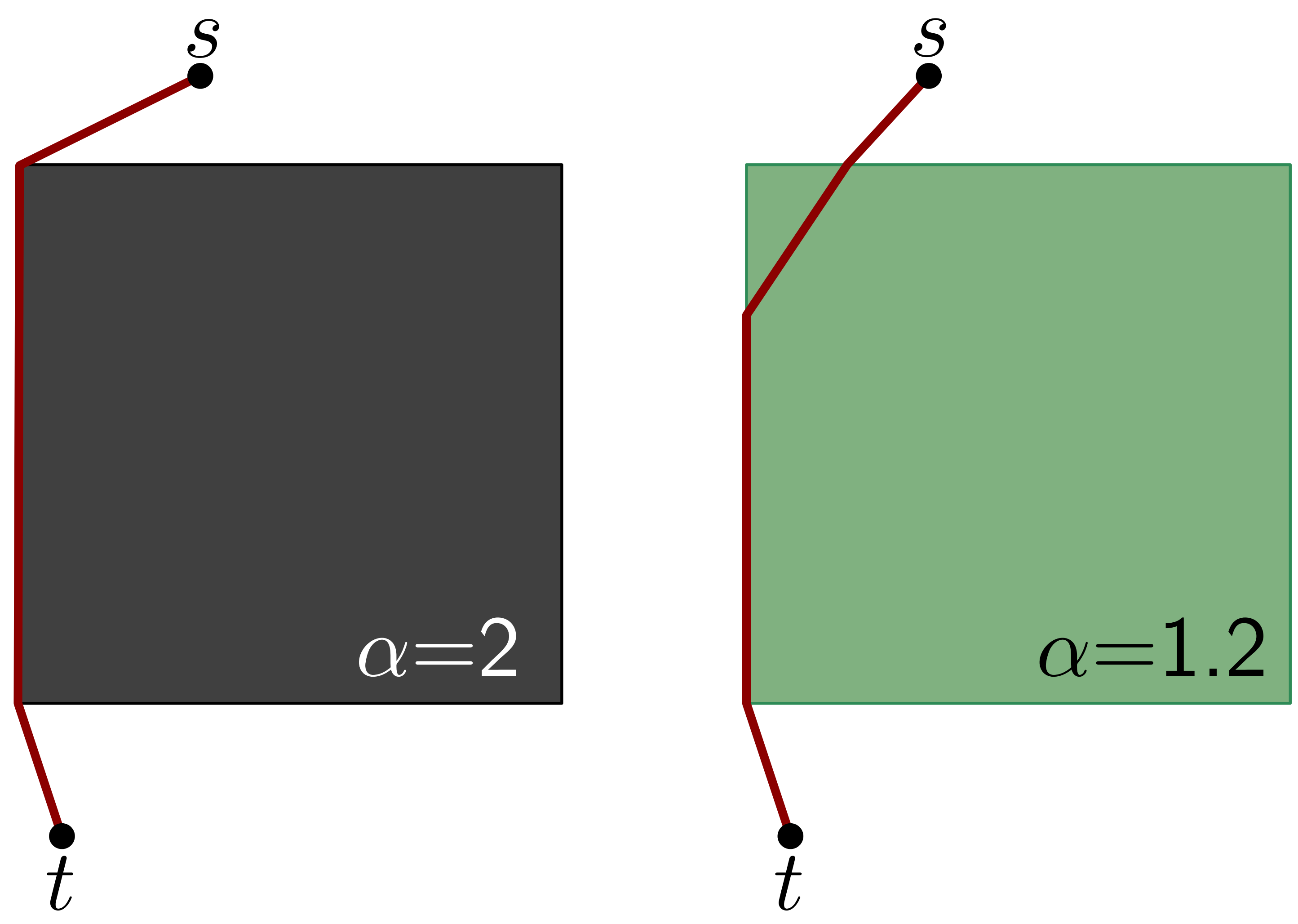

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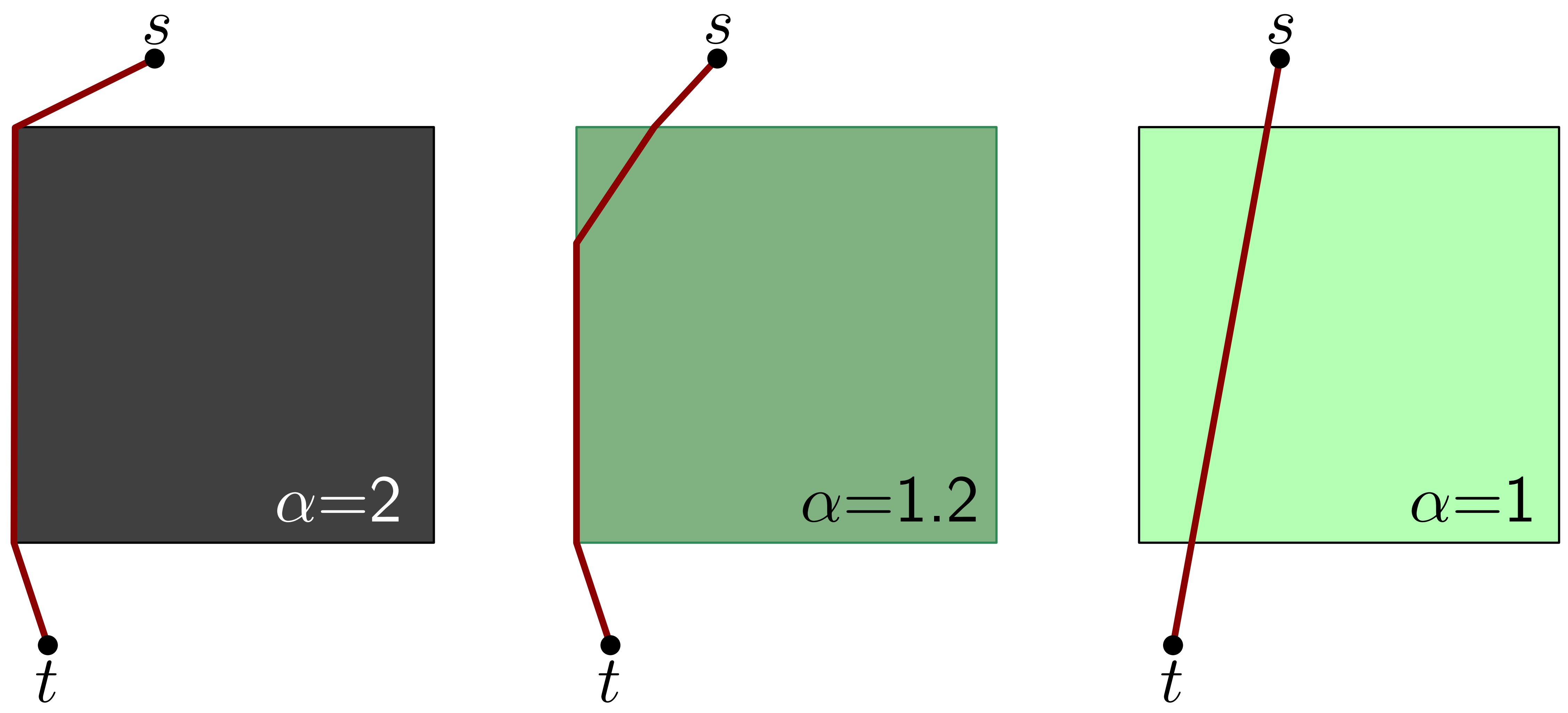

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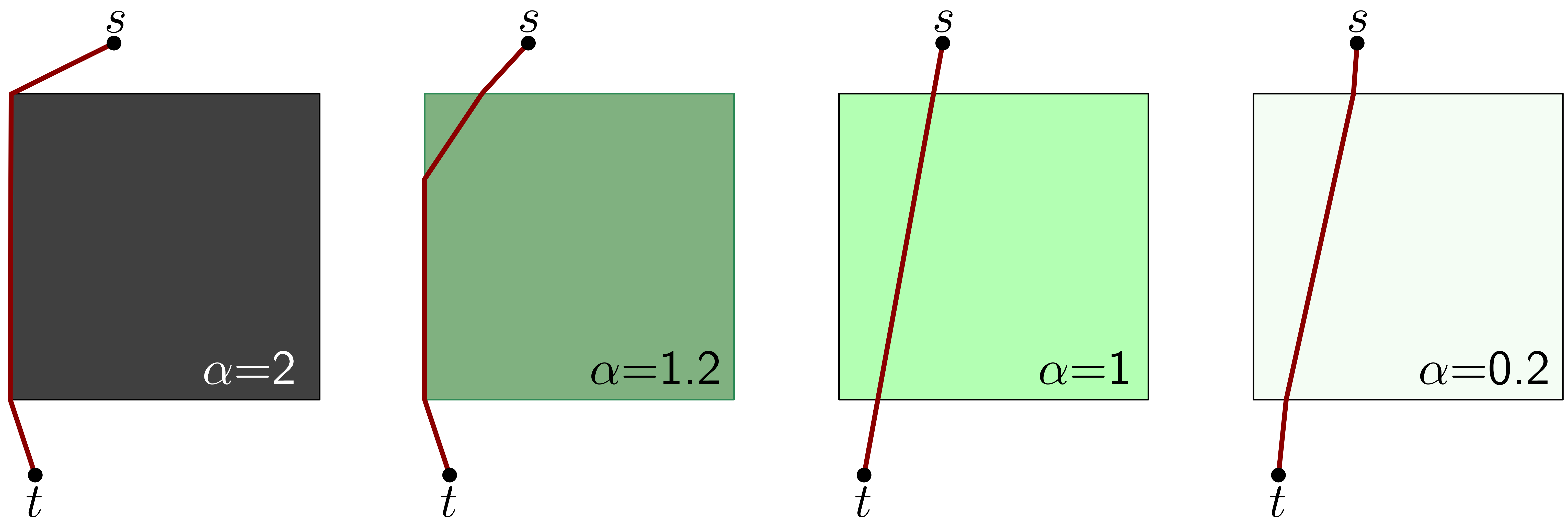

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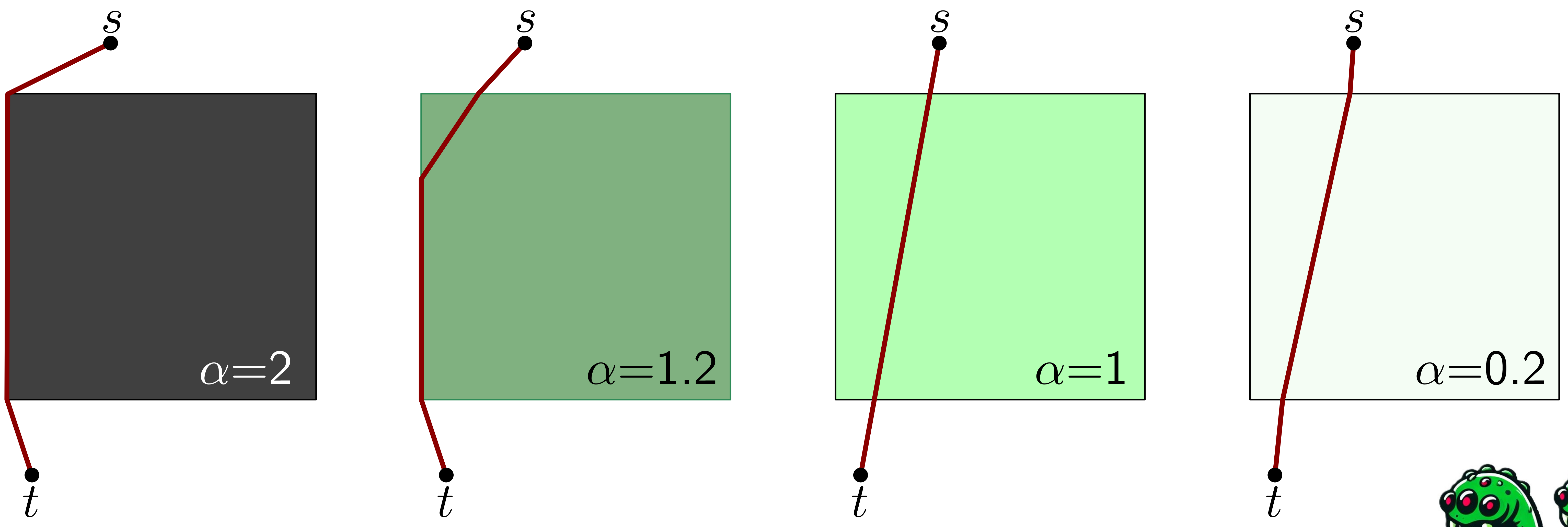

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Particular case of the *Weighted Region Problem*

Weighted Region Problem (WRP)

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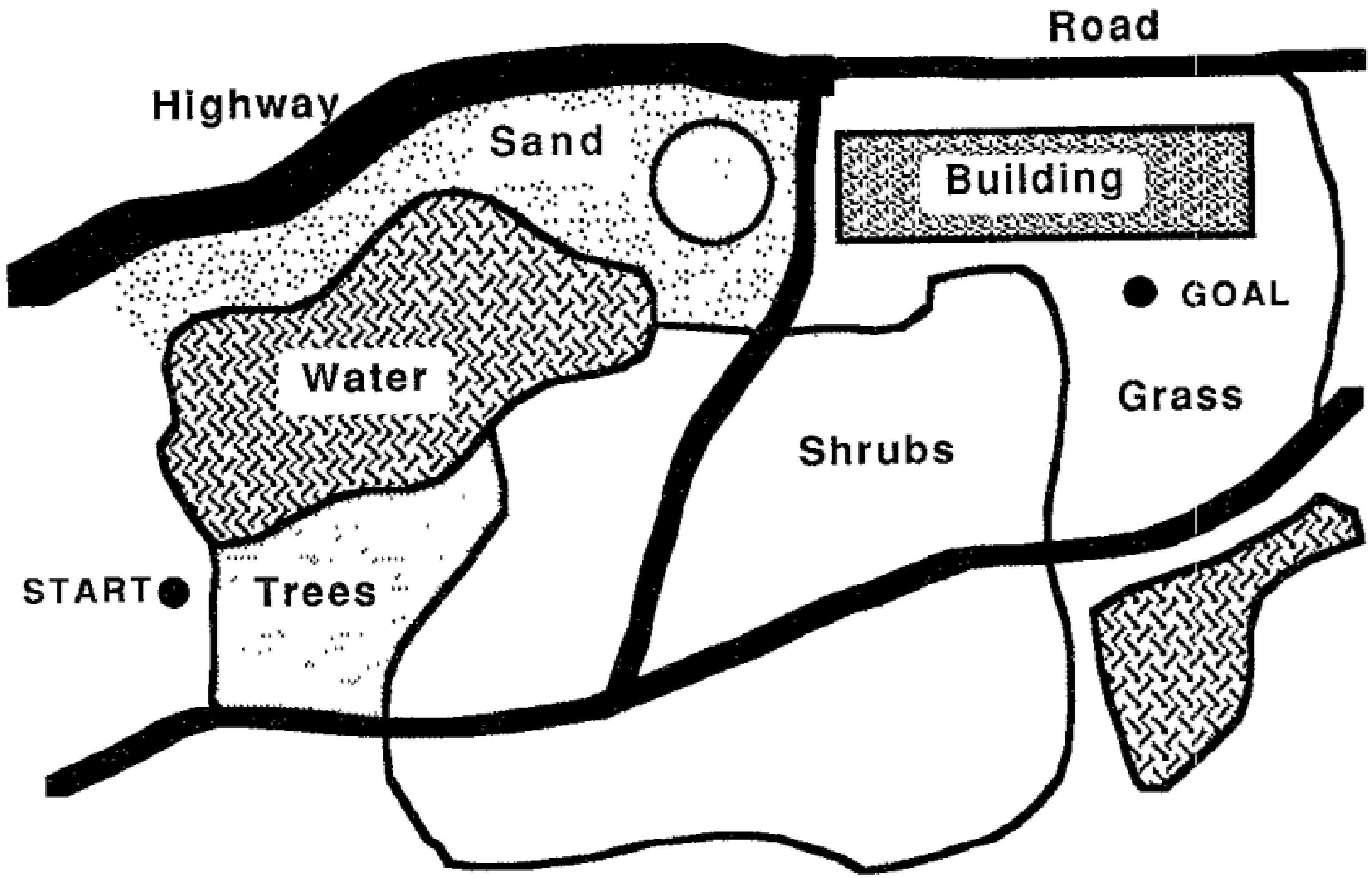

J. S. B. Mitchell, C. H. Papadimitriou. *The weighted region problem: finding shortest paths through a weighted planar subdivision*. Journal of the ACM, Vol. 38, pp 18–73, **1991**

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General problem

(arbitrary non-negative weights, arbitrary regions)

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ε: desired approximation factor

n: number of region vertices

N: max integer coordinate of any region vertex

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Proof uses only **3 weights**!

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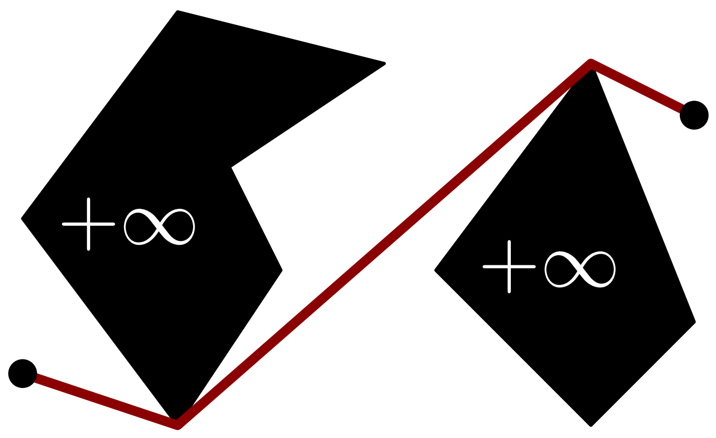

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  $\rightarrow$ exact algorithms for polygonal (and even curved) obstacles

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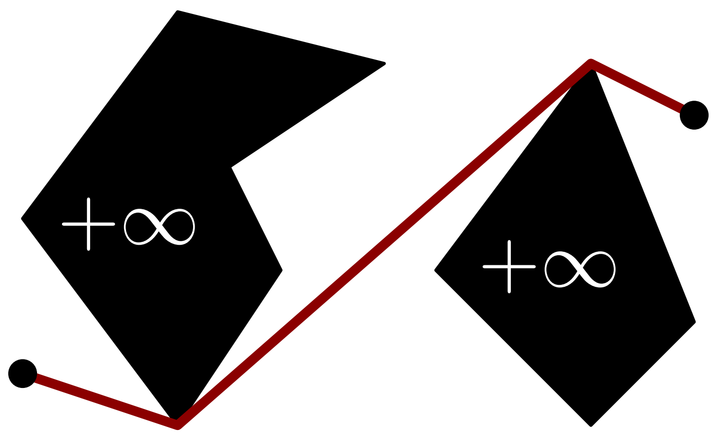

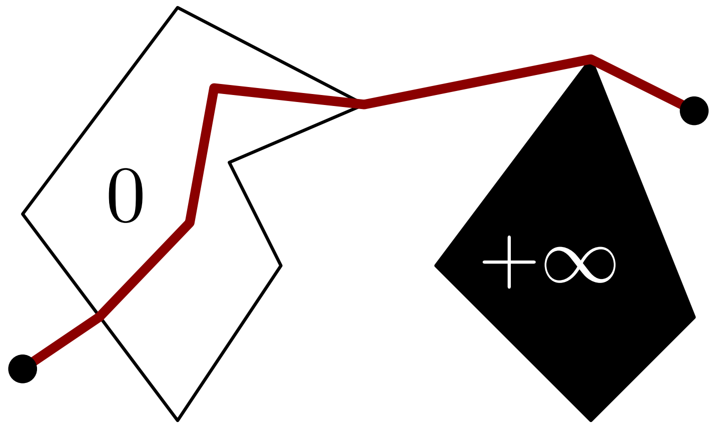

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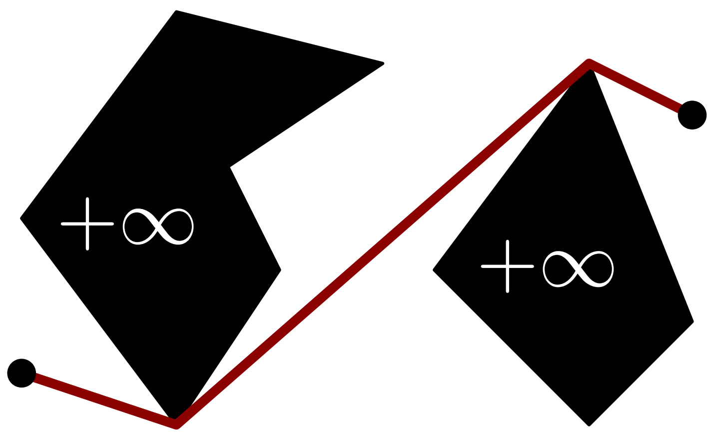

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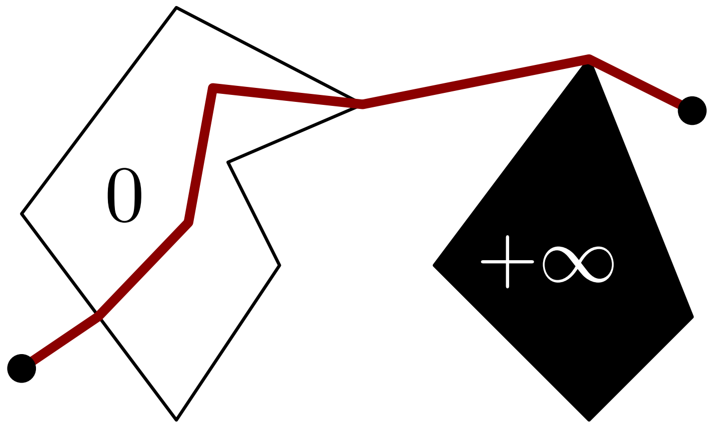

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- weights $\{1, \alpha\}$

  $\rightarrow$ exact algorithms for obstacles that are parallel strips

Our results

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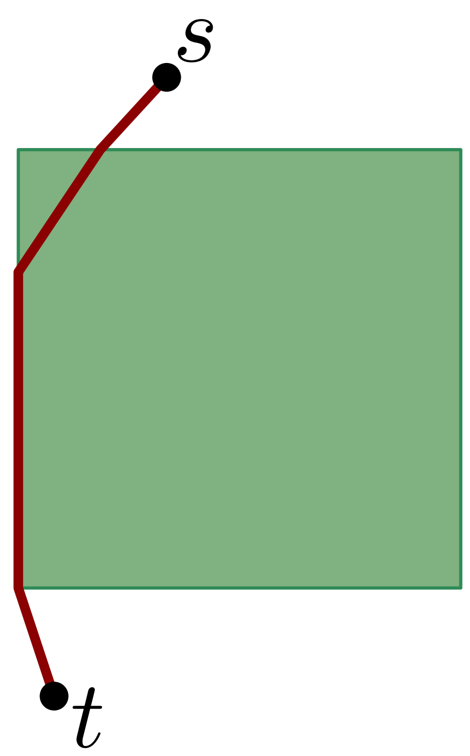
One square/rectangle with weight α



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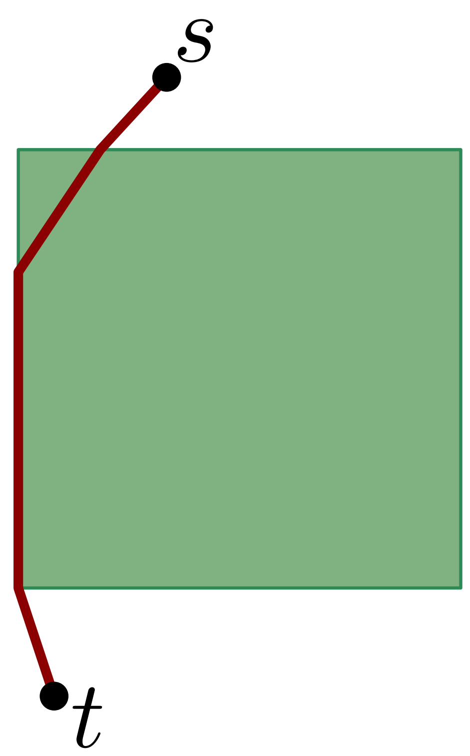

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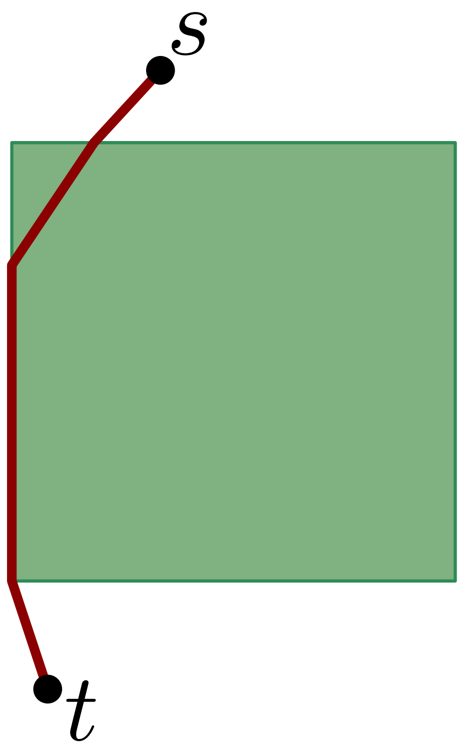
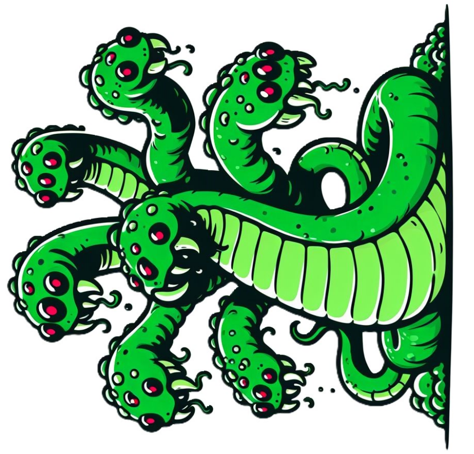


Unsolvable in the Algebraic Computation Model over the Rational Numbers ($ACM\mathbb{Q}$)



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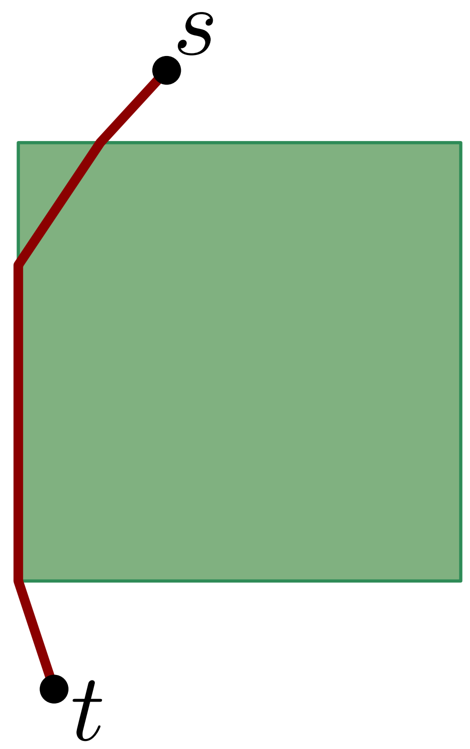

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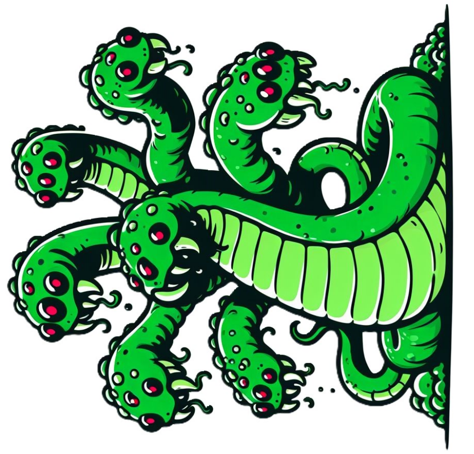

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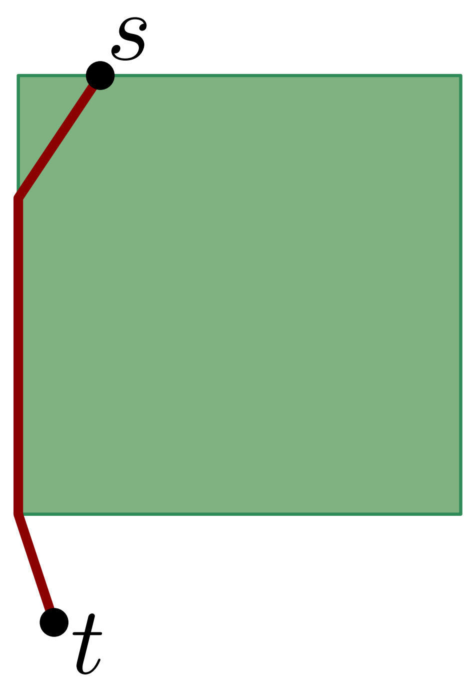

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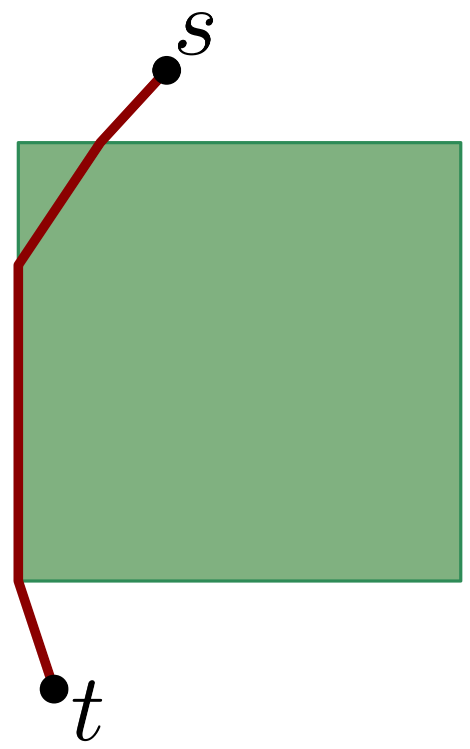

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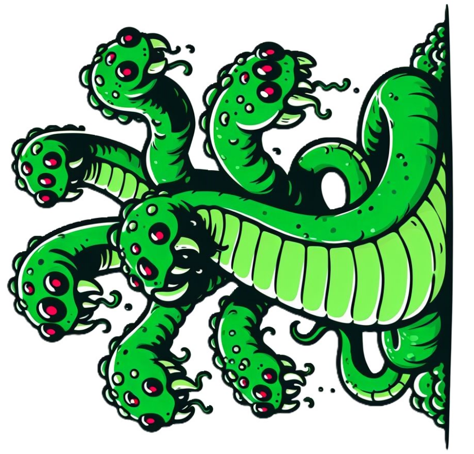

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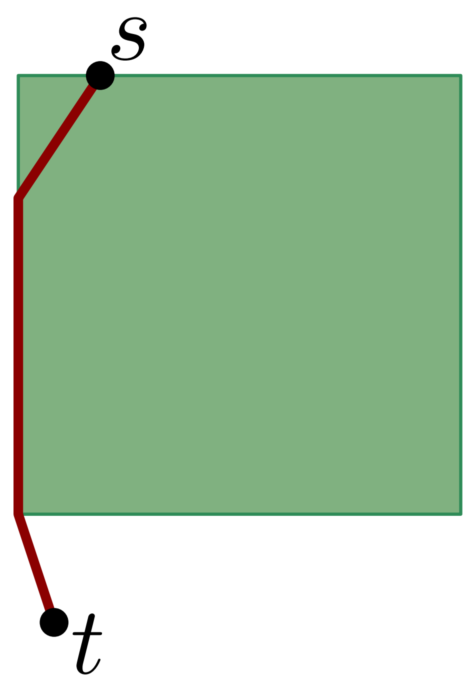

We can characterize and compute all possible types of shortest paths

s on boundary: *all* shortest path types

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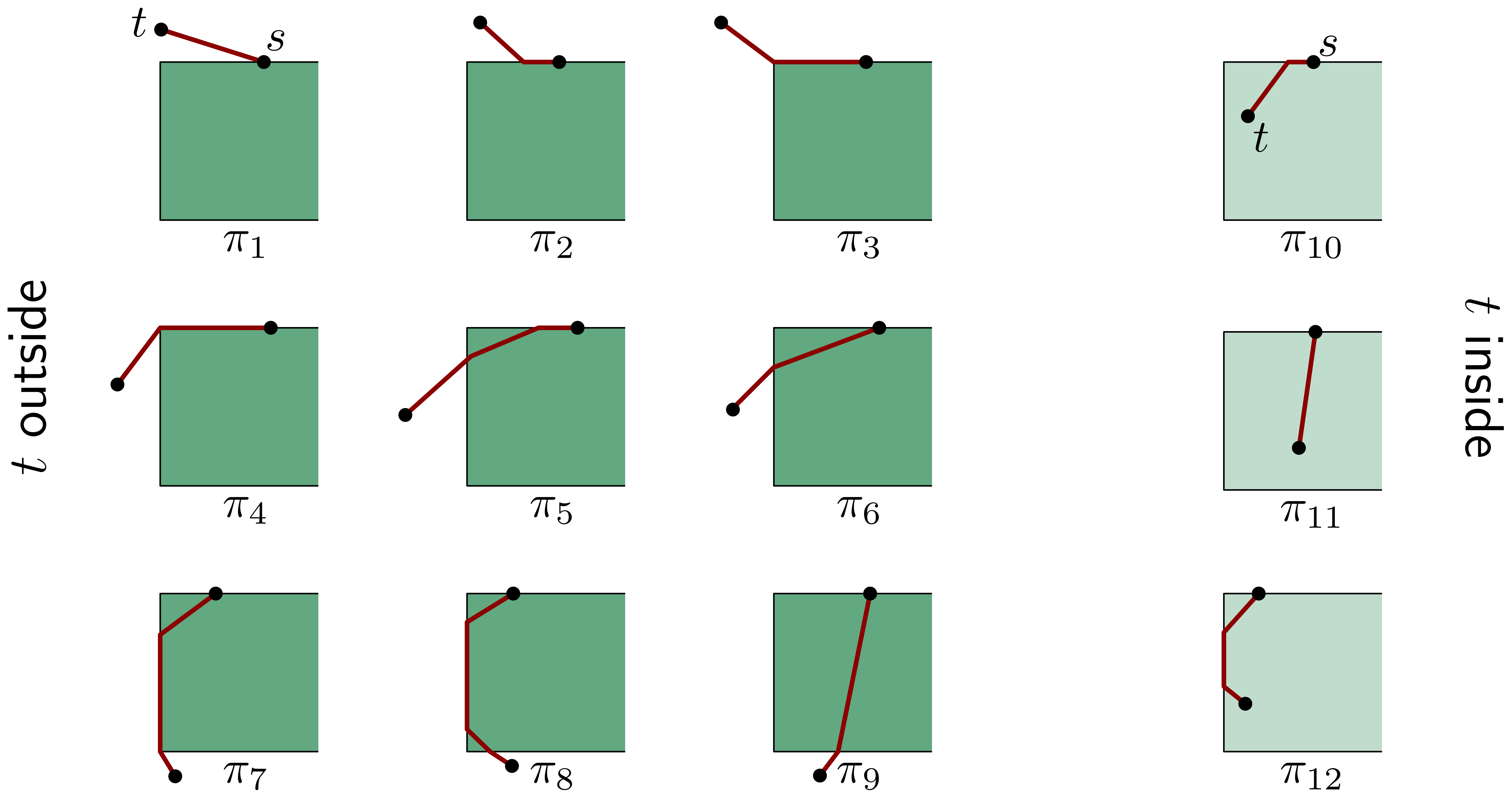

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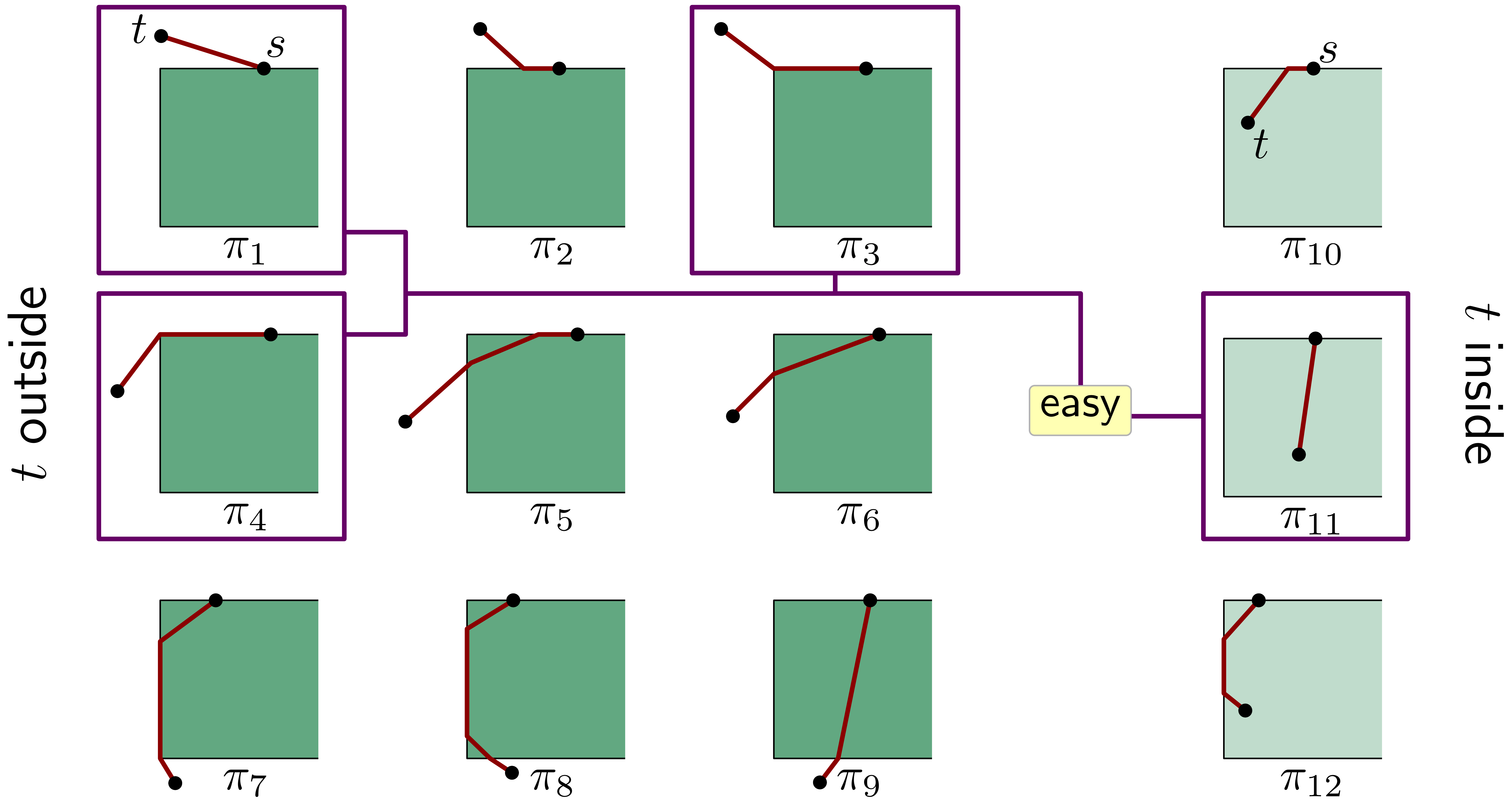
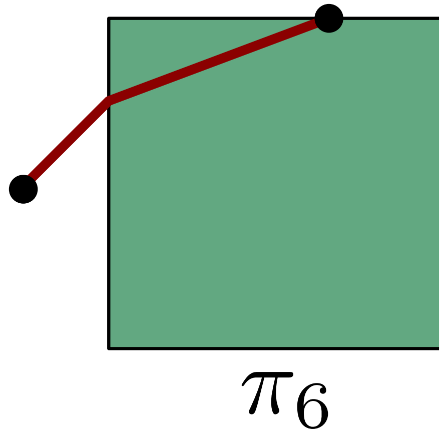


Example: one type of shortest path



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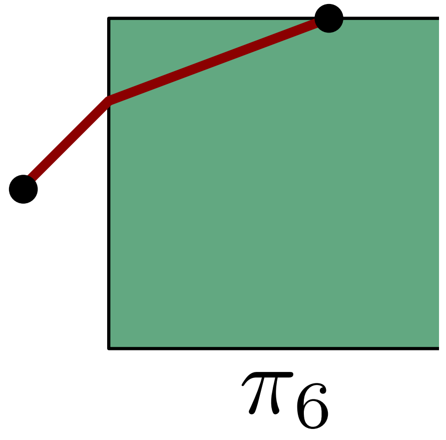

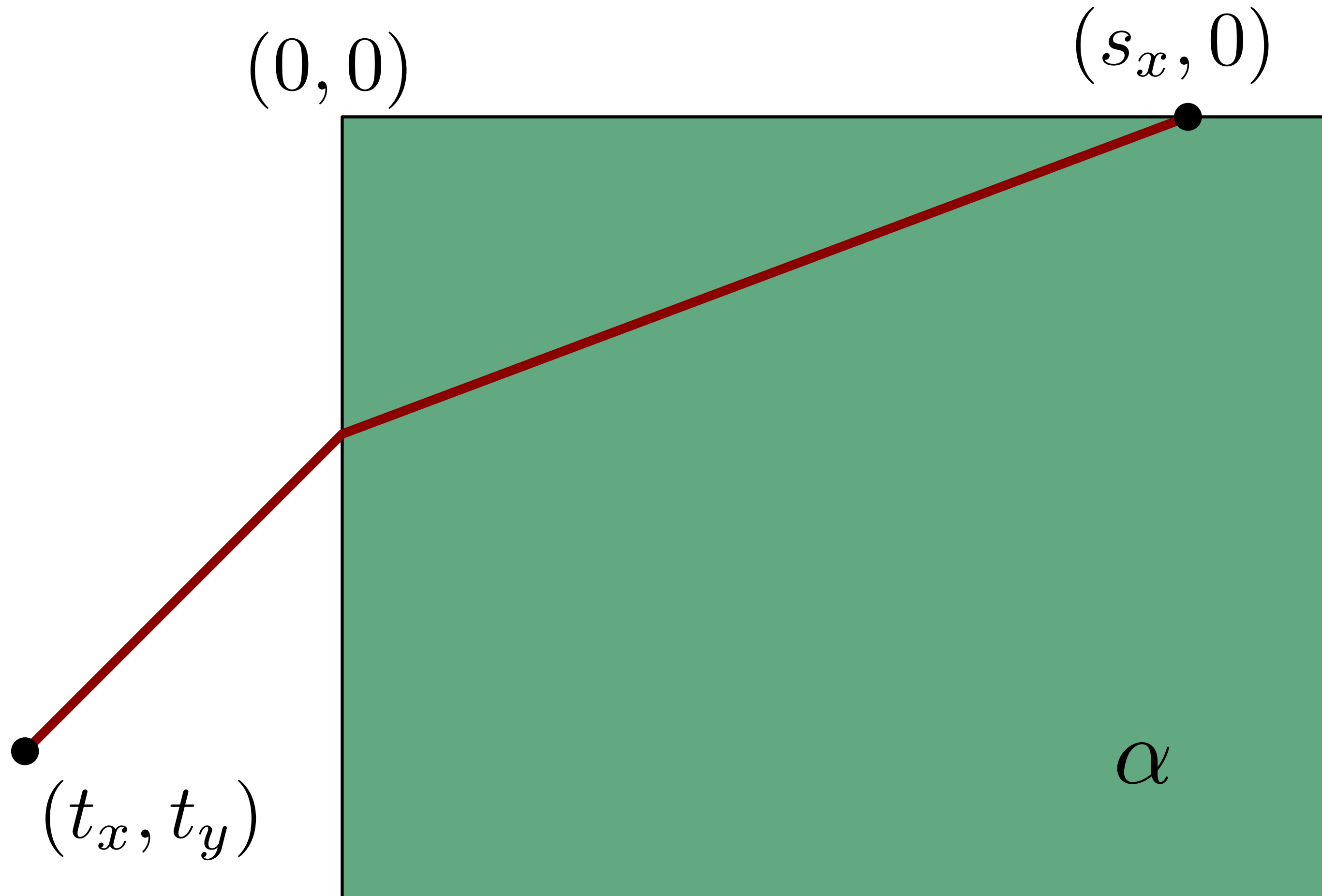

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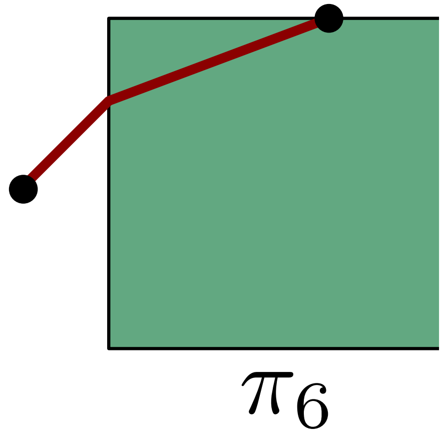

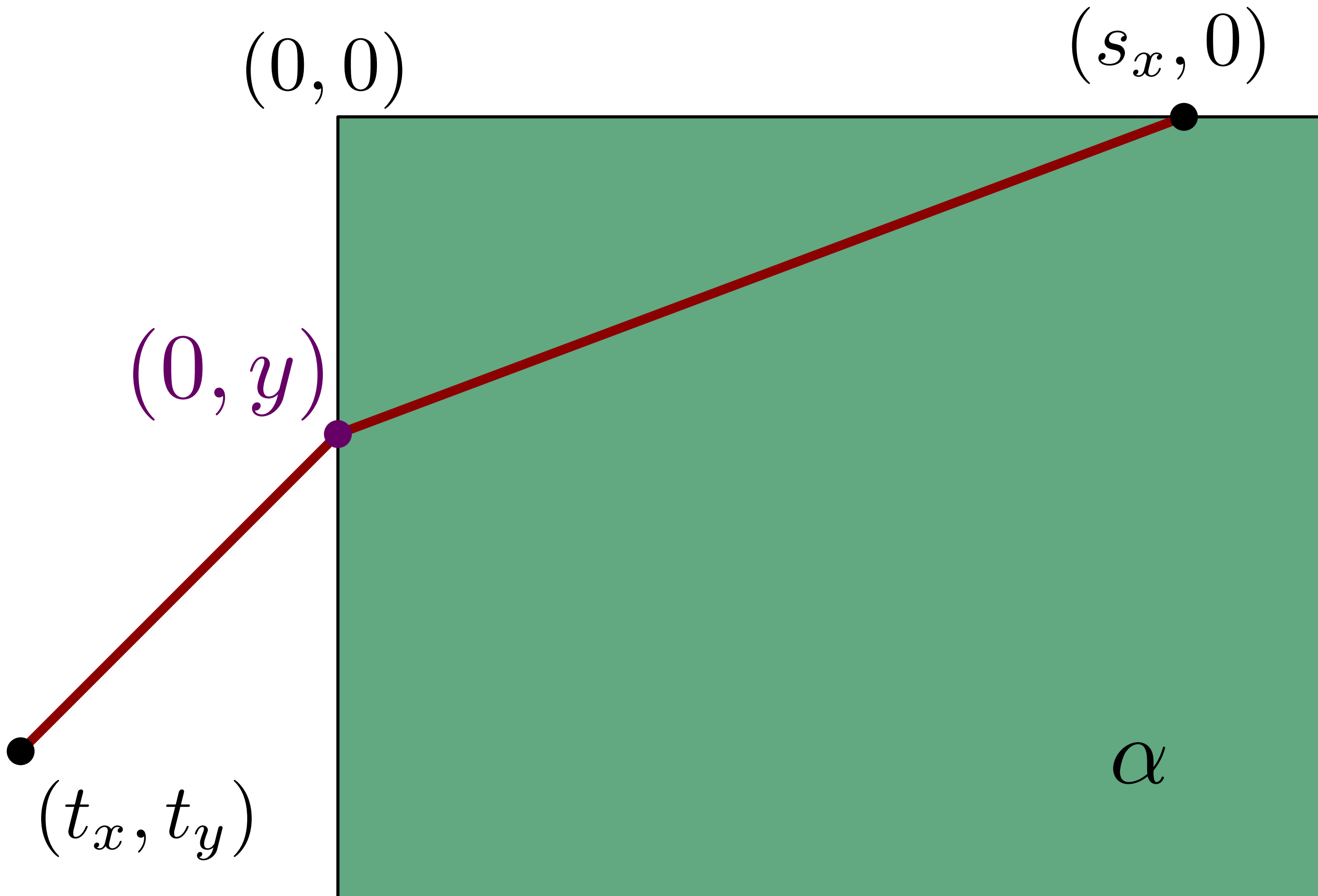

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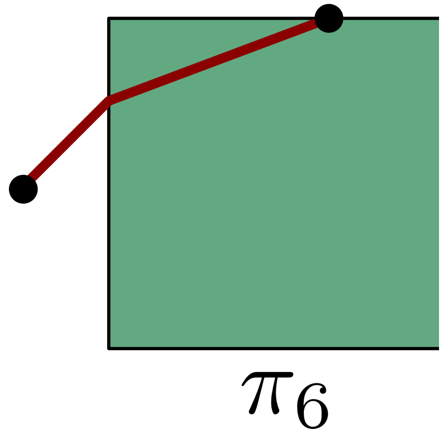

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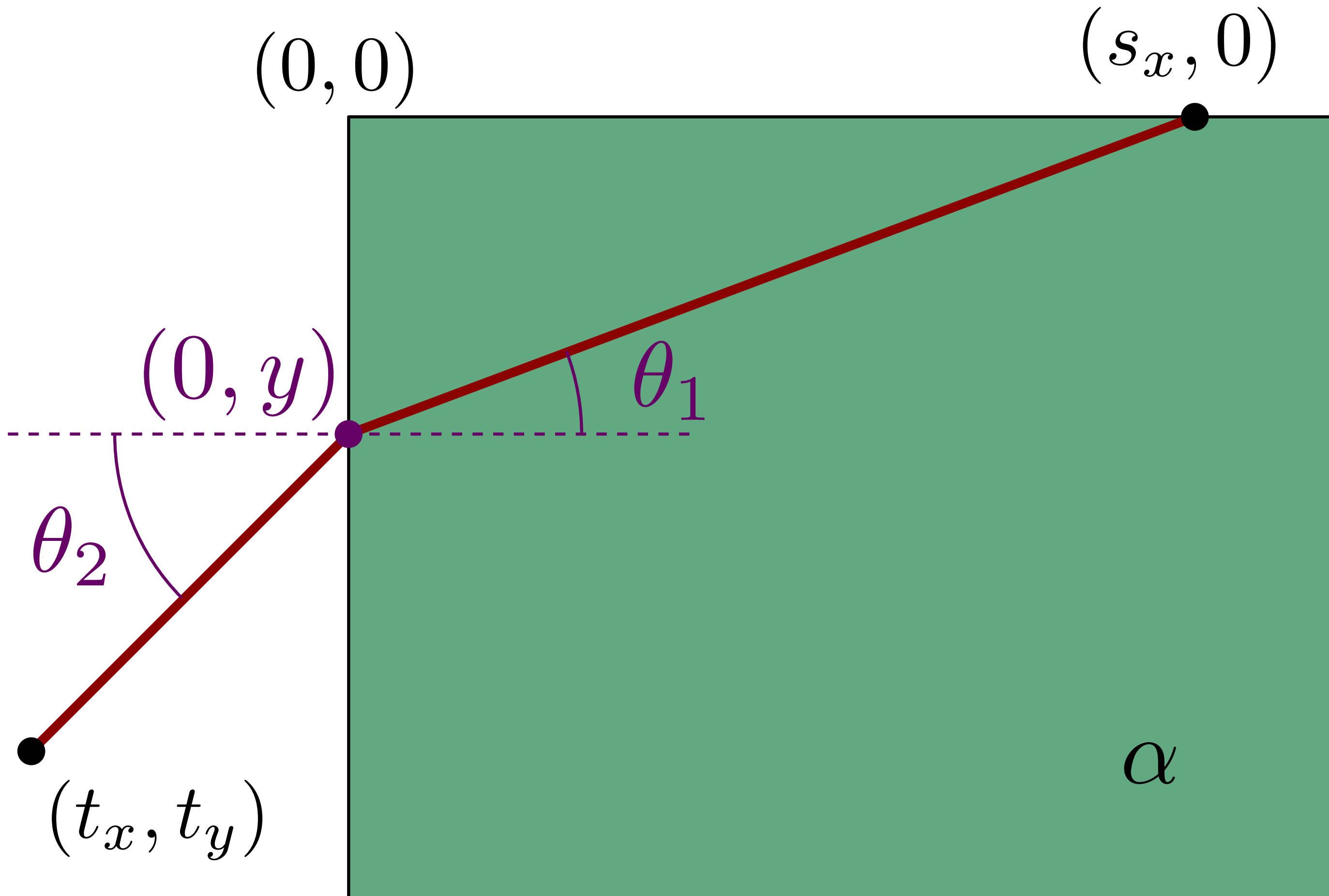

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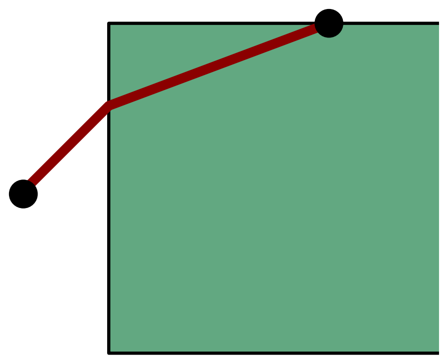

π_6

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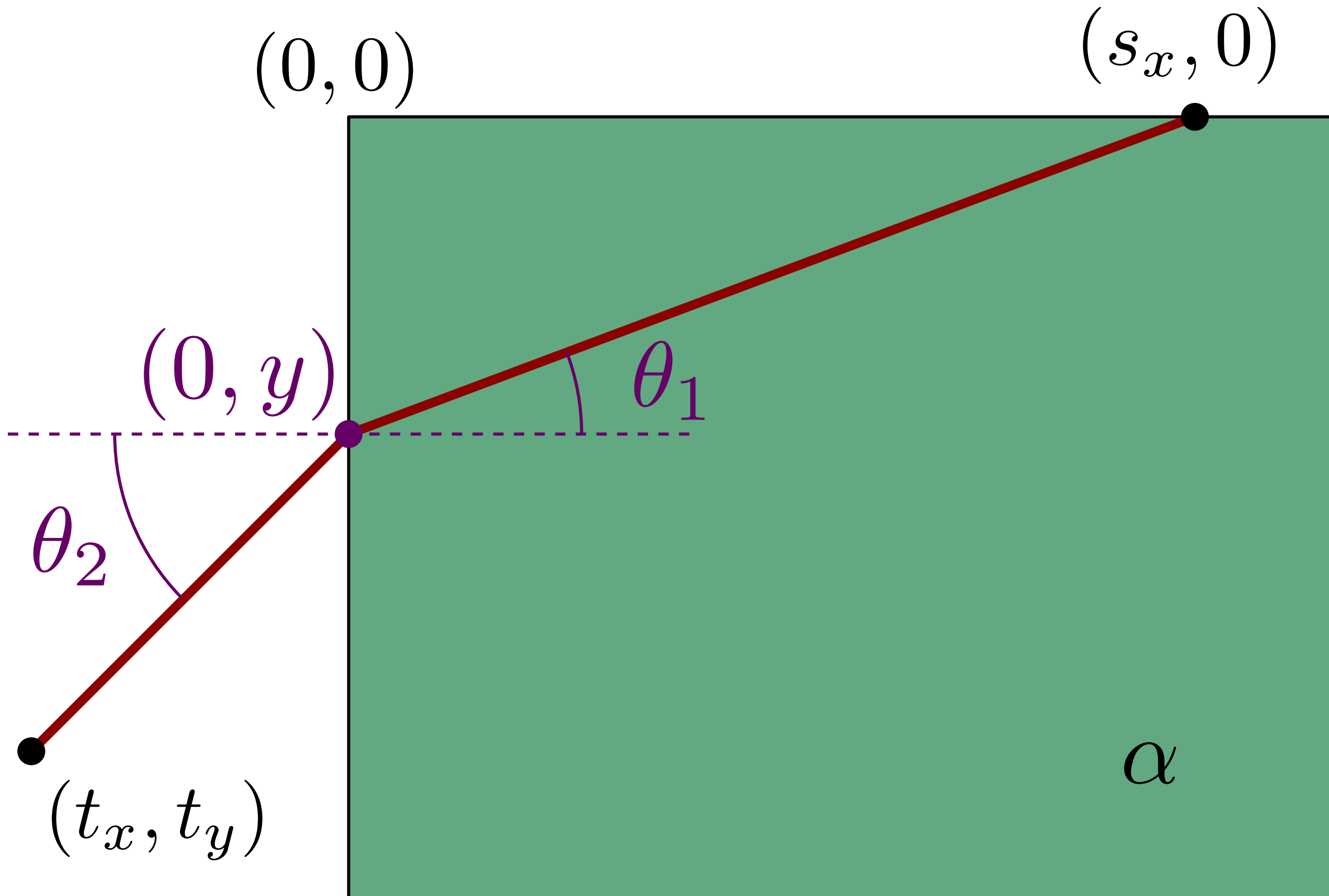

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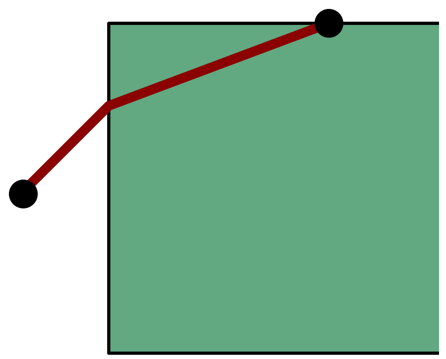

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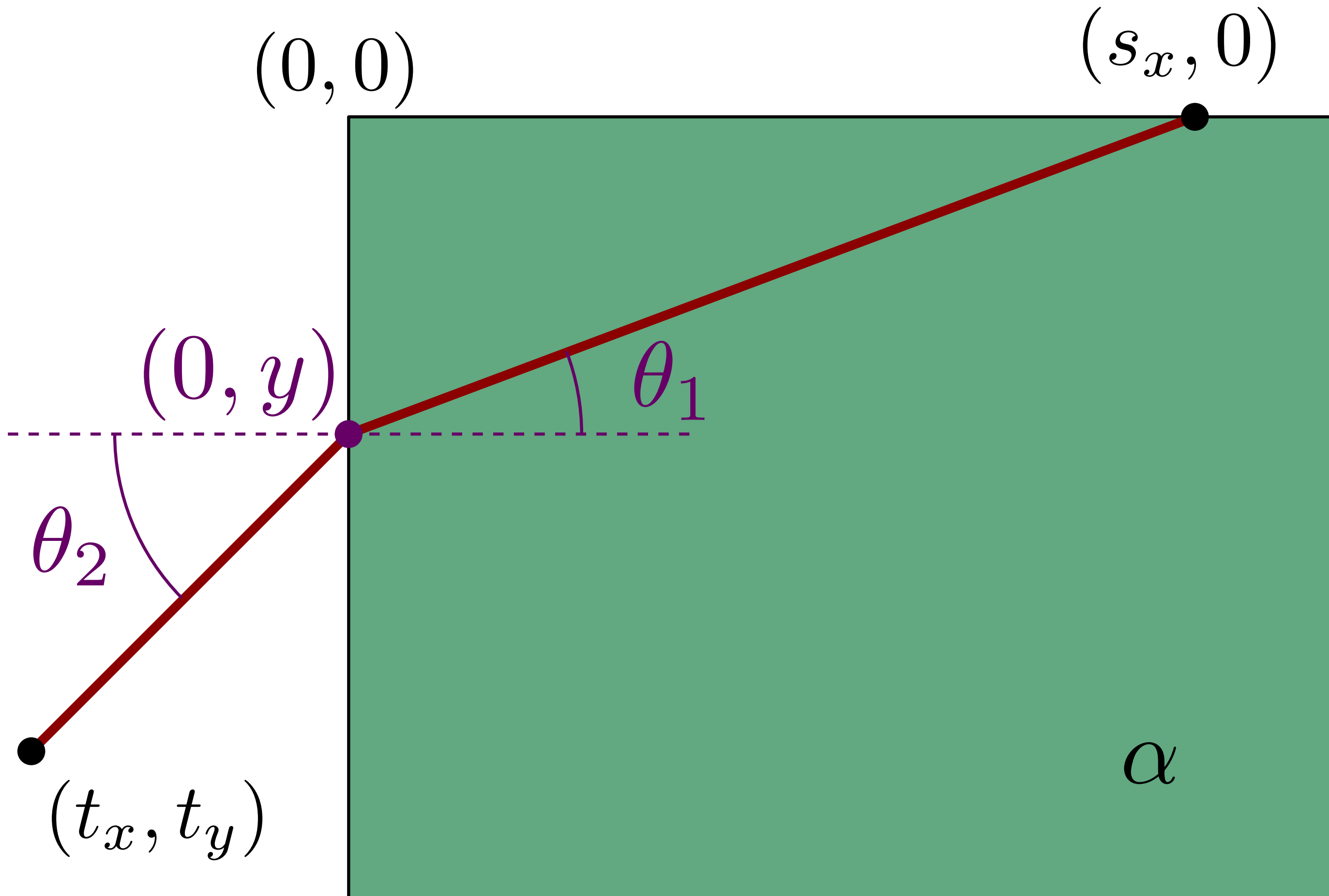
$$\Rightarrow (\alpha^2 - 1)y^4 - 2t_y(\alpha^2 - 1)y^3 + [\alpha^2 t_x^2 + (\alpha^2 - 1)t_y^2 - s_x^2]y^2 + 2s_x^2 t_y y - s_x^2 t_y^2 = 0$$



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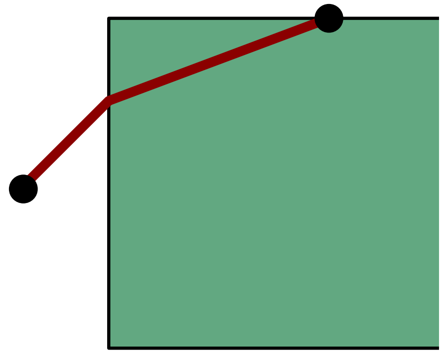

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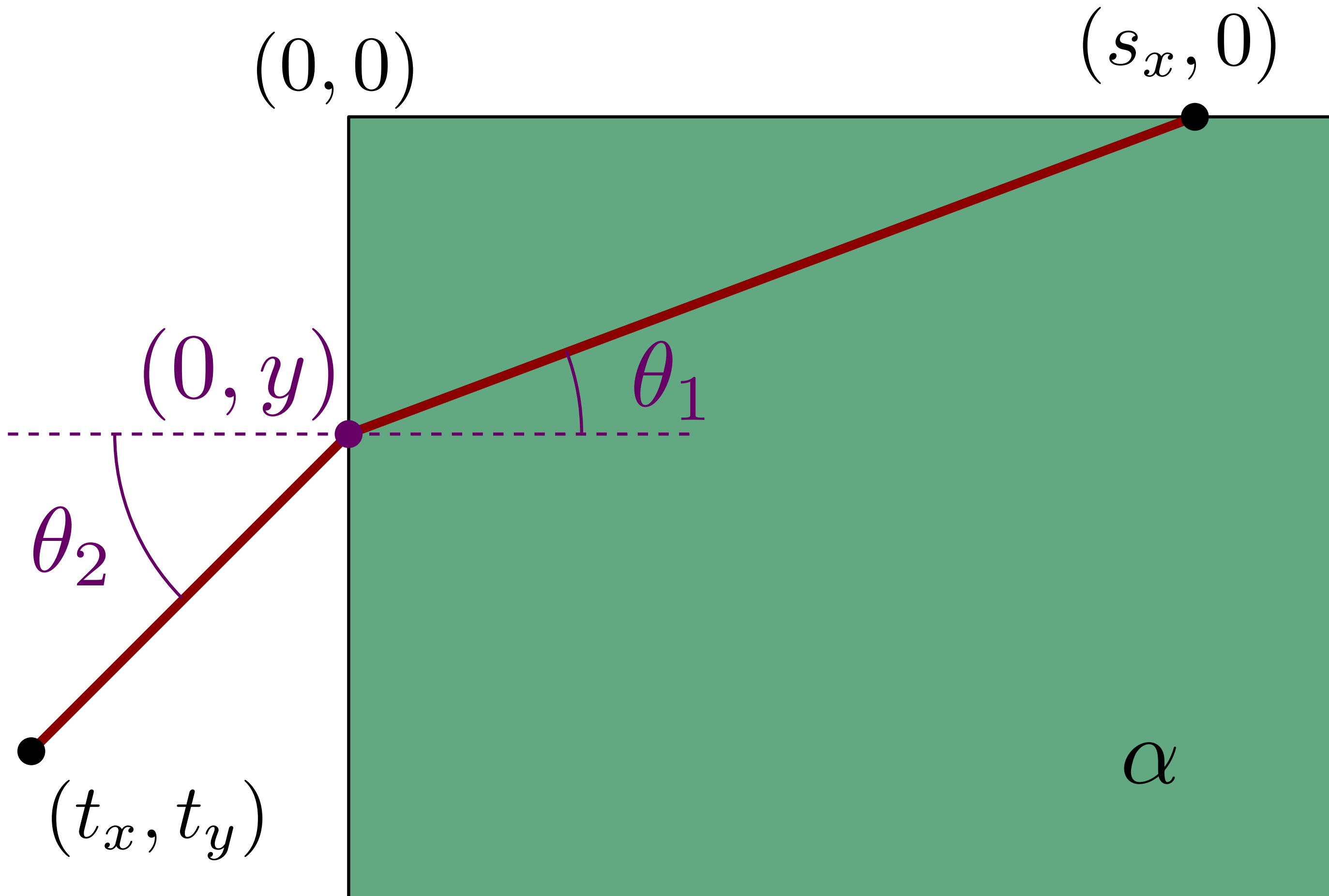
- Thus the length of the path is

$$\alpha\sqrt{s_x^2 + y^2} + \sqrt{t_x^2 + (t_y - y)^2}$$

where y is the unique solution to (1) in interval $(t_y, 0)$



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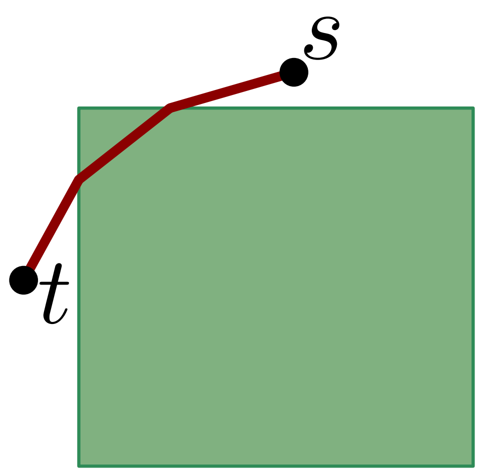

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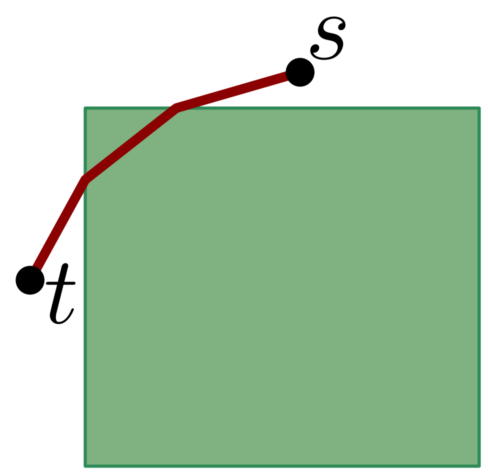

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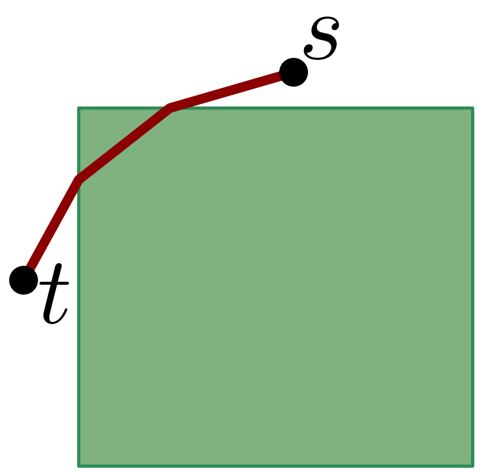

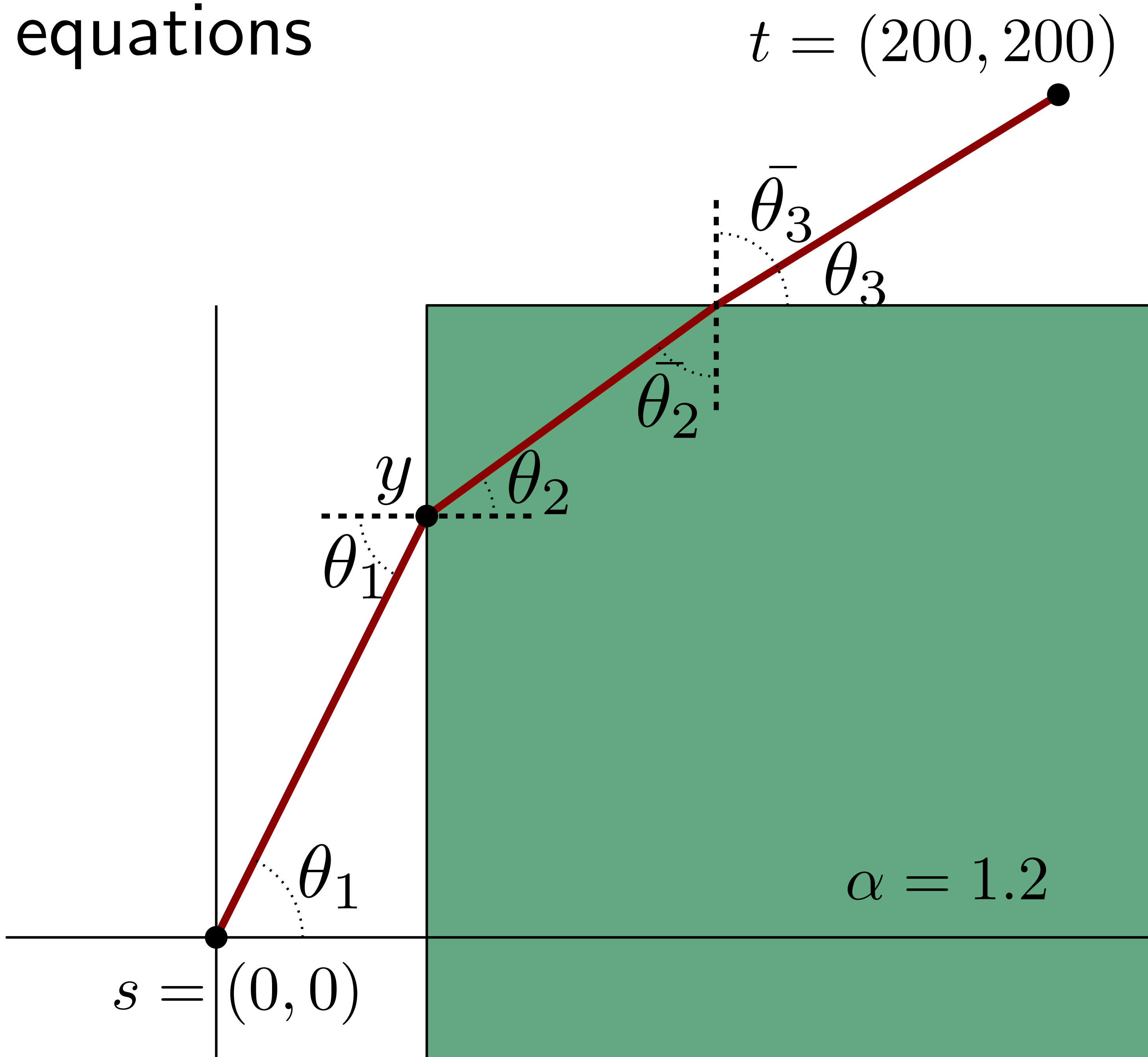

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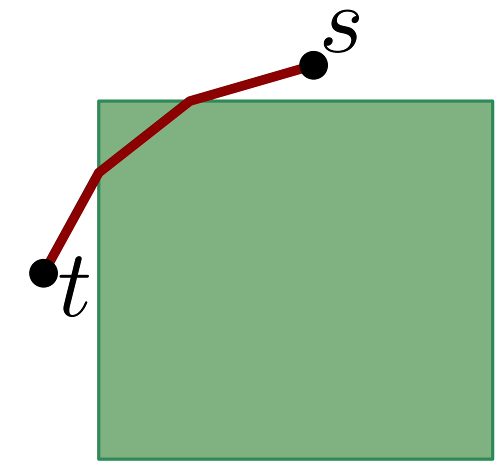
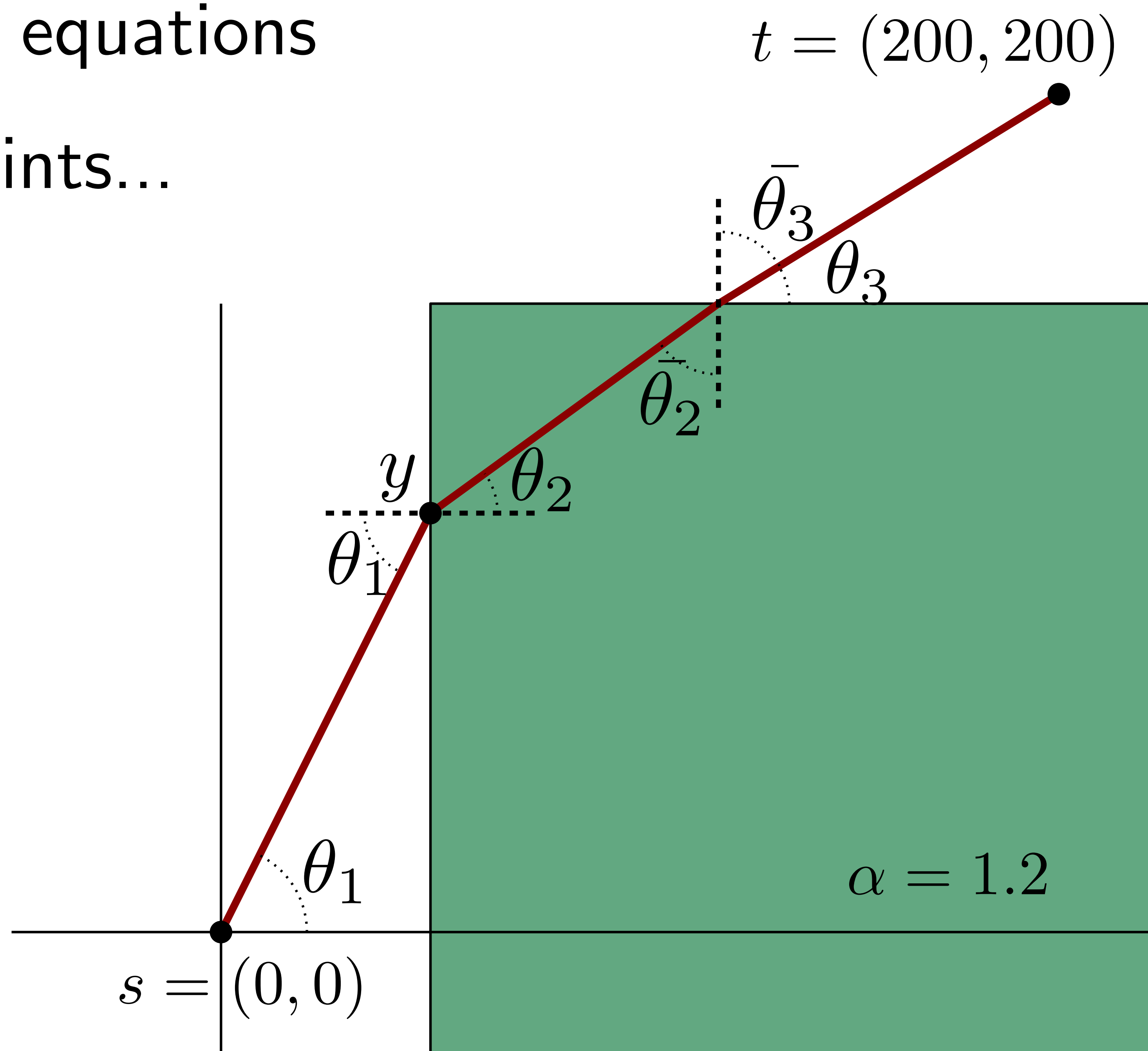
- Complication: this shortest path bends twice...
  ...in a way that leads to more complicated equations
- Applying Snell's law at the two bending points...
  ...and a lot of algebraic manipulations...





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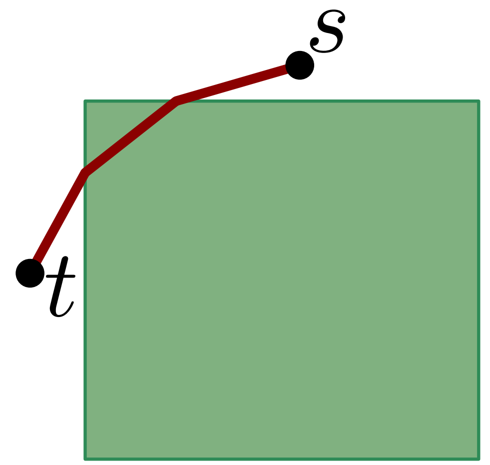
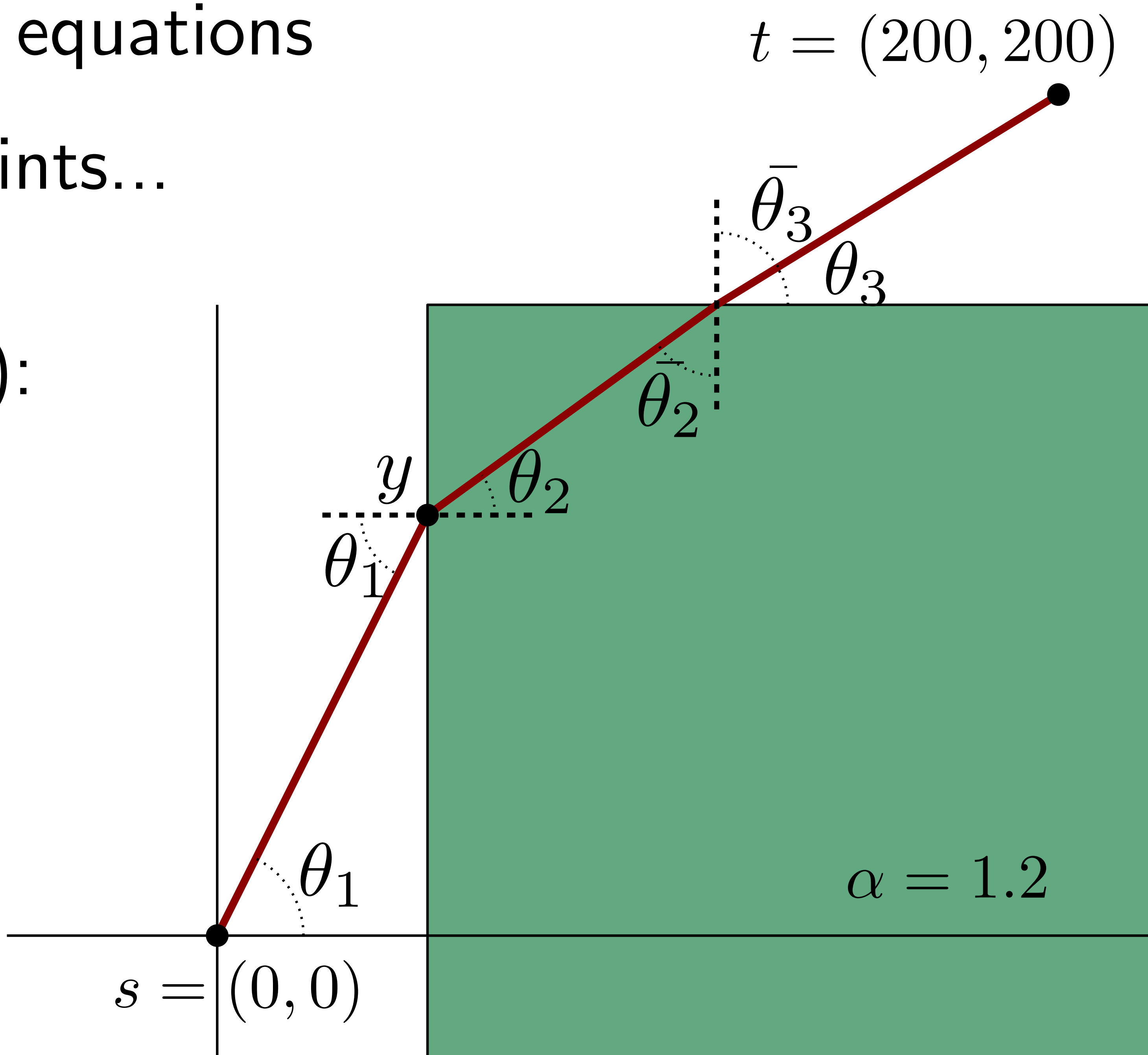


- Similar types of shortest paths, but a few are special
- Complication: this shortest path bends twice...
  ...in a way that leads to more complicated equations
- Applying Snell's law at the two bending points...
  ...and a lot of algebraic manipulations...
  ...leads to this equation (where $x = \sin\theta_1$):



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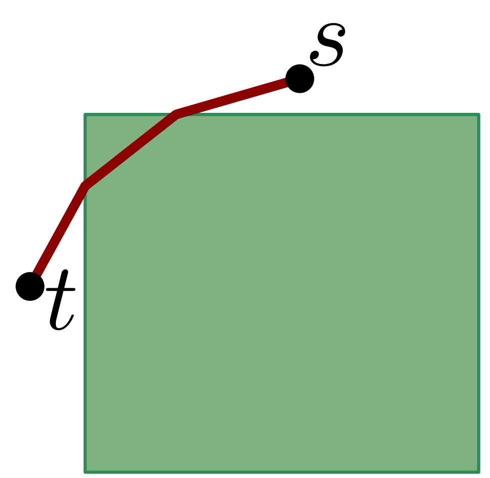

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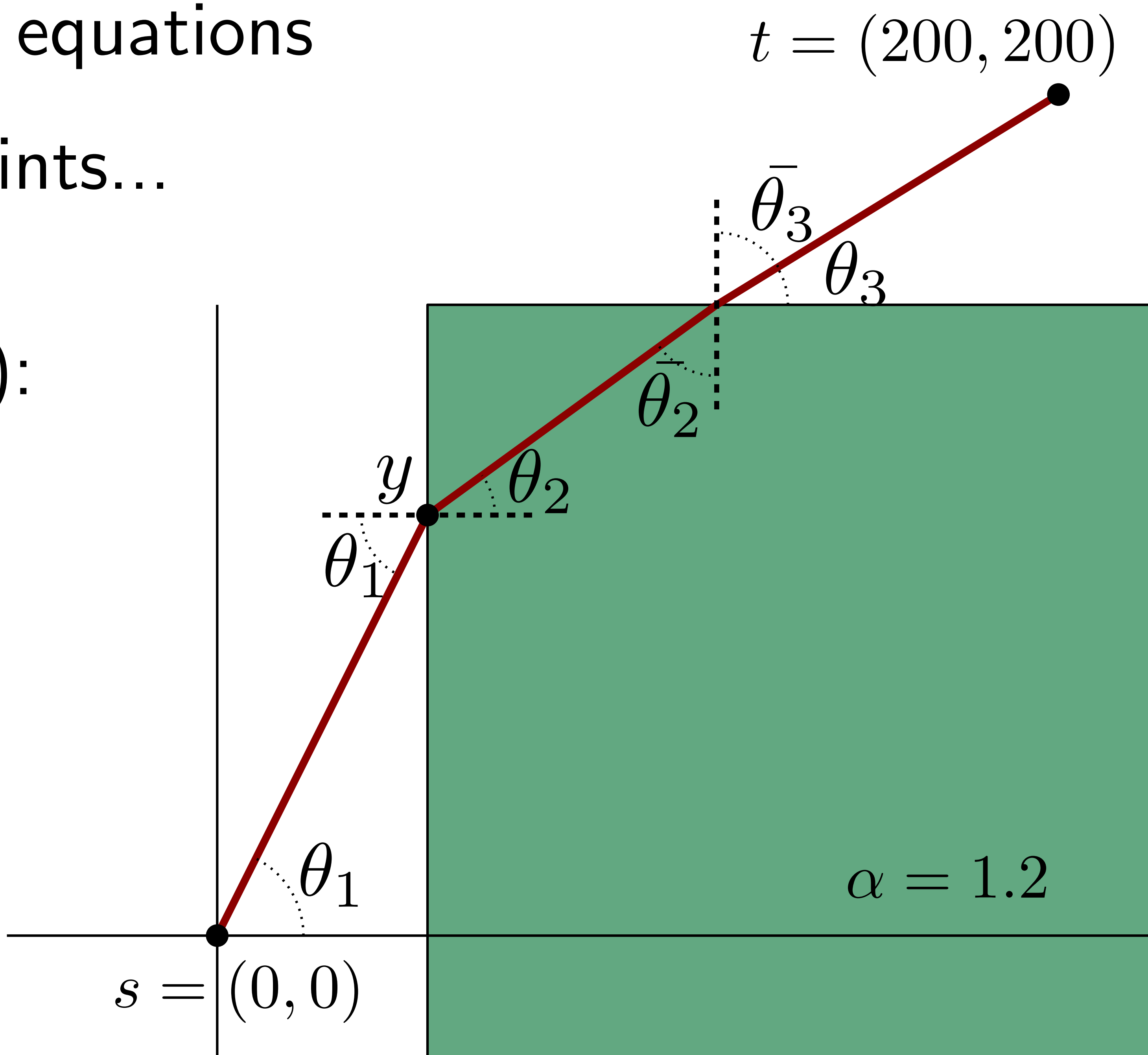
$$\sqrt{\alpha^2 - x^2}\left(\frac{3}{x} - \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\alpha^2+x^2}}\right) = 3$$

- The equation is equivalent to a degree-11 polynomial



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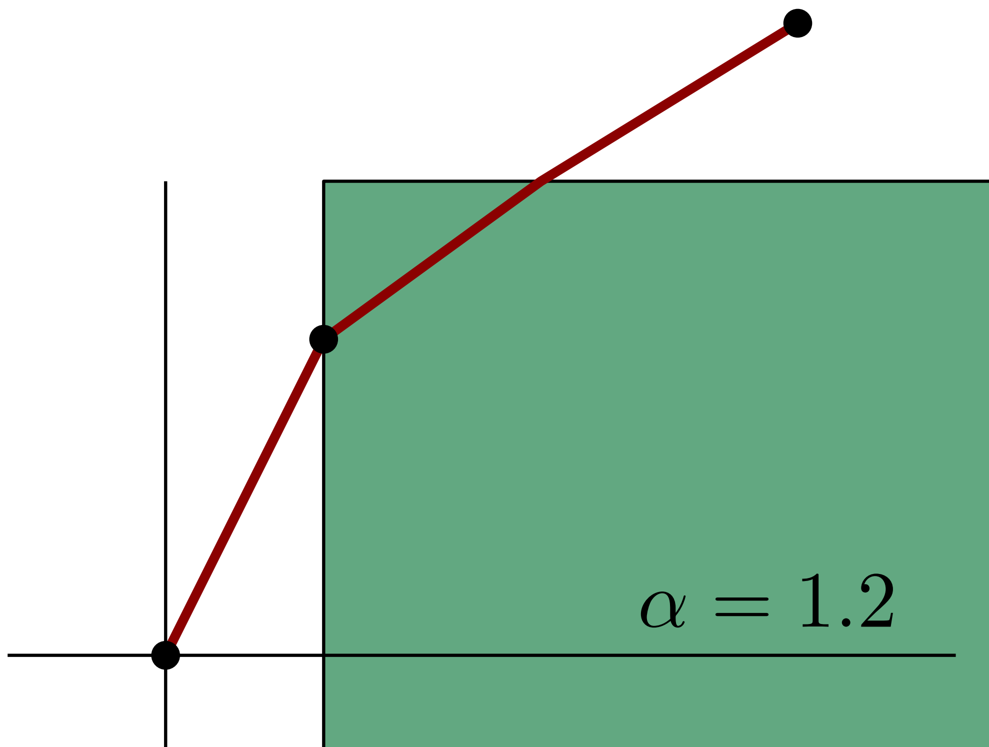
- So, to compute the shortest path for this instance, we need the roots of a complicated polynomial



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- So, to compute the shortest path for this instance, we need the roots of a complicated polynomial

$$
\begin{aligned}
p(x) = &- 5602195930320001 + 93511401766200000x - 713160370741499900x^2 \\
&+ 3259398736514250000x^3 - 9869397269940000000x^4 + 20717559301050000000x^5 \\
&- 30701172521250000000x^6 + 32082903984375000000x^7 - 23159988281250000000x^8 \\
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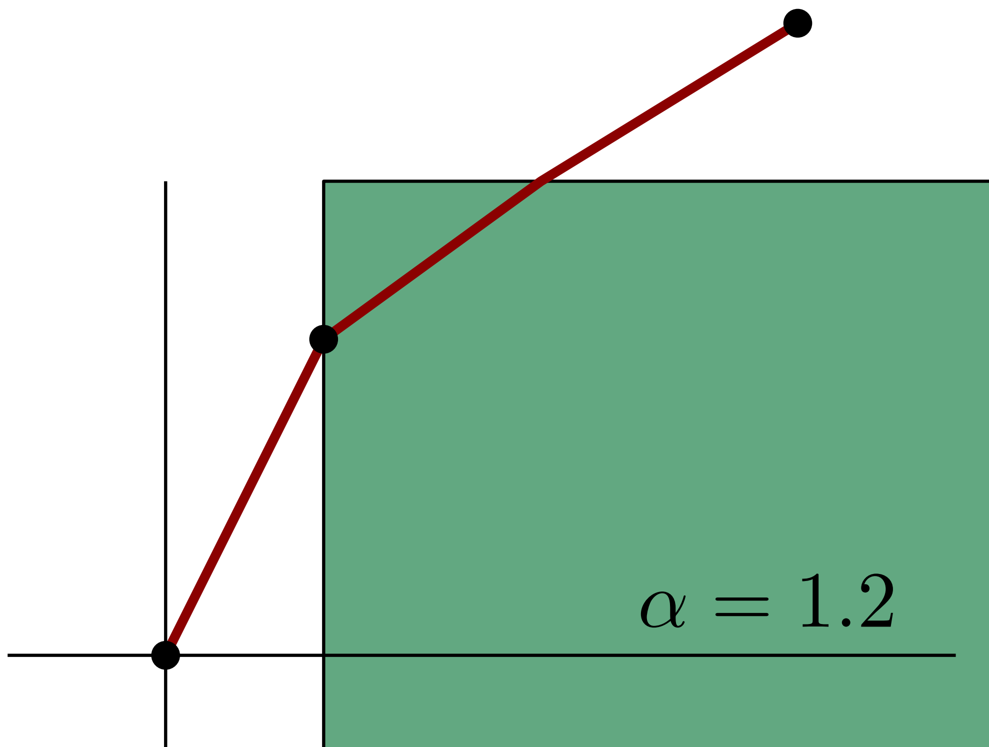

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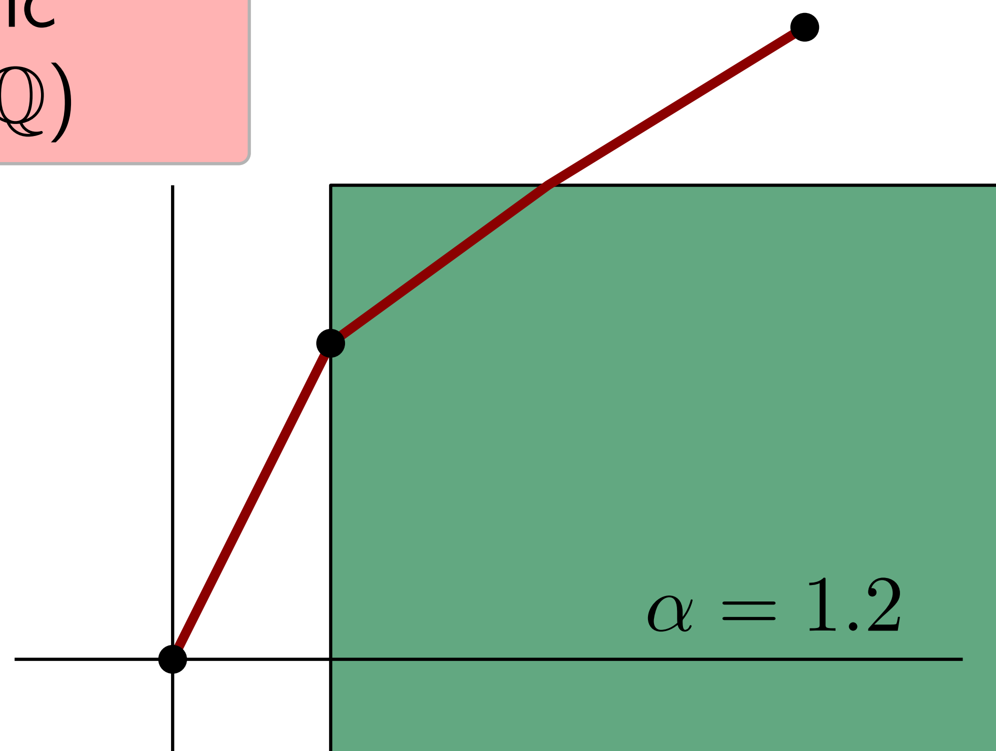

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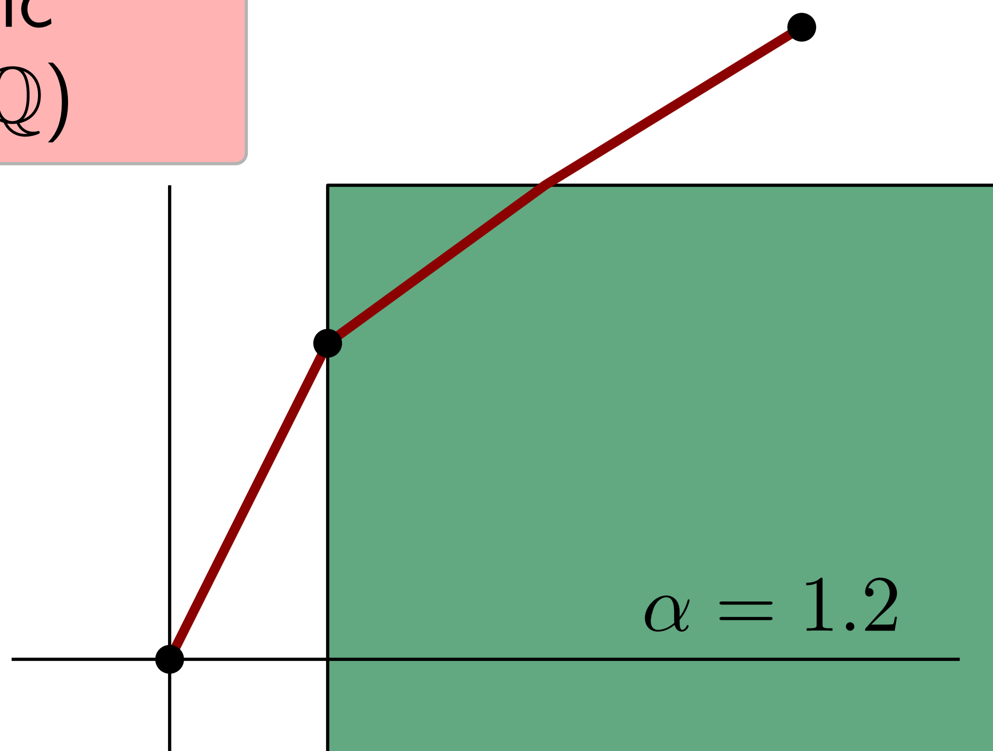
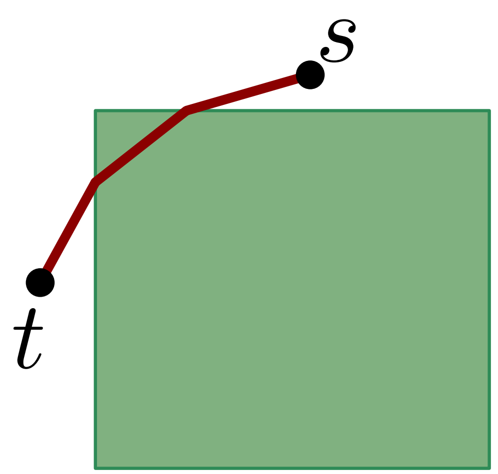
Therefore, the same happens to the WRP with one region (a quadrant), and two arbitrary weights



Conclusions

- Even for one single rectangular region, with two weights, the WRP is unsolvable



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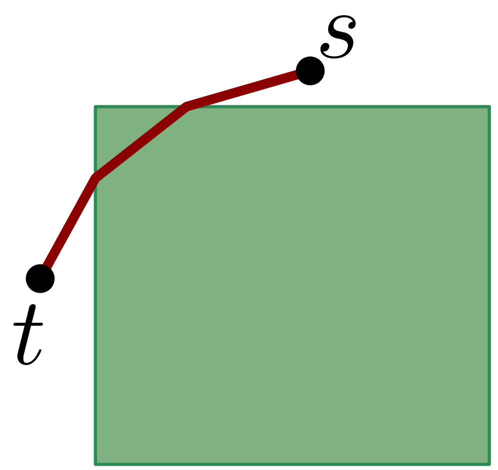
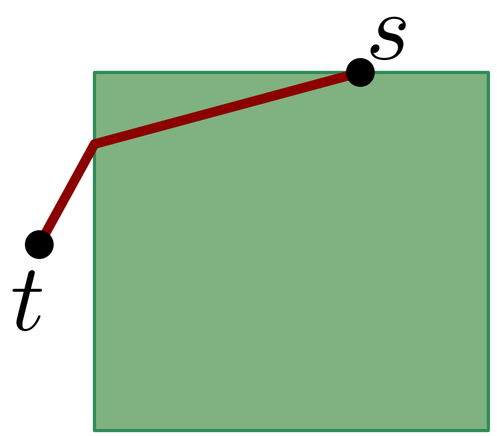


- Solvable if s is on the boundary

  ...we worked out every single equation



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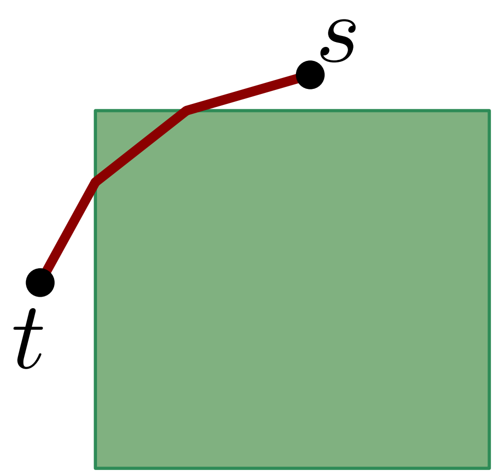

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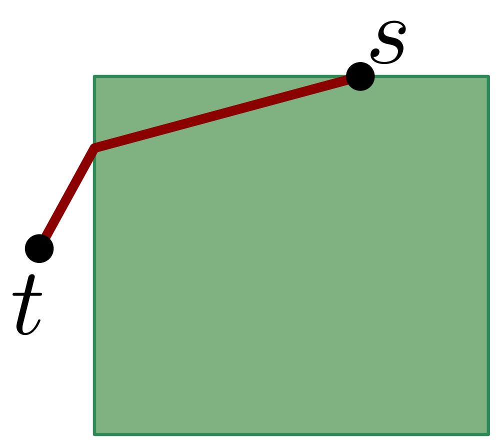
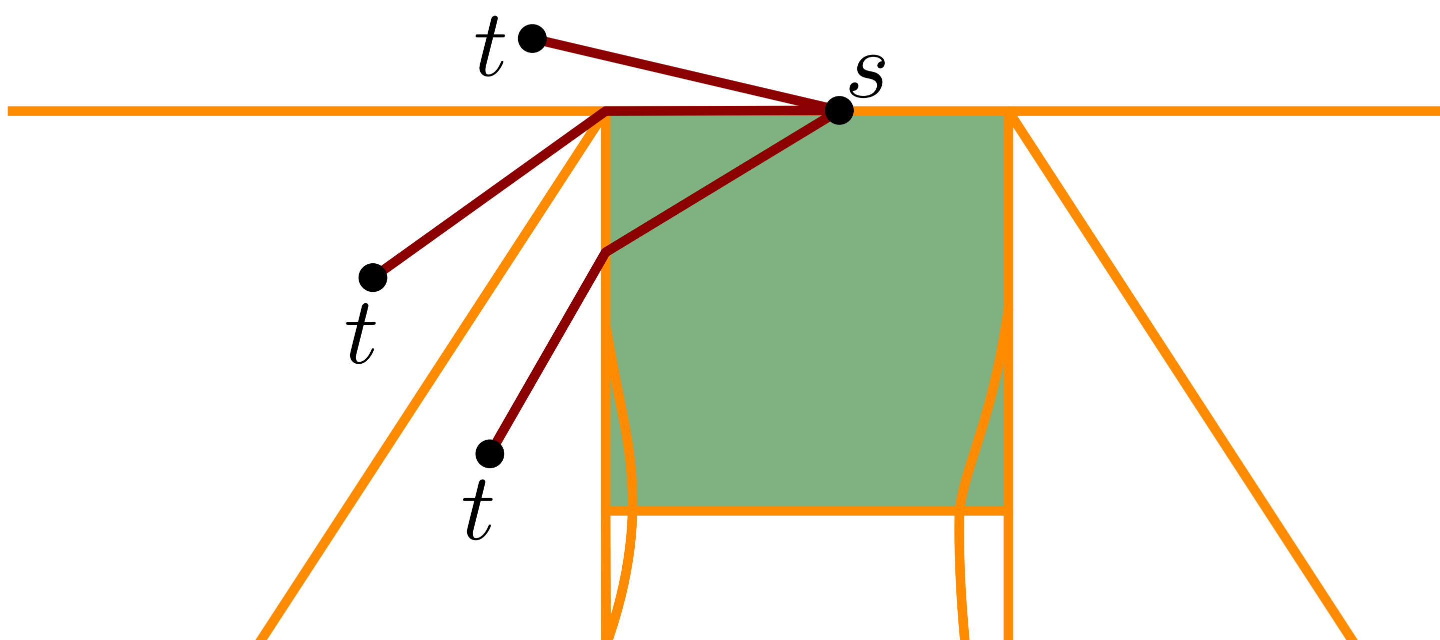


- For s on the boundary, we can *almost* compute the Shortest Path Map

  ...but not quite



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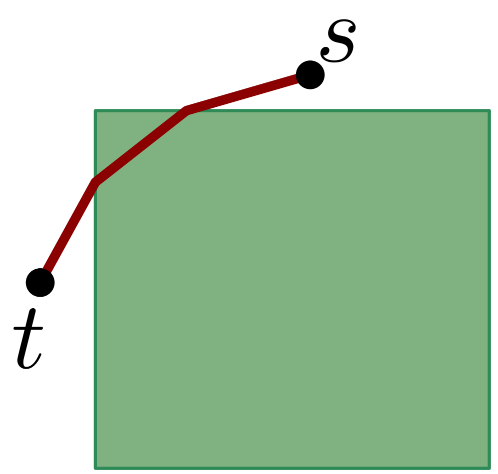

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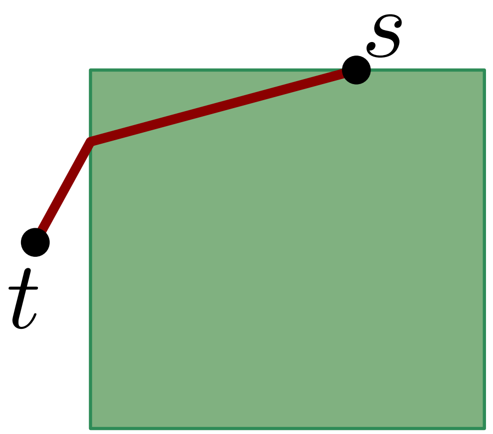

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It's difficult to beat the monster!