Clustering with Few Disks to Minimize the Sum of Radii

(accepted at SoCG '24)

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Introduction

Clustering in the Plane

Given: Set of points P in the plane

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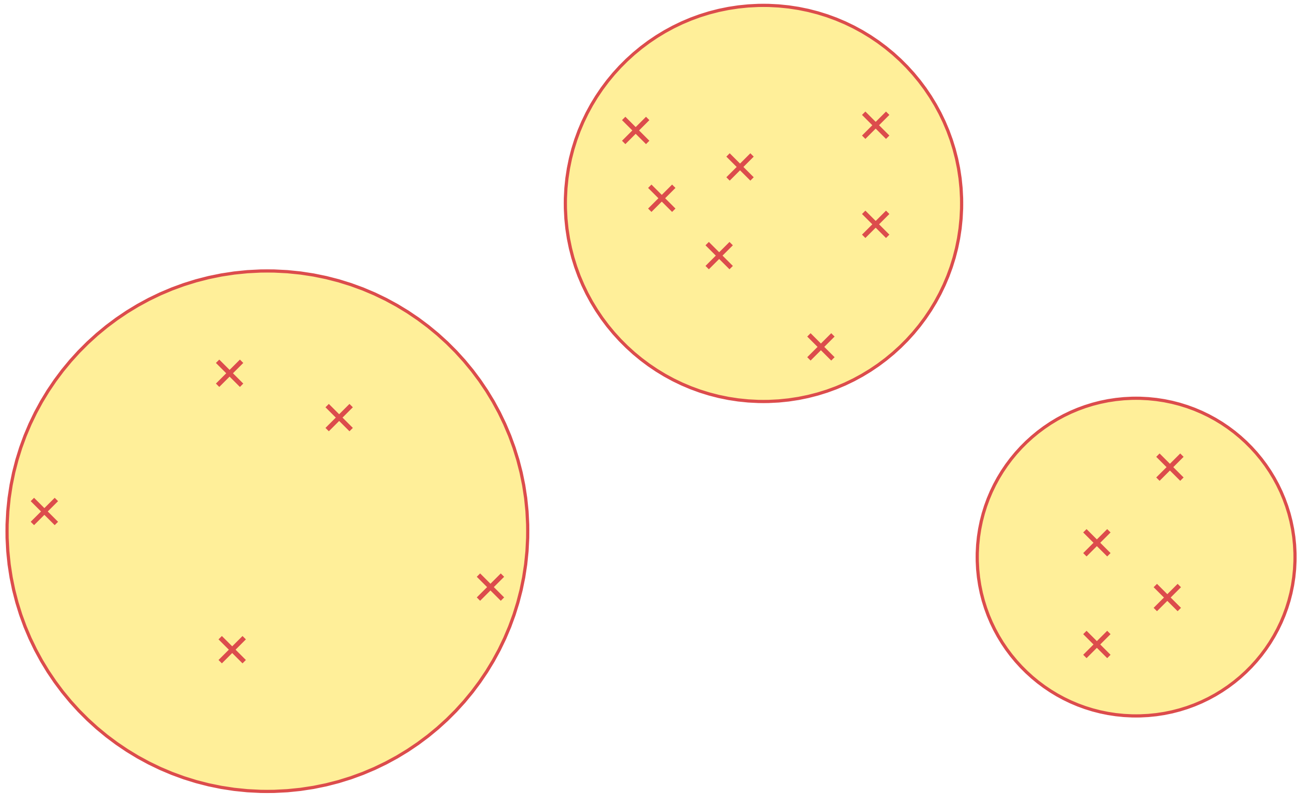
Goal: Cover P with k disks, minimizing the *objective*



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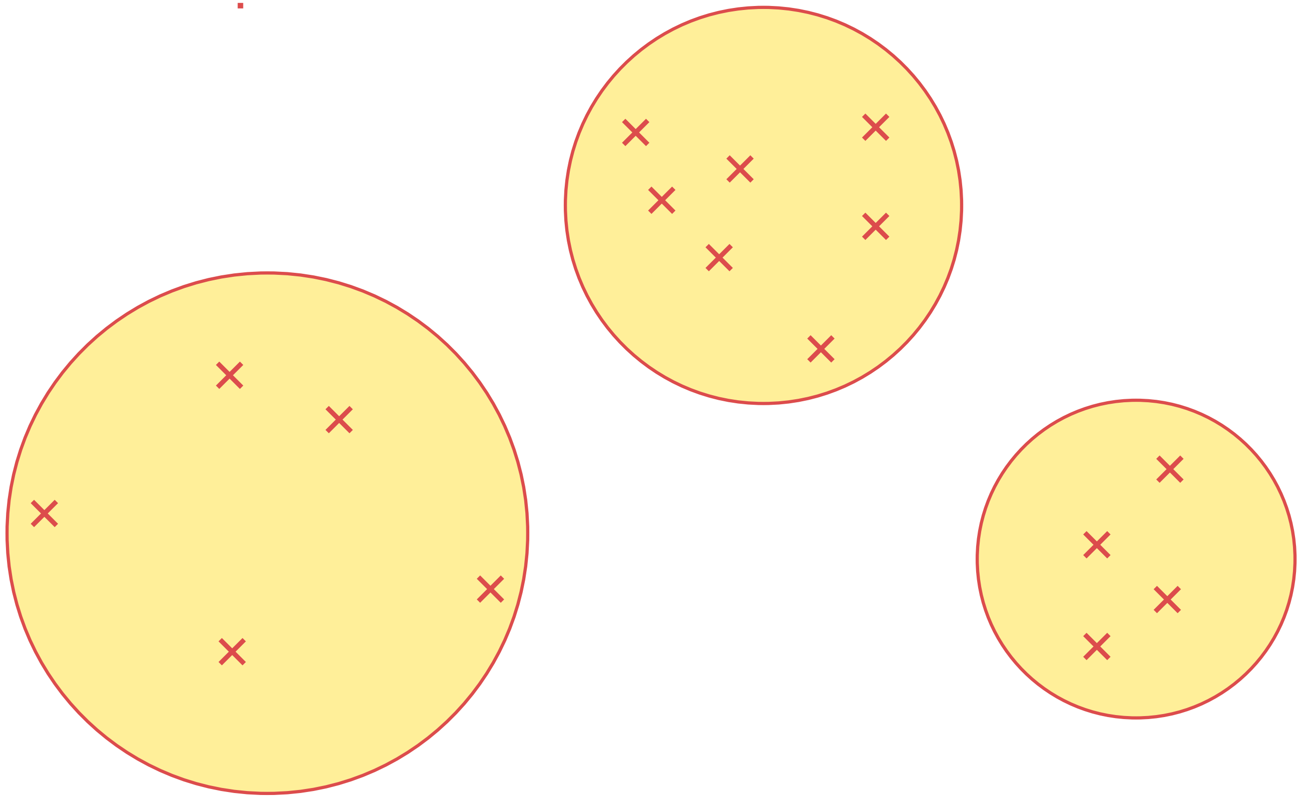

Clustering in the Plane

k-Center

Problem: Find disks $C_1, \ldots, C_k$ covering P minimizing

$$\max_i r(C_i).$$

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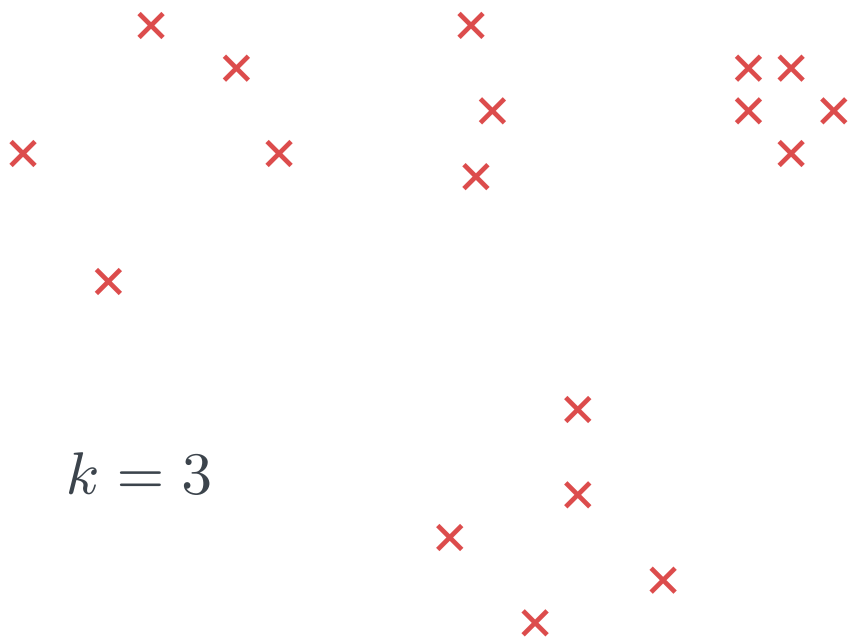

$k = 3$

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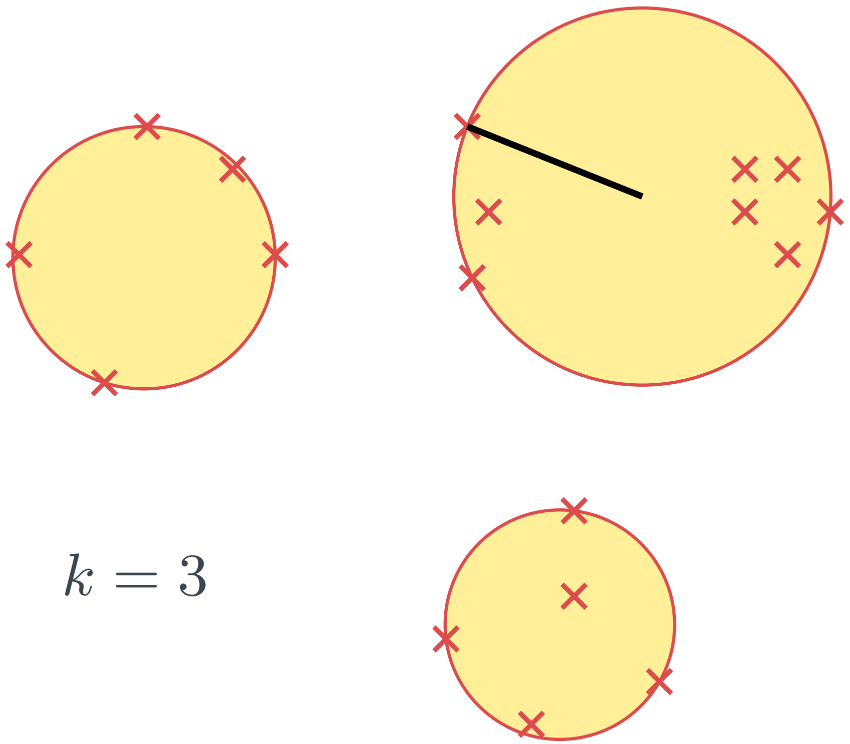

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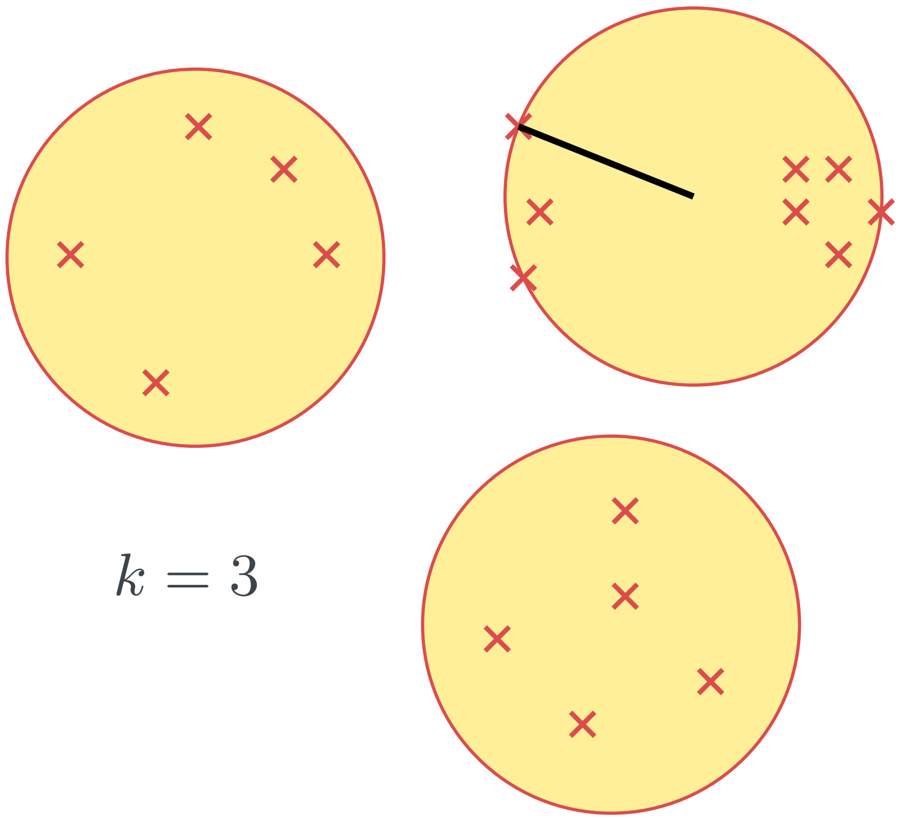

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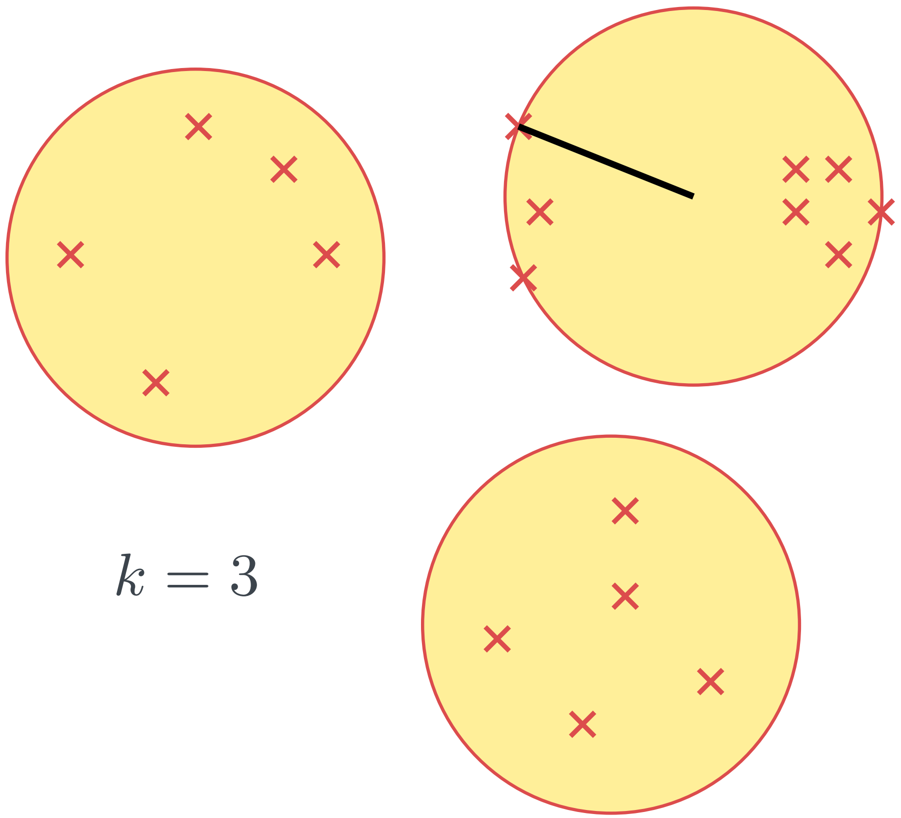

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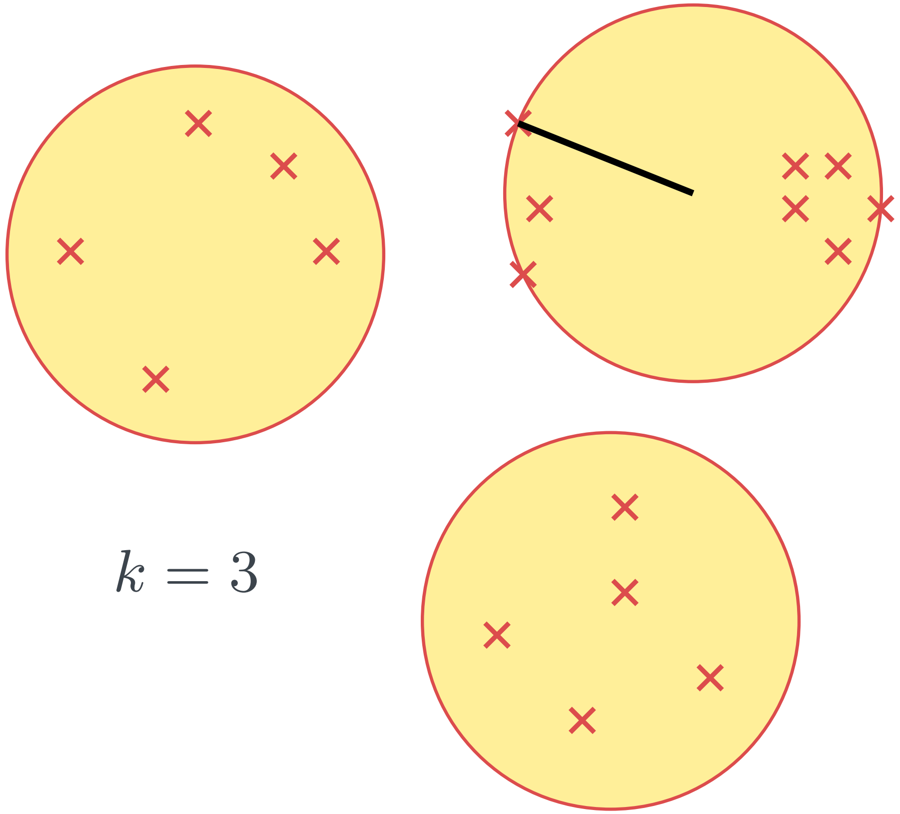

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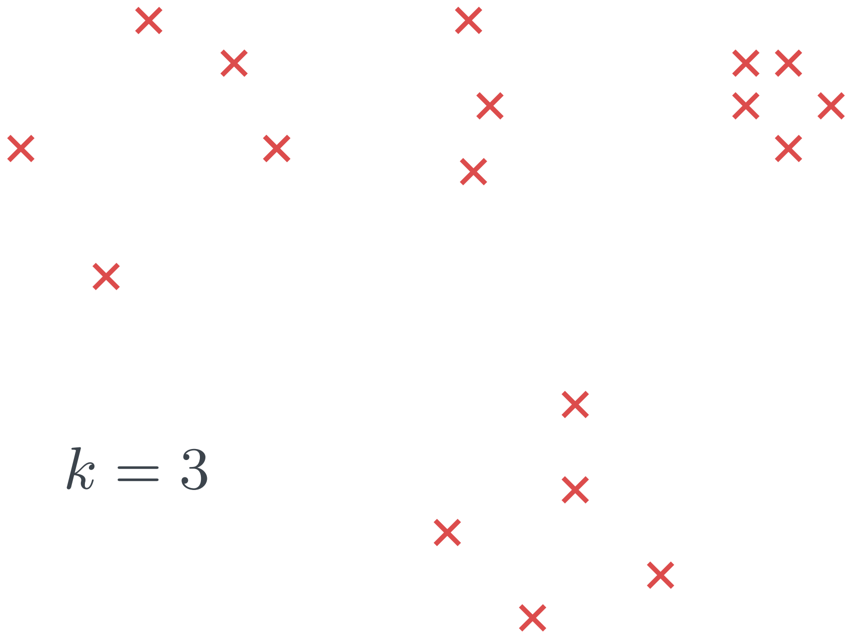

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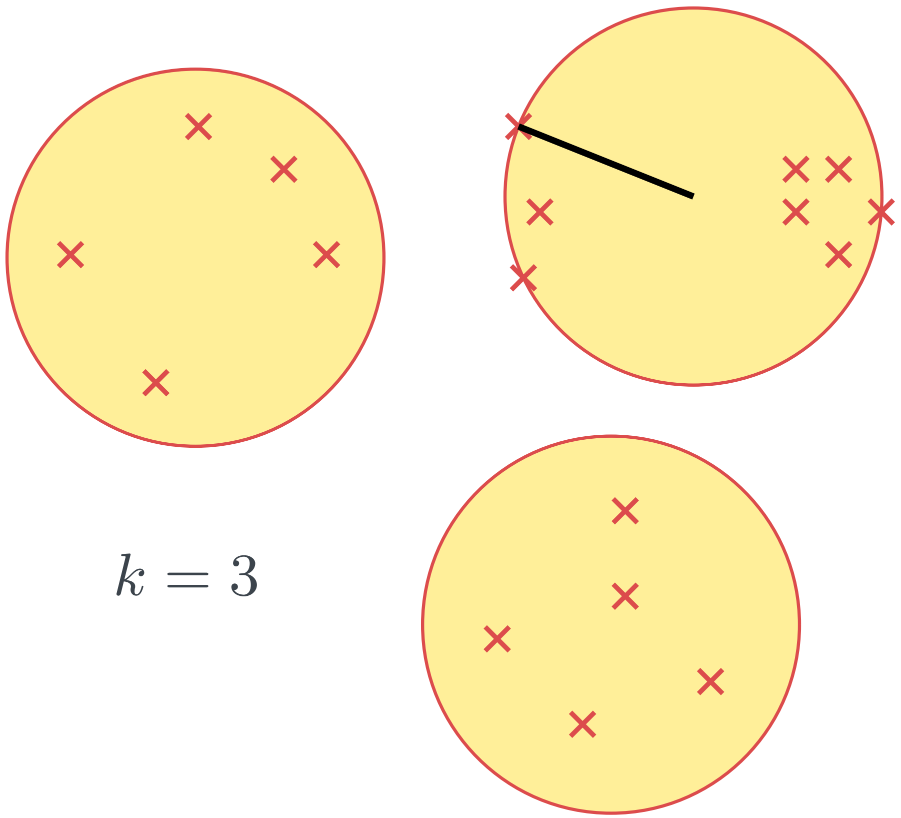

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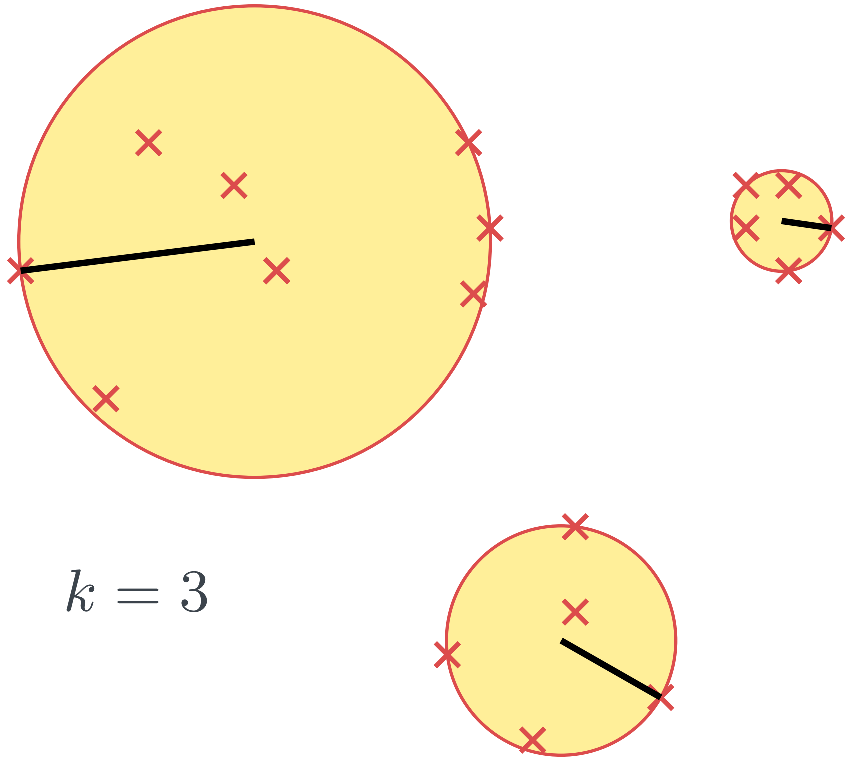
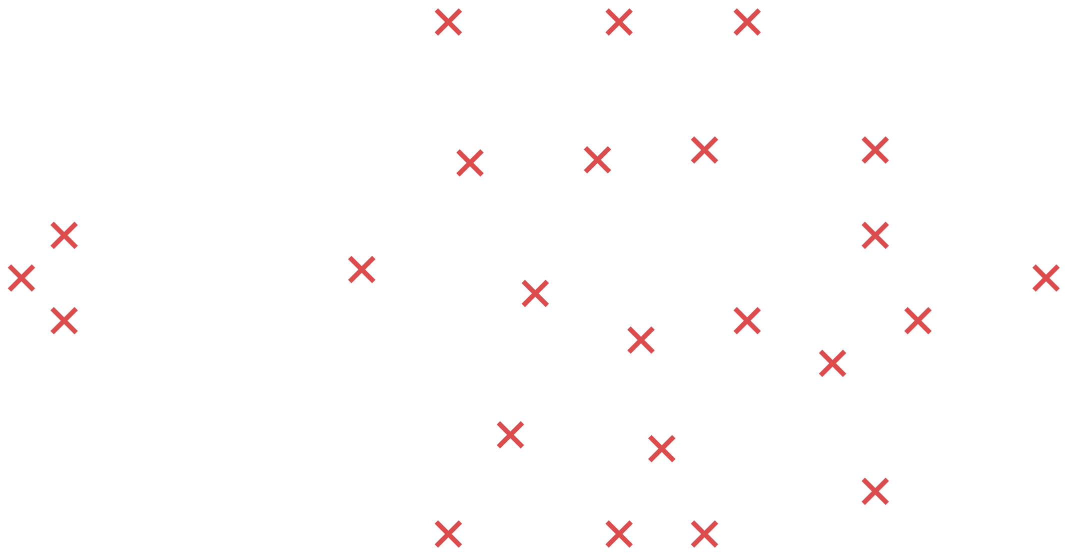


Clustering in the Plane

k-Center vs. k-MinSumRadius



Clustering in the Plane

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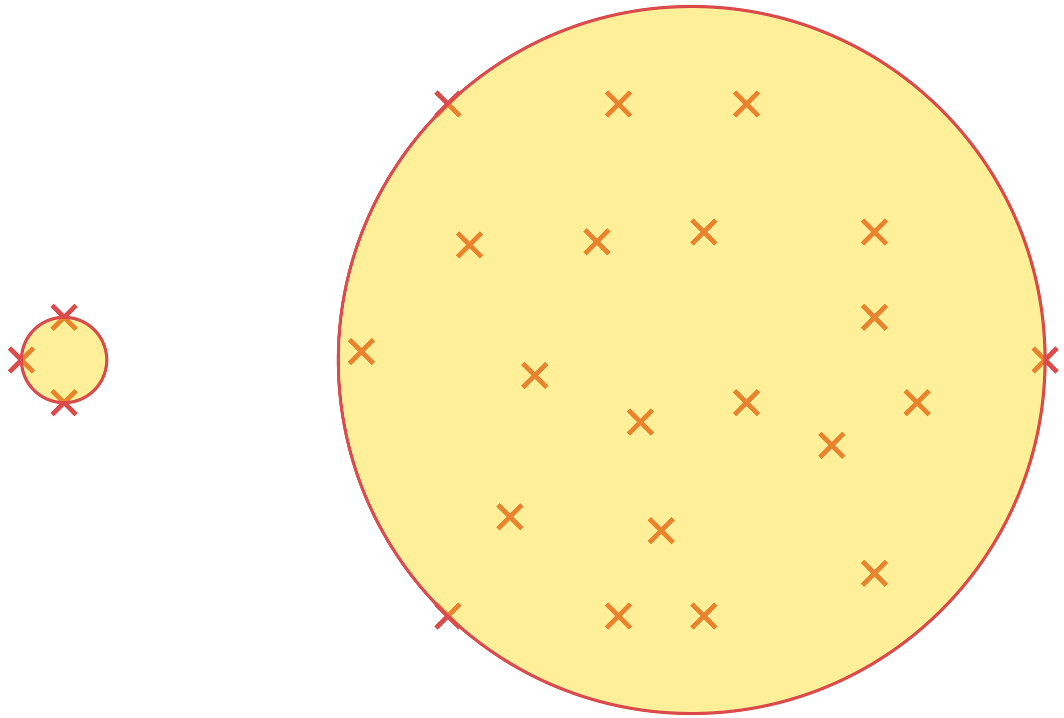
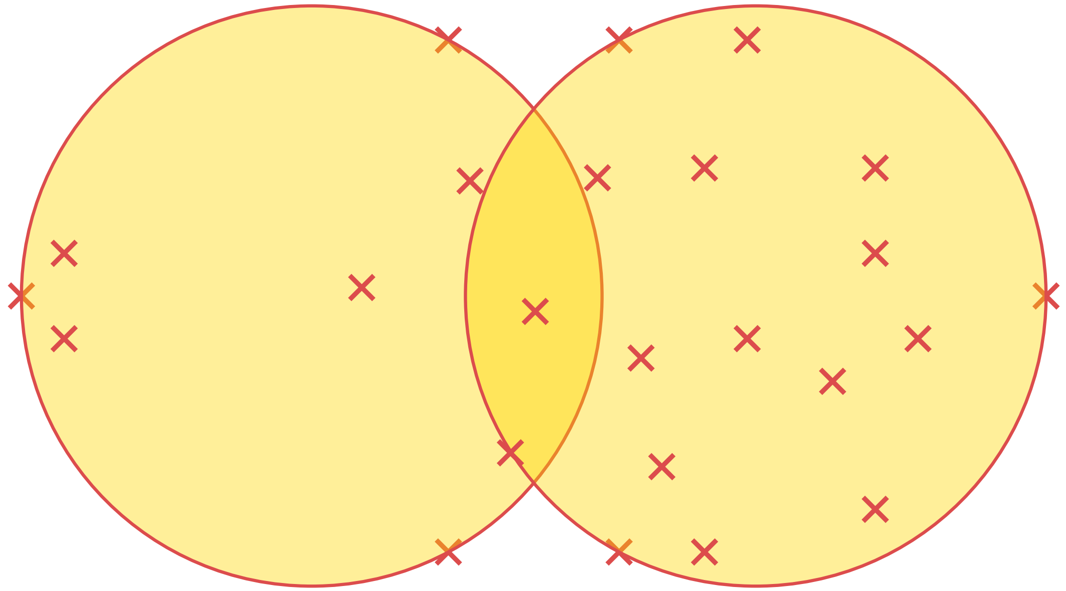


k-MinSumRadius solution

Clustering in the Plane

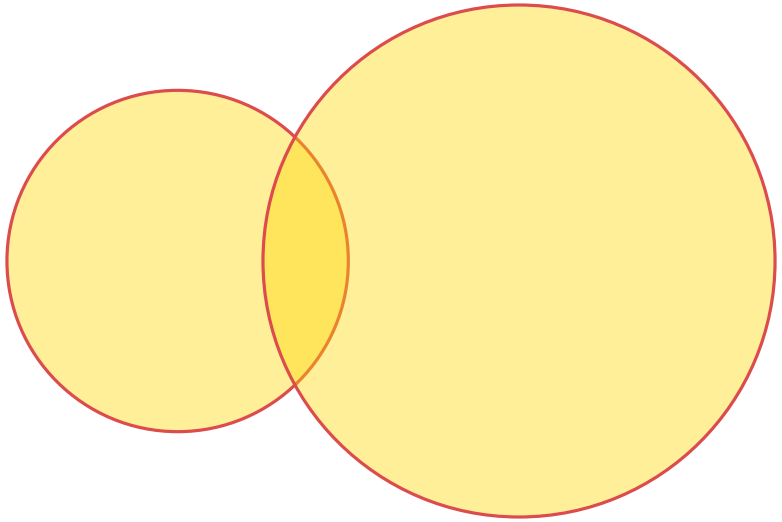
k-Center vs. k-MinSumRadius



k-Center solution

Clustering in the Plane

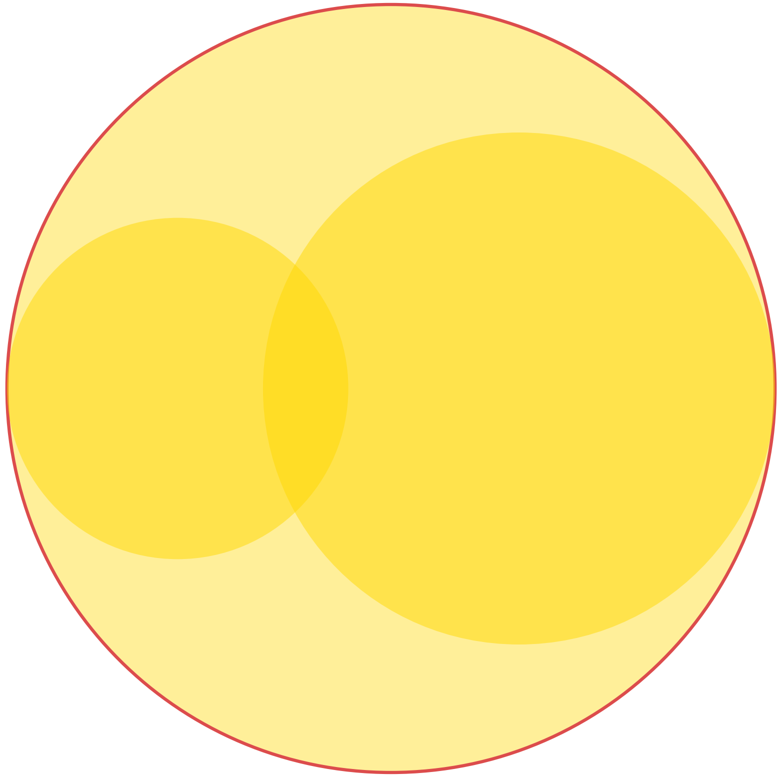
k-Center vs. k-MinSumRadius



k-MinSumRadius clusters do **not** intersect

Clustering in the Plane

k-Center vs. k-MinSumRadius



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Clustering in the Plane

k-Center vs. k-MinSumRadius

k-Center

NP-hard with k part of input [Megiddo, Supovit '84]

k-MinSumRadius

$\mathcal{O}(n^{881})$ algorithm [Gibson, Kanade, Krohn, Pirwani, Varadarajan '12]

Clustering in the Plane

k-Center vs. k-MinSumRadius

k-Center

NP-hard with k part of input [Megiddo, Supovit '84]

$k = 2$:

near-quadratic:

- [Eppstein '92] [Katz, Sharir '93] [Agarwal, Sharir '94] [Jaromczyk, Kowaluk '94]

near-linear:

- [Sharir '96] [Eppstein '97] [Chan '99] [Wang '20] [Cho, Oh '20]

k-MinSumRadius

$\mathcal{O}(n^{881})$ algorithm [Gibson, Kanade, Krohn, Pirwani, Varadarajan '12]

$k = 2$:

$\mathcal{O}(n^2 \log^2 n)$ algorithm [Eppstein '92]

Our Results

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k-MinSumRadius Algorithms

	$k = 2$	$k = 3$ *
$d = 2$		
$d > 2$		

* We omit $\log \log$ factors here

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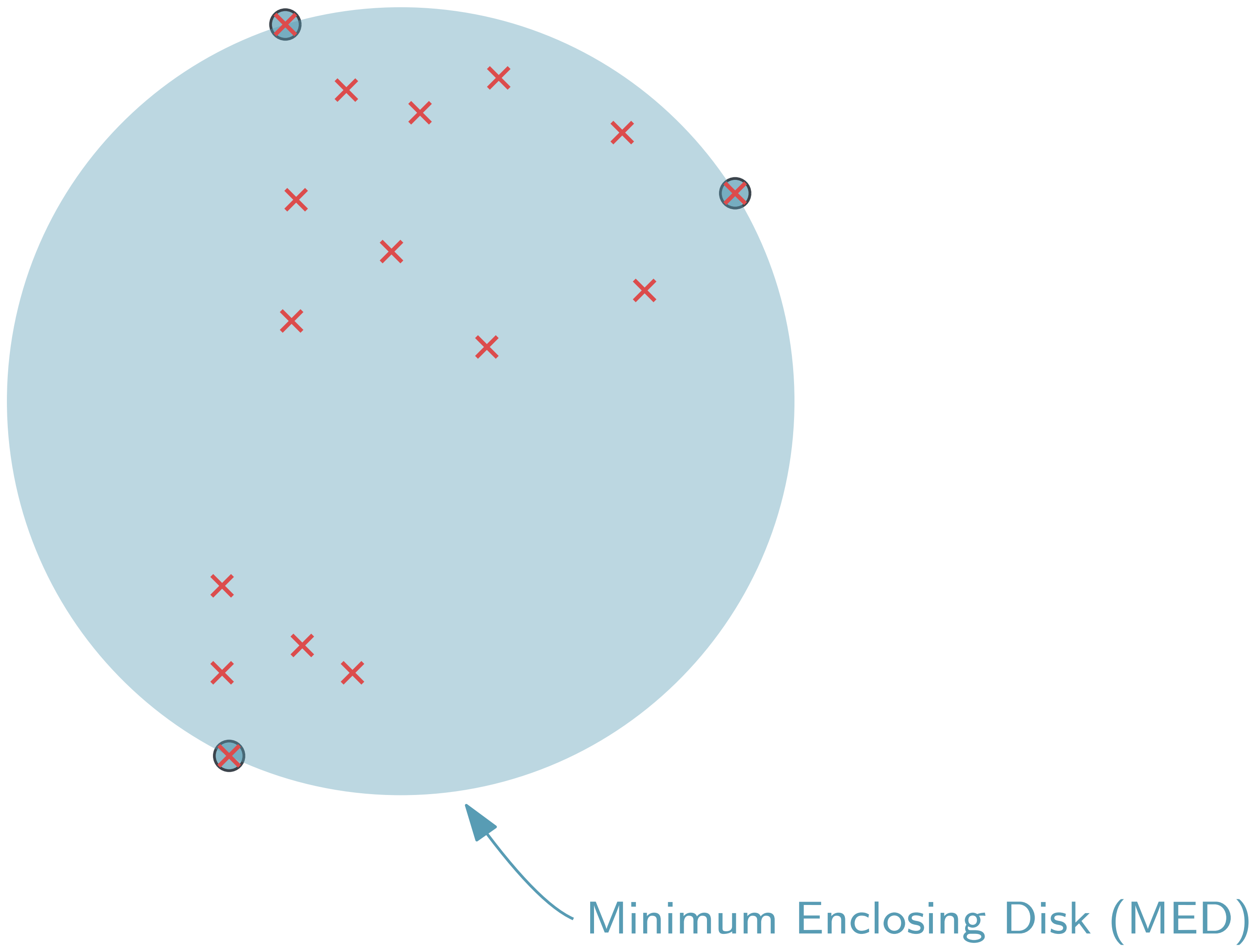
Structural Lemmas

Lemmas

Lemmas



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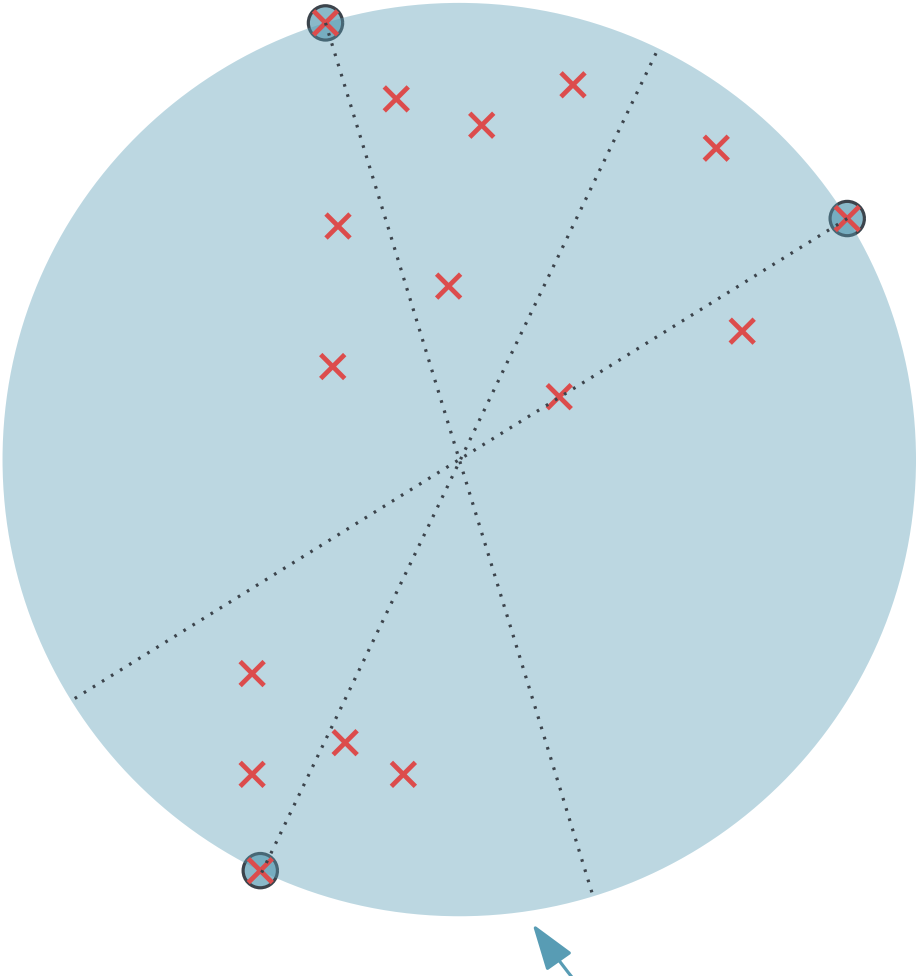
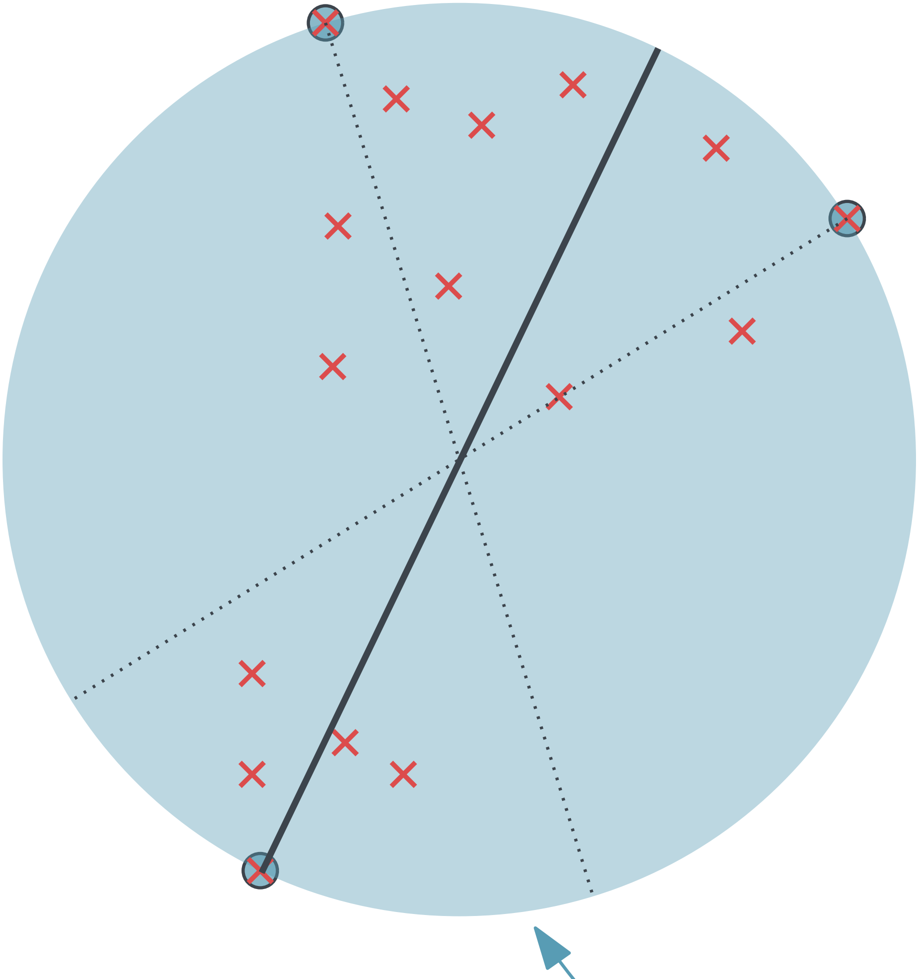


Minimum Enclosing Disk (MED)

Lemmas



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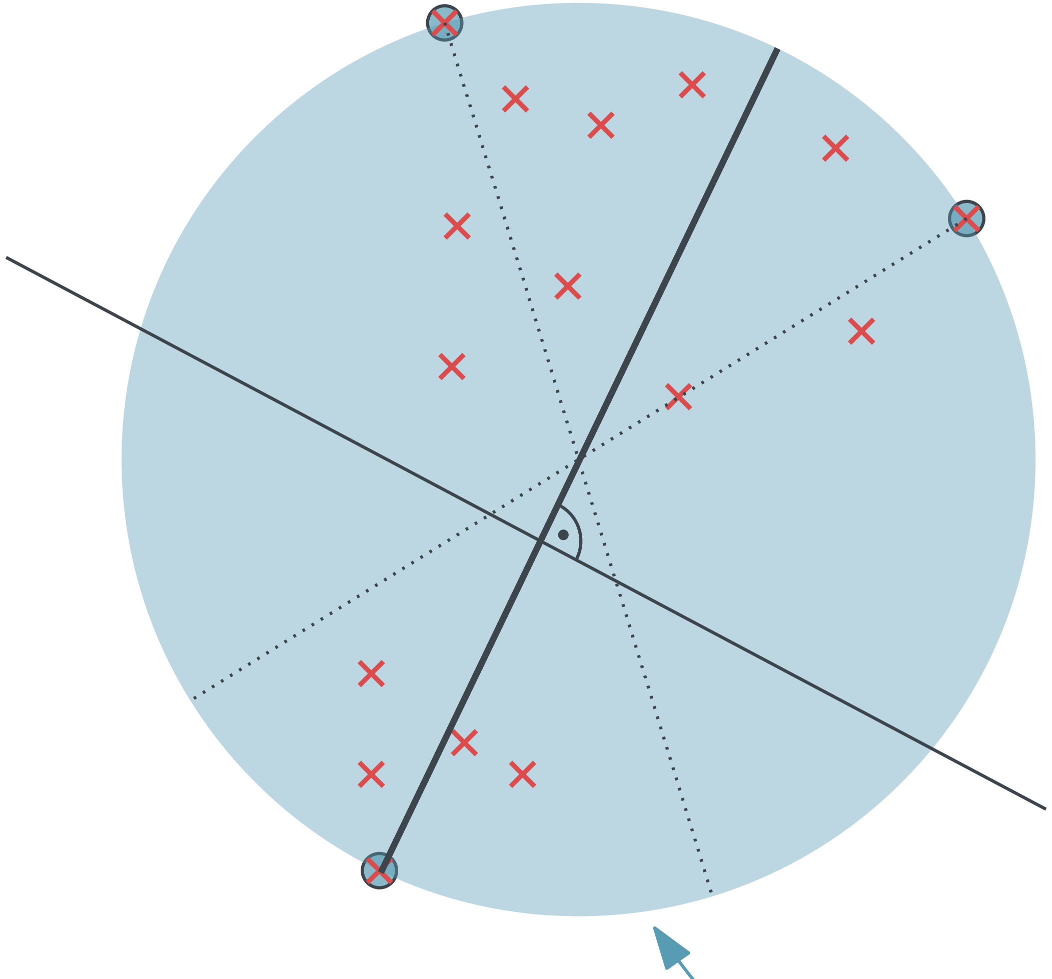

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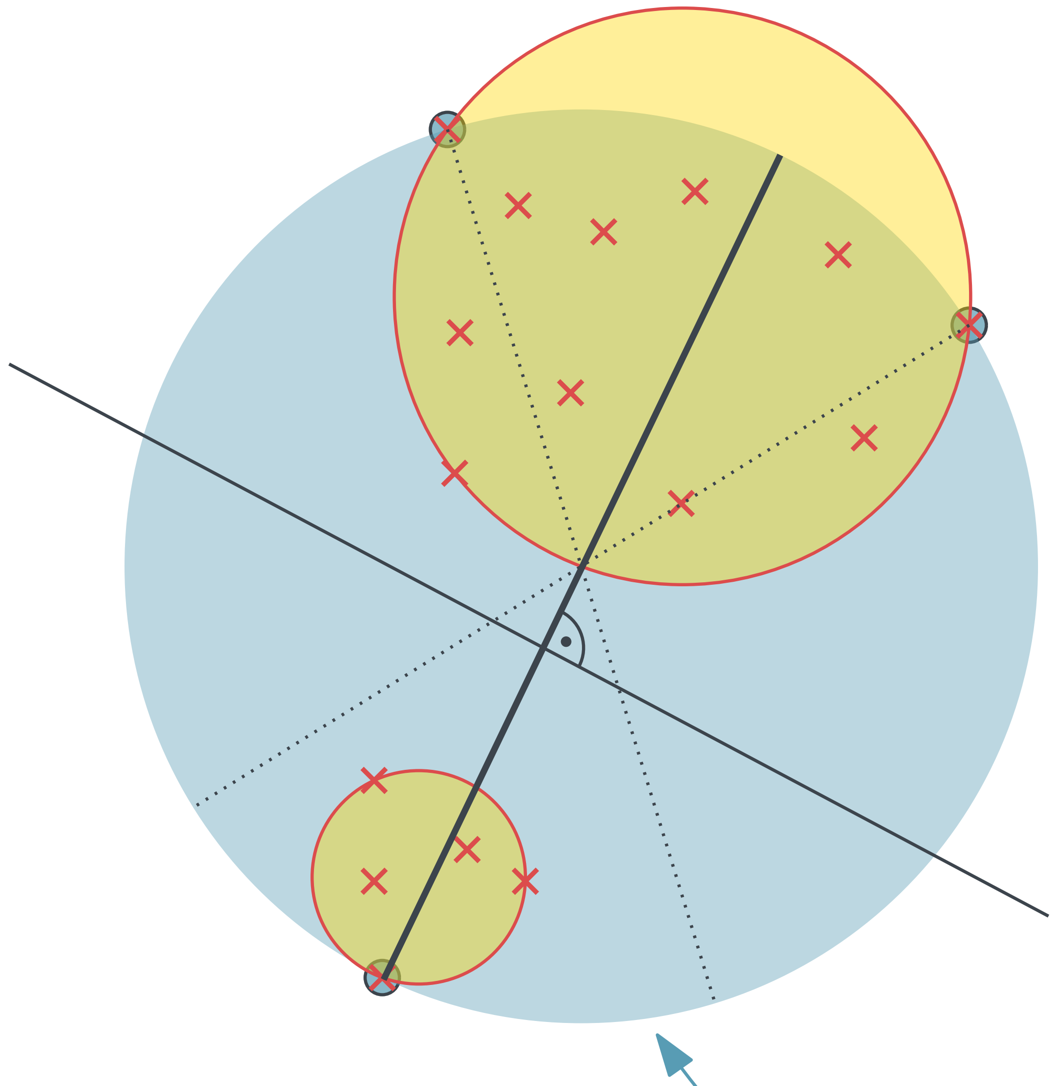

Minimum Enclosing Disk (MED)

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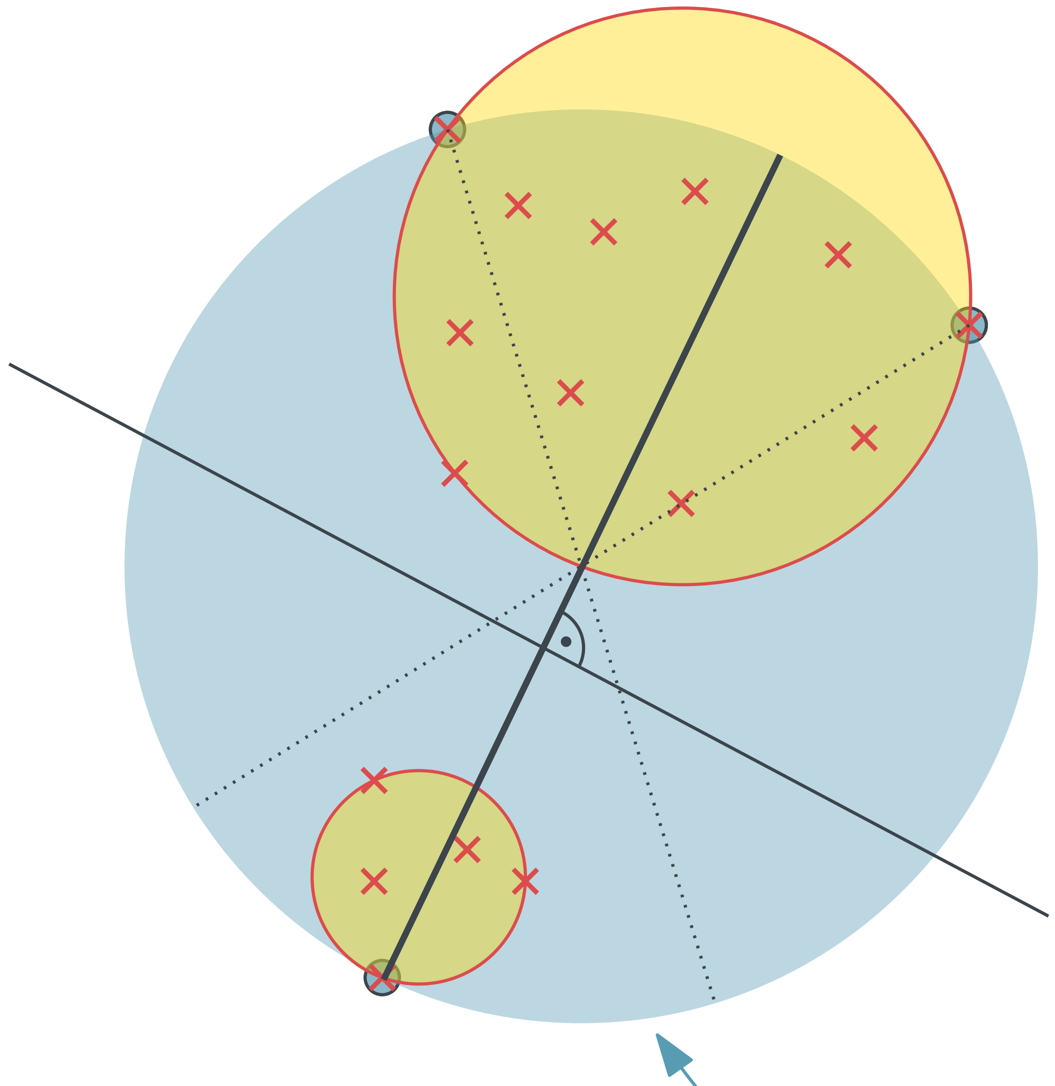
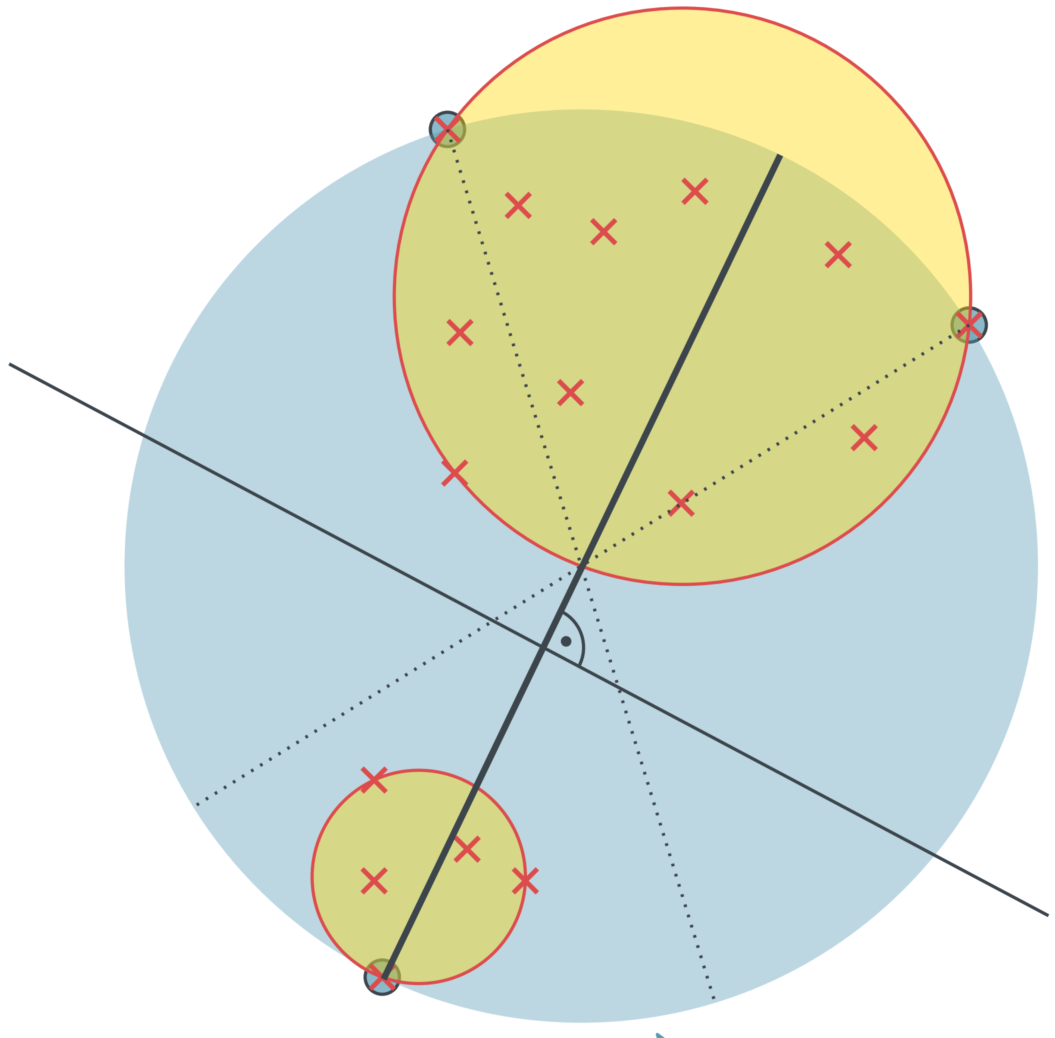


Minimum Enclosing Disk (MED)

Lemma

Given a 2-MinSumRadius instance, an orthogonal (——) to one of the diagonals (⋯⋯) separates the clusters in an optimal solution.

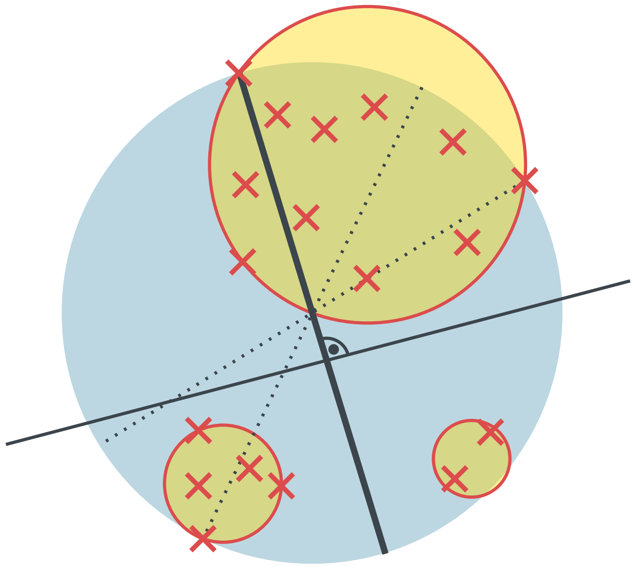
Lemmas



Minimum Enclosing Disk (MED)

Lemma

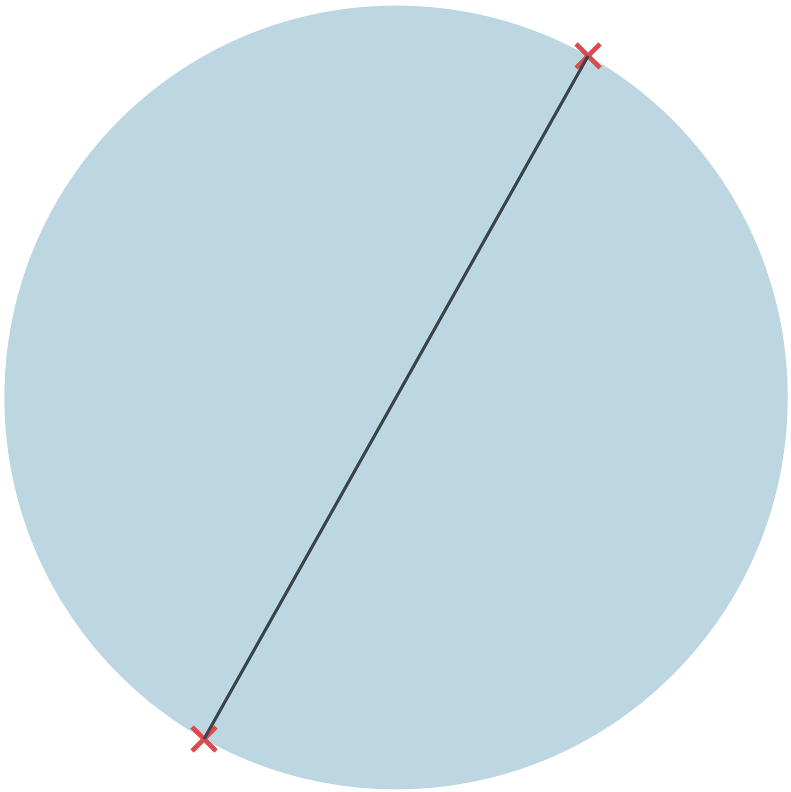
Given a ~~2~~ 3-MinSumRadius instance, an orthogonal (——) to one of the diagonals (⋯⋯) separates ~~the clusters~~ one cluster from the other two in an optimal solution.



Proof Sketch

$$k = 2$$

Case 1: two points define MED



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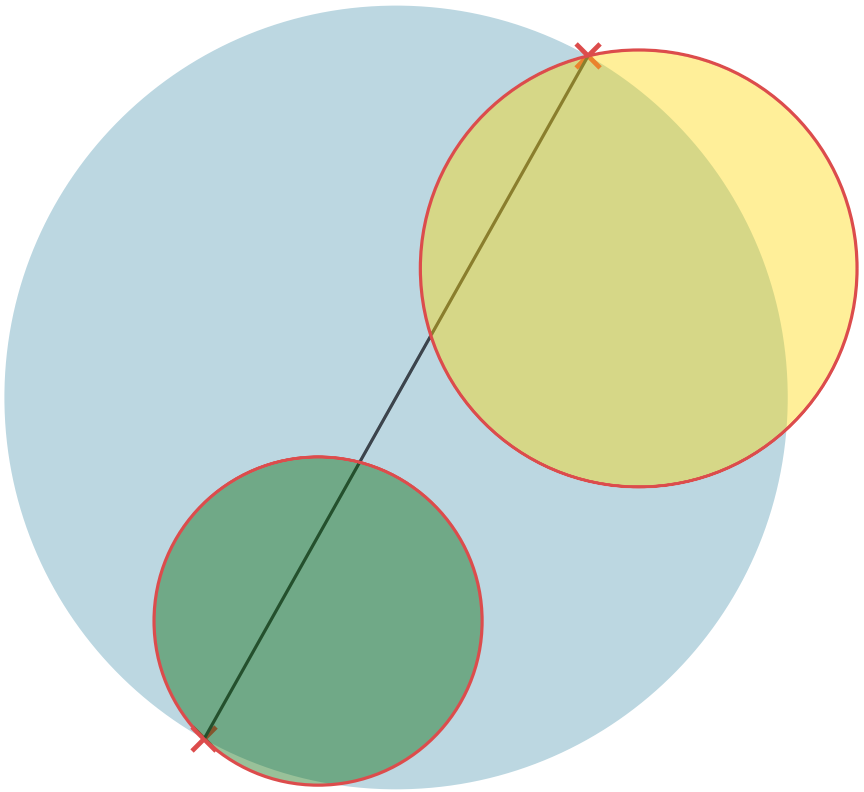

$\rightarrow$ assume points are in different clusters

Proof Sketch

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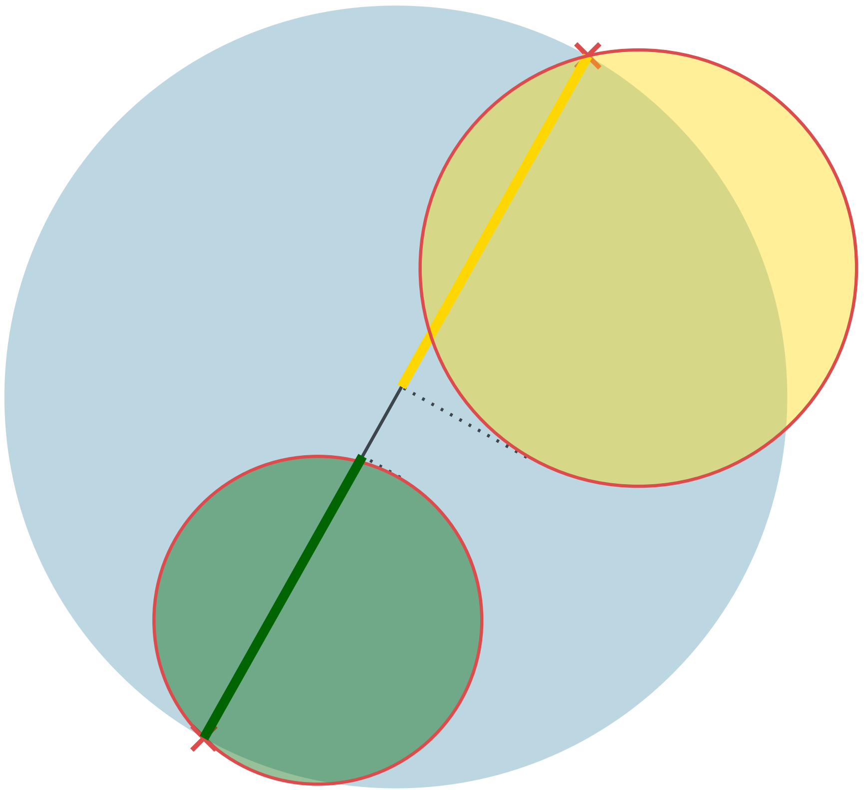

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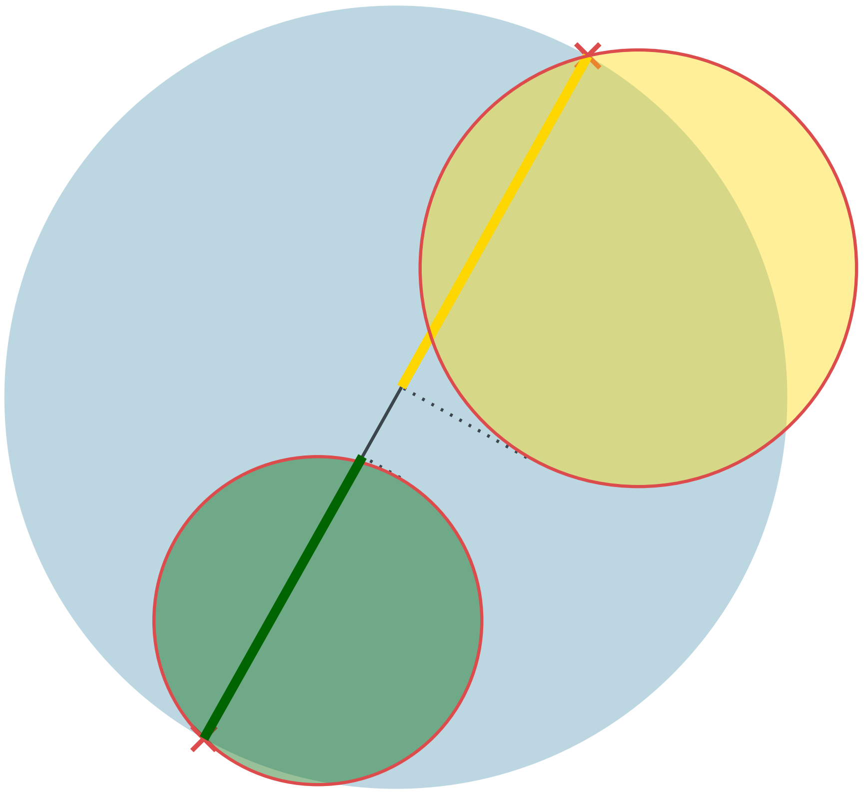

/ and / intersect

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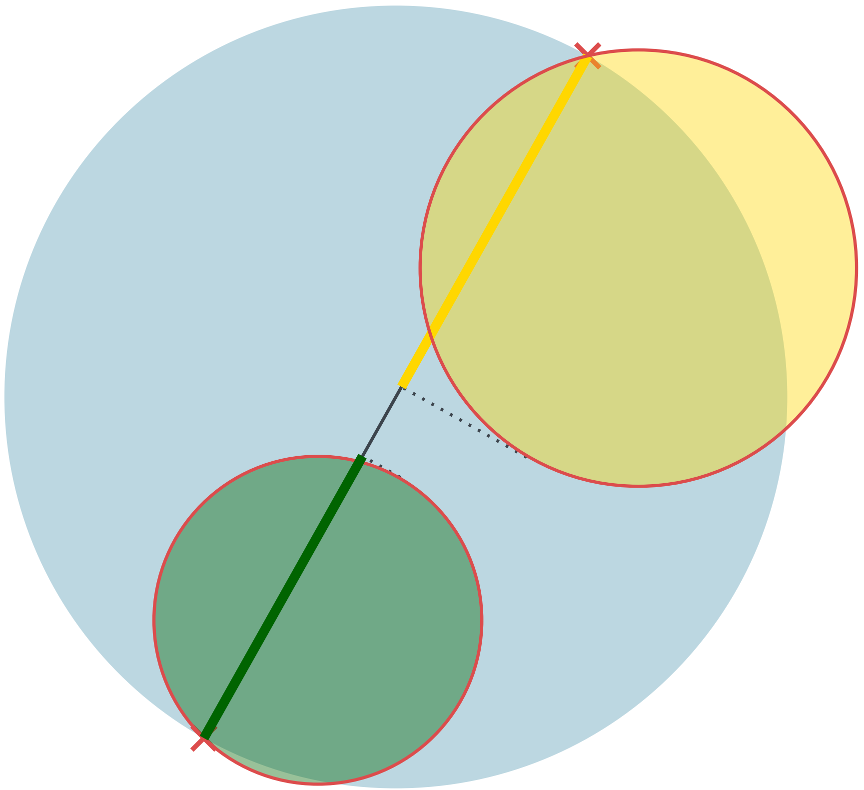

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$\implies$ len(—) $+$ len(—) $>$ len(—) $\lightning$

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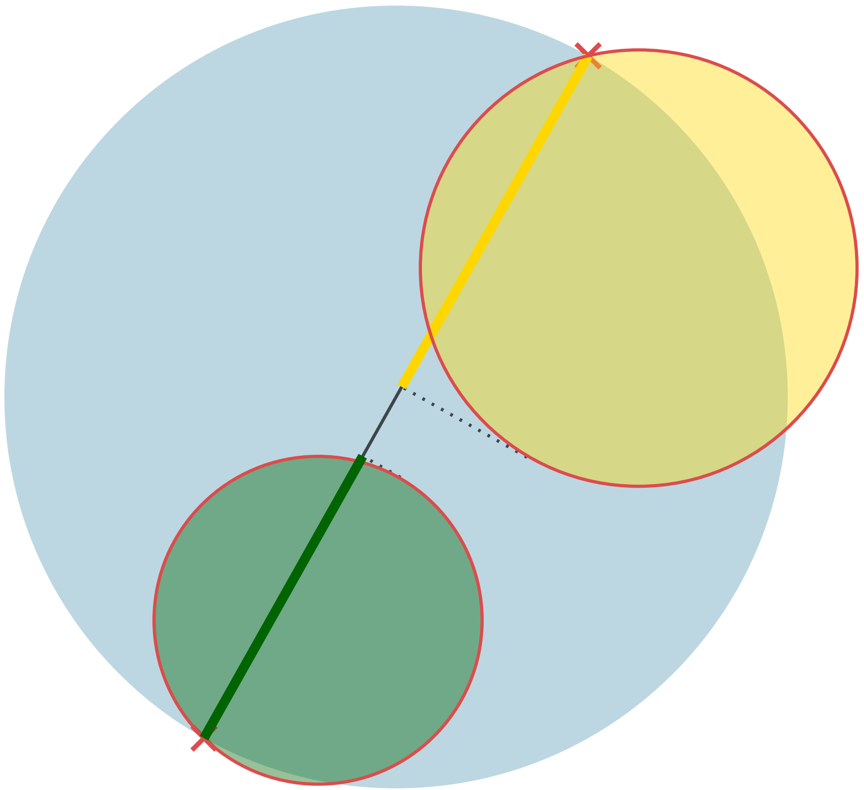
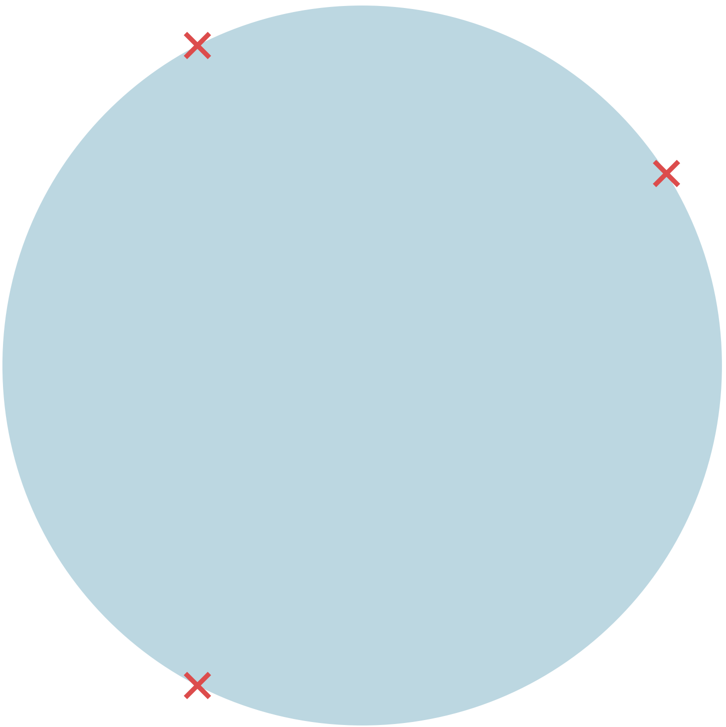


/ and / intersect

$\Longrightarrow$ len(—) + len(—) > len(—) ↯

Case 2: three points define MED



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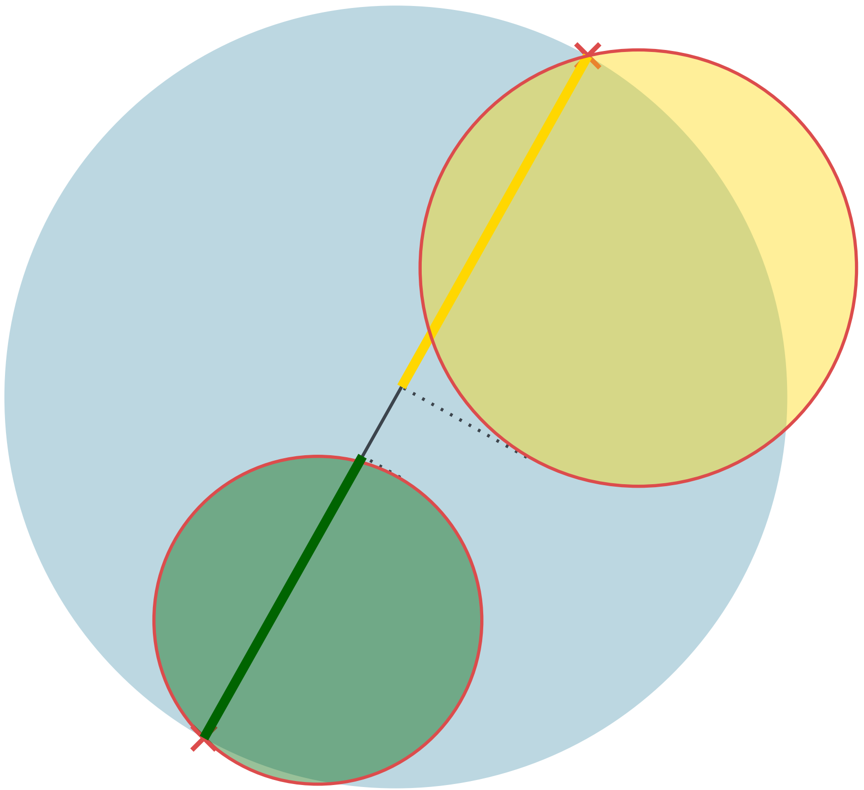

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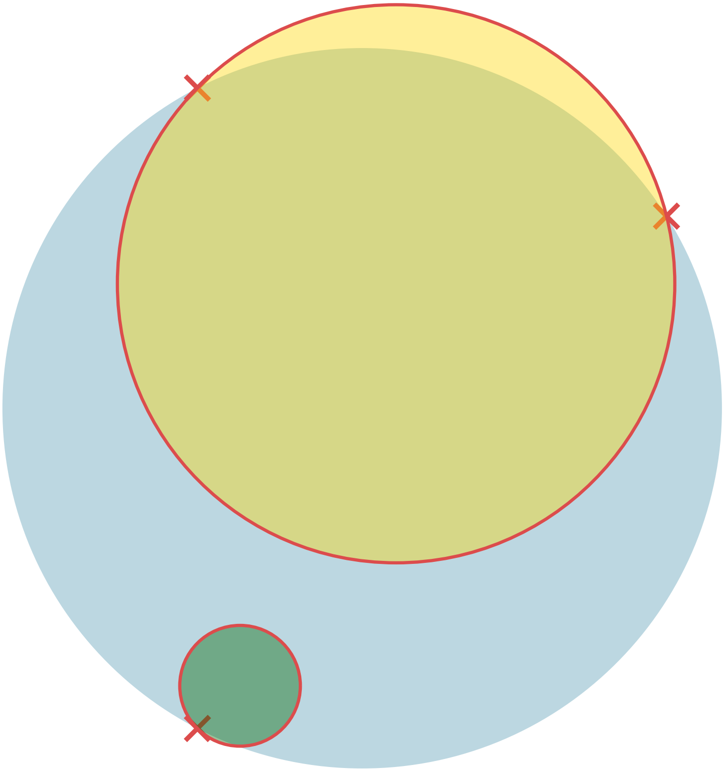

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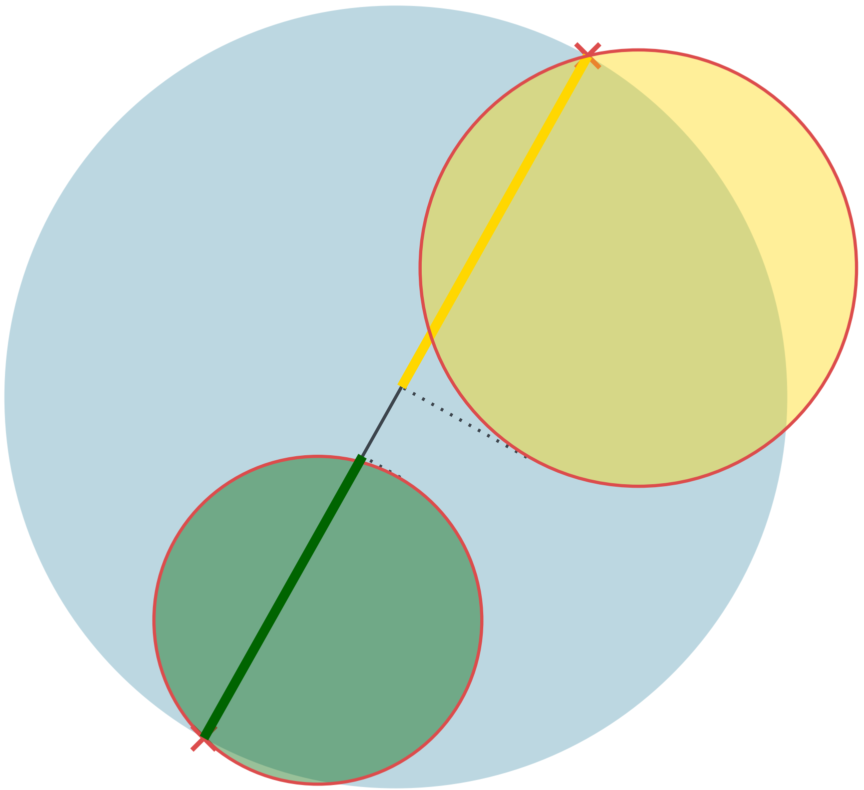

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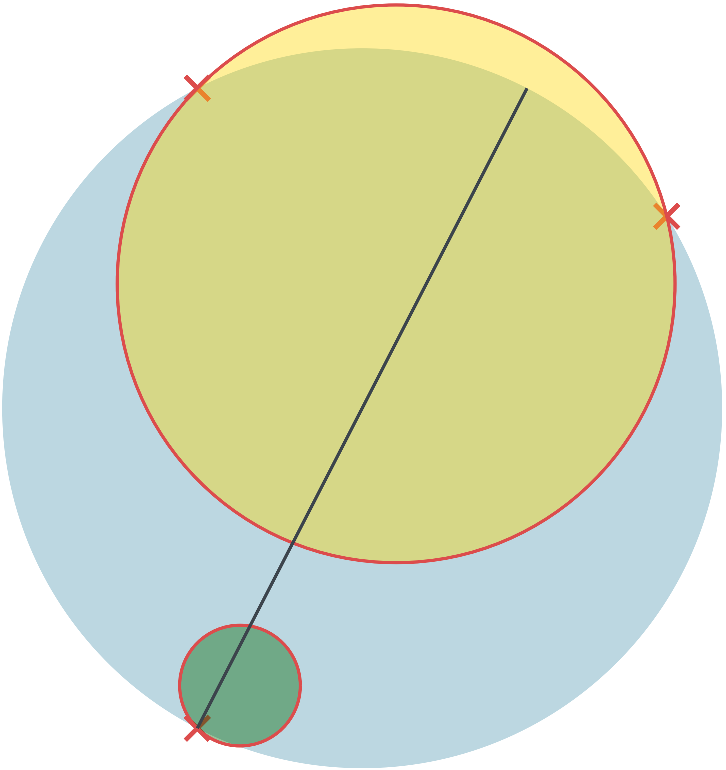

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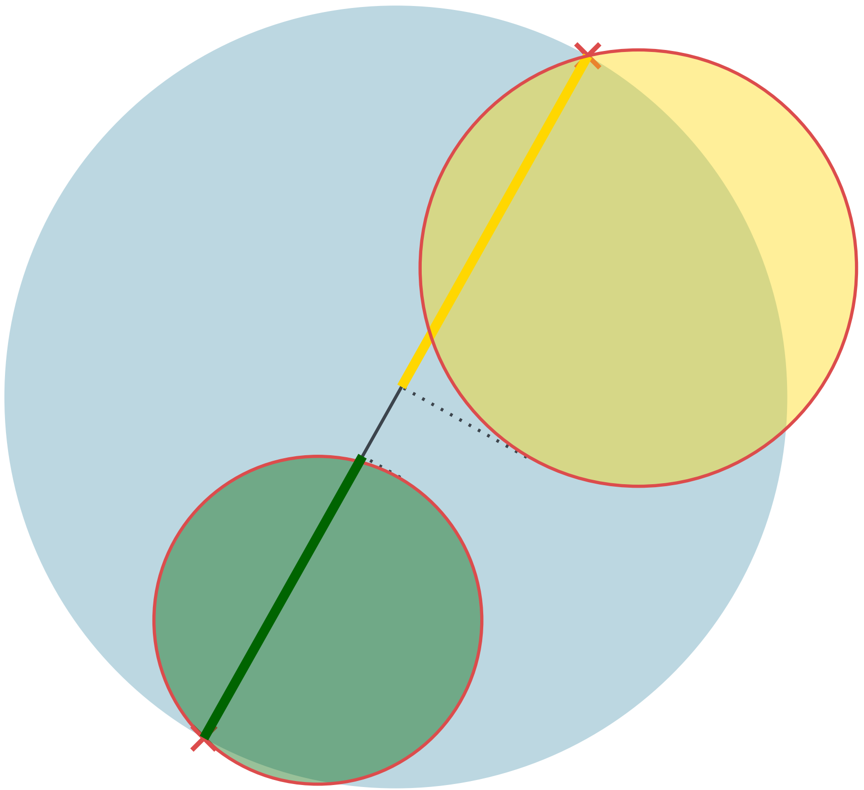

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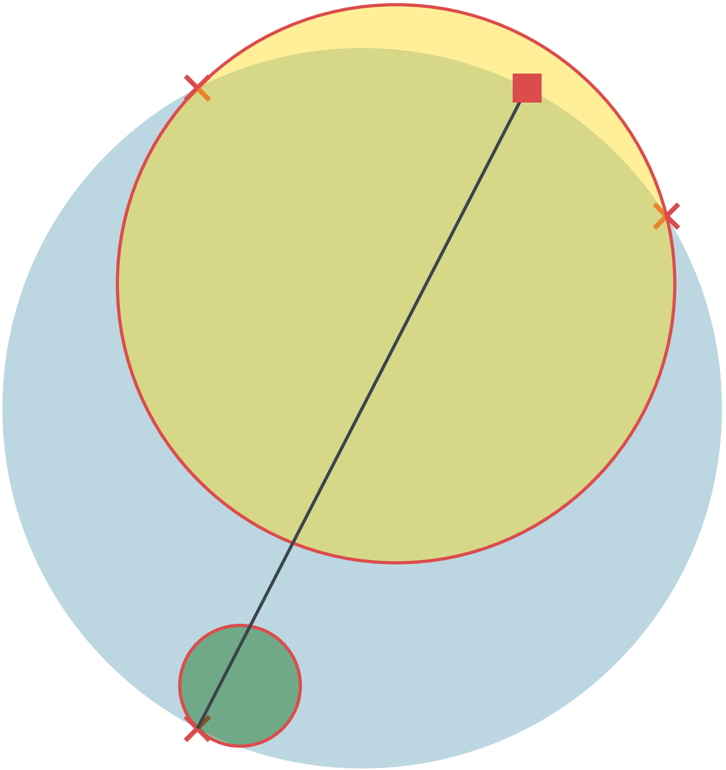

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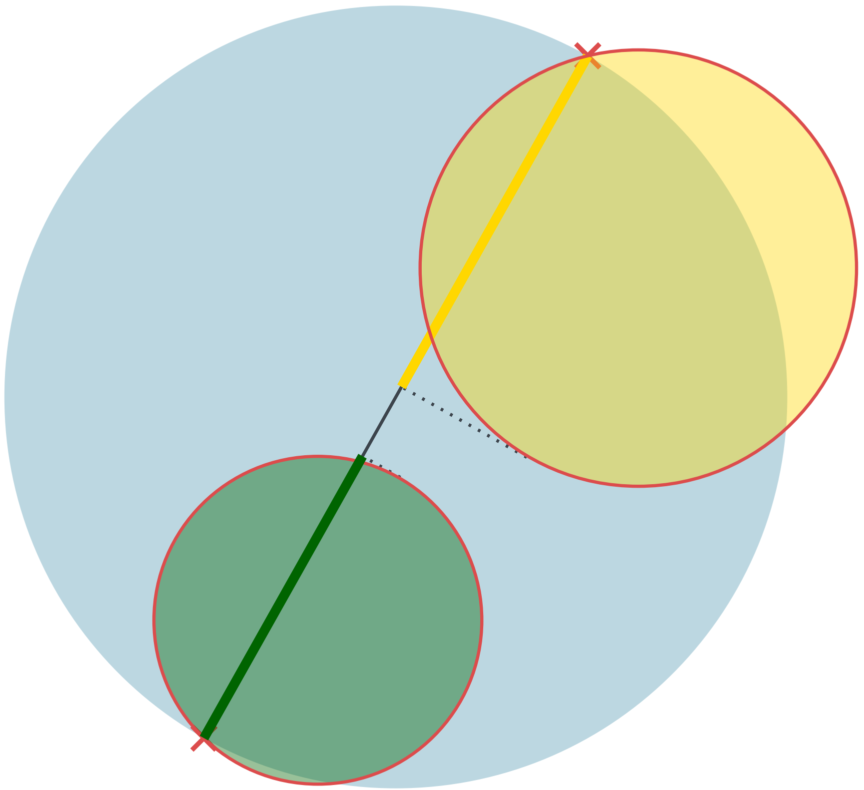

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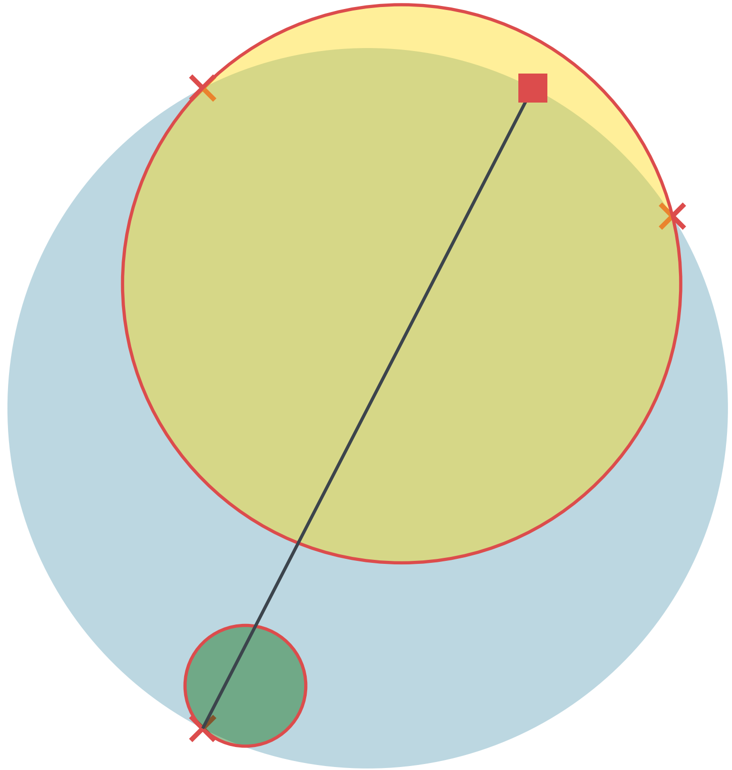


■ ∈ ○ $\implies$ reduction to **Case 1**

Algorithms

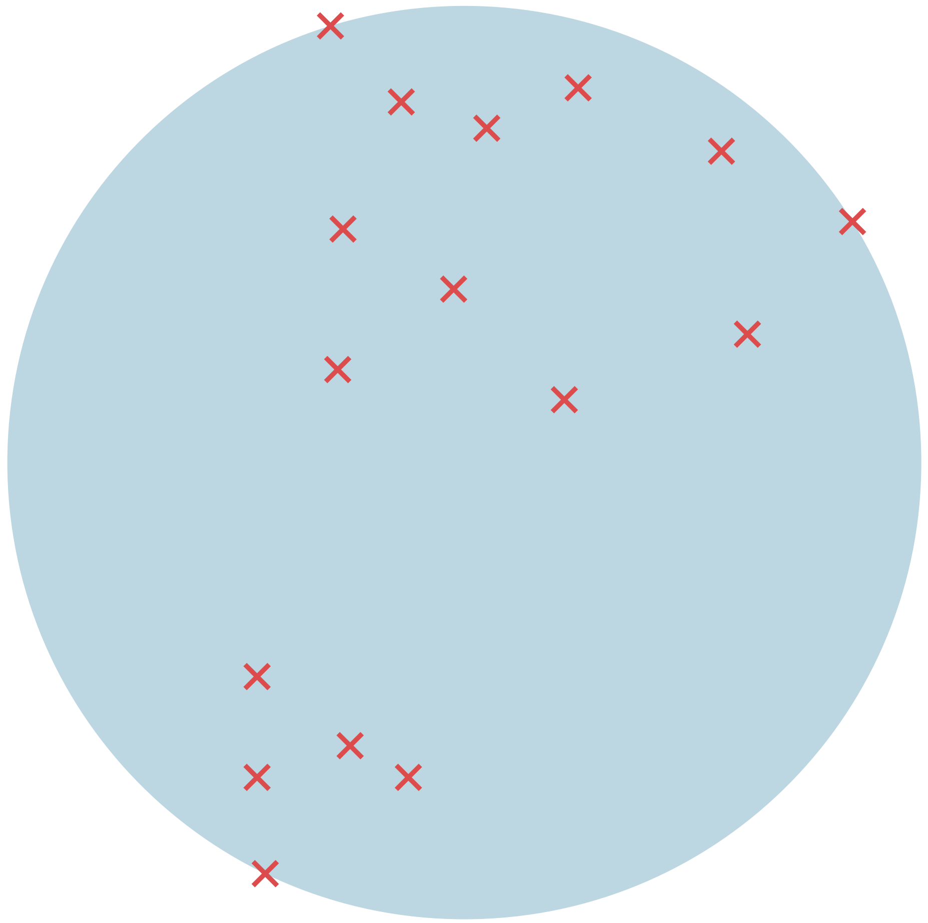
Algorithm

$k = 2$

Running time:

Algorithm

$k = 2$



Running time:

$O(n)$

Algorithm

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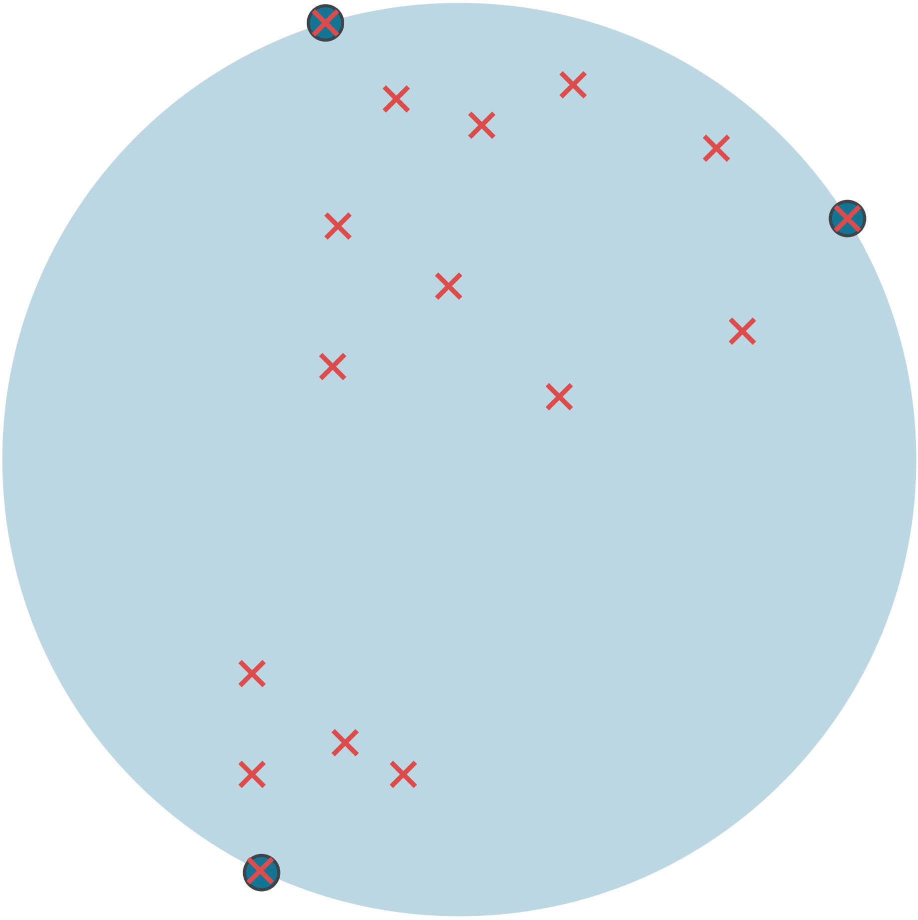

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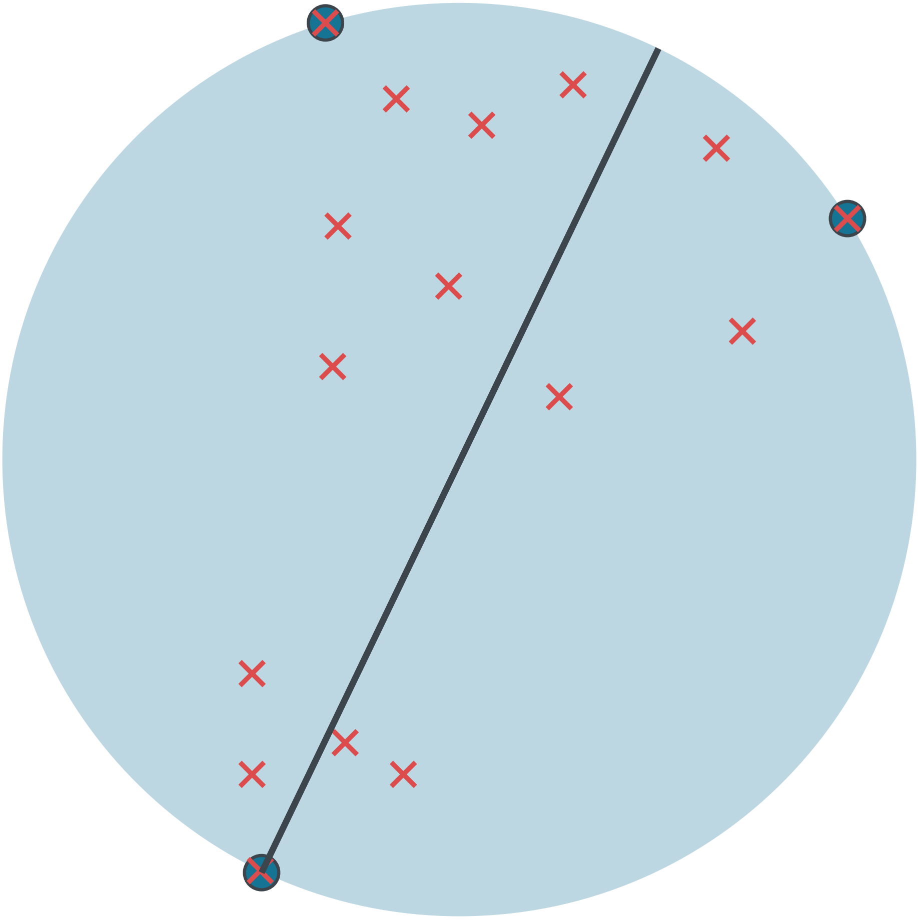
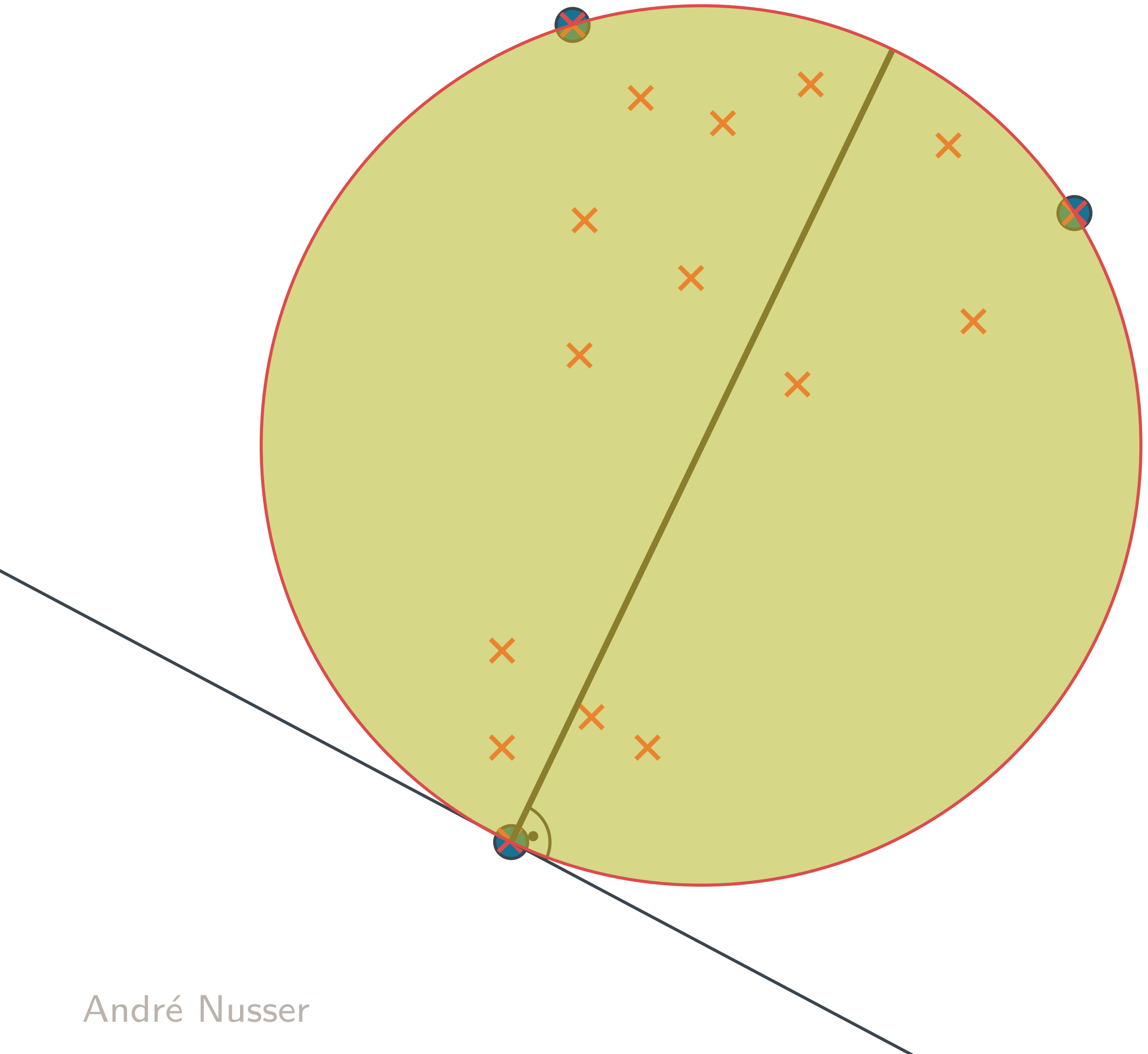


Running time:

$O(n) + 3 \cdot$

Algorithm

$$k = 2$$



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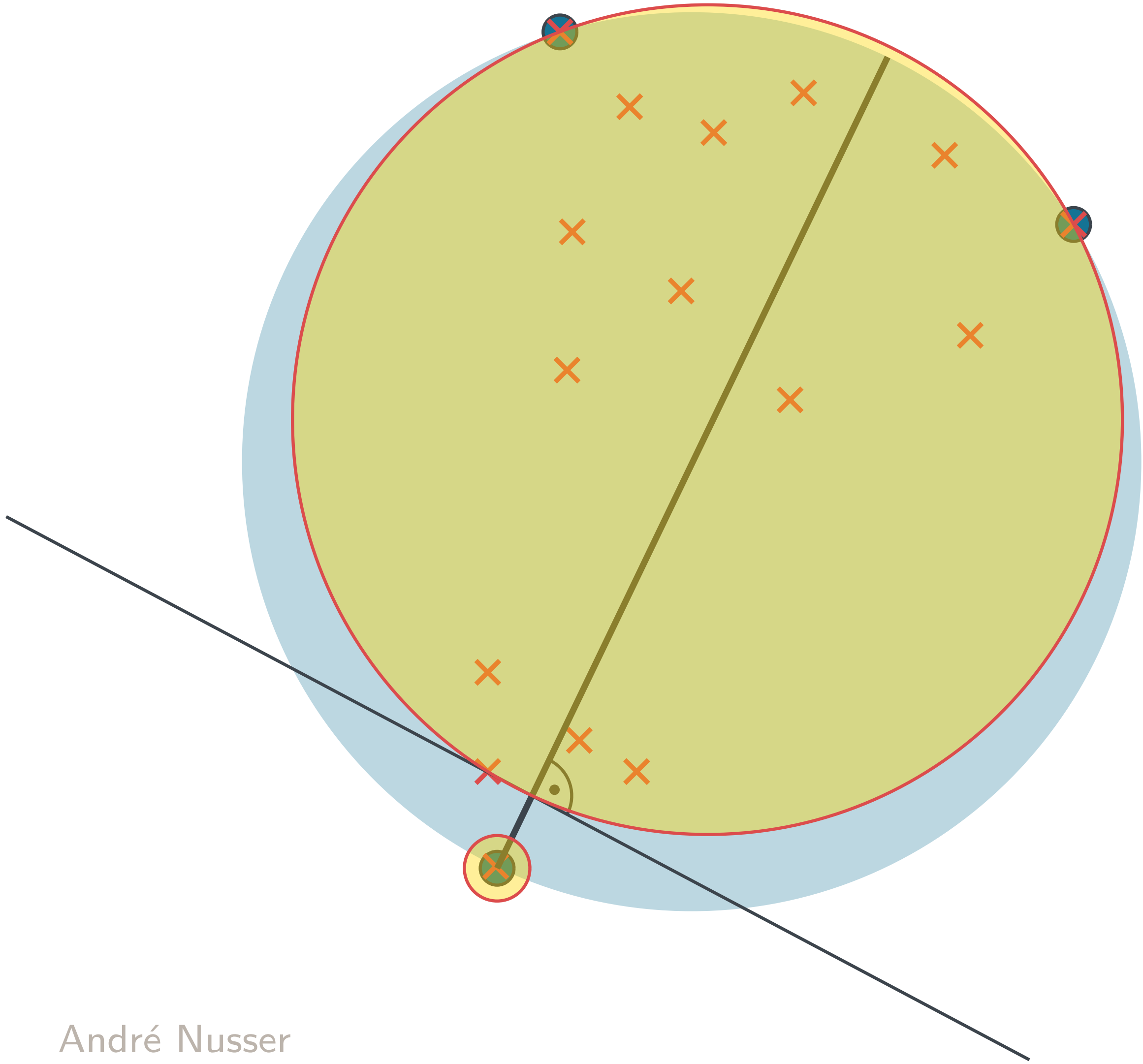

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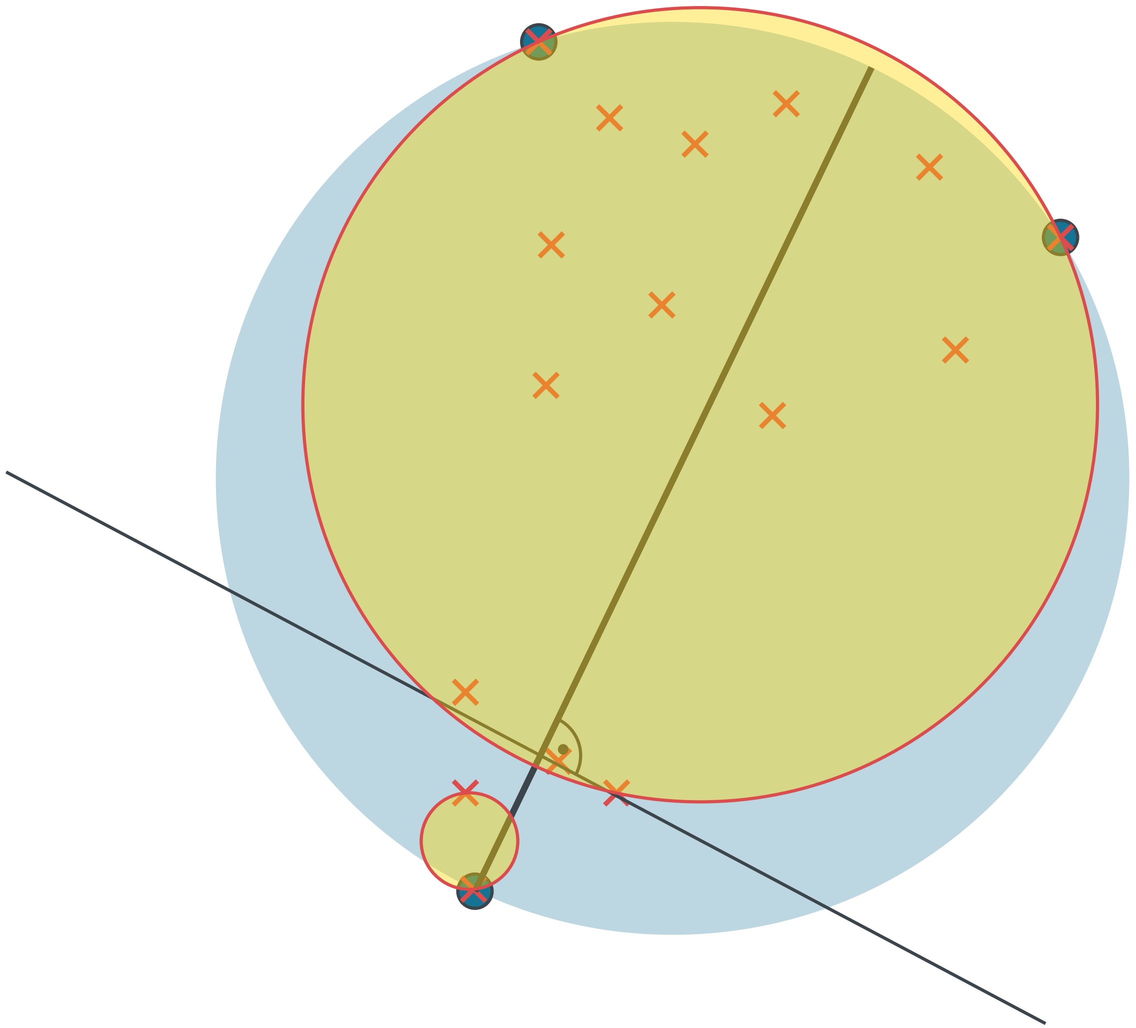

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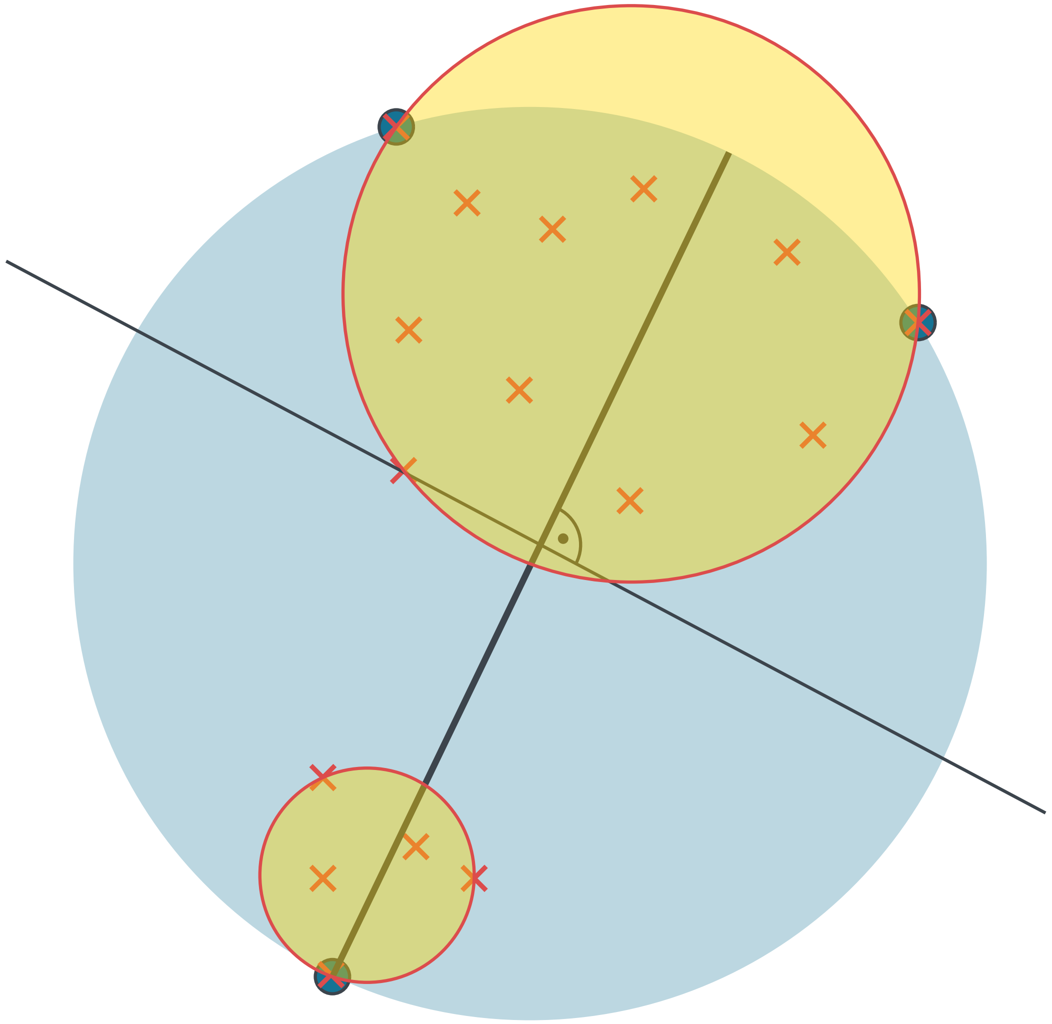
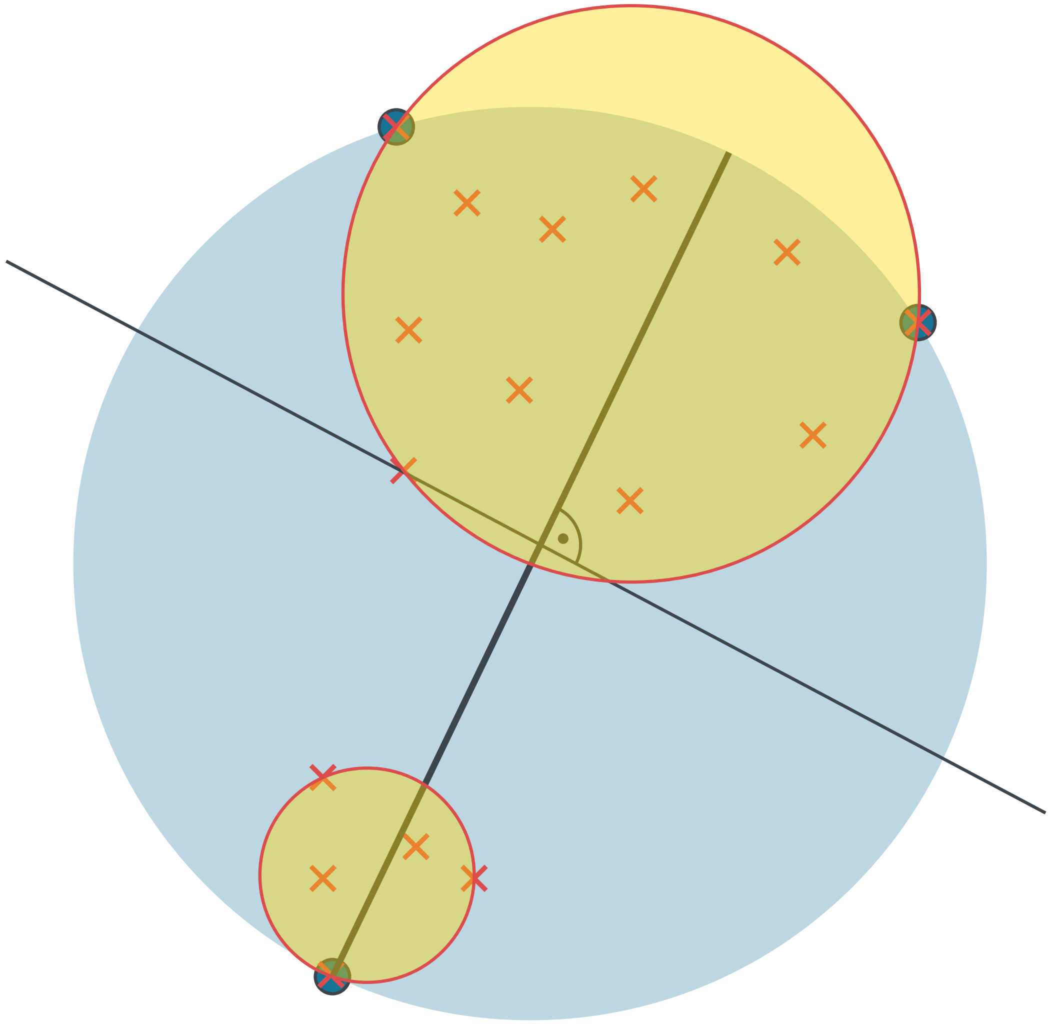


Running time:

$O(n) \; + \; 3 \cdot n \cdot$

Algorithm

$k = 2$



*

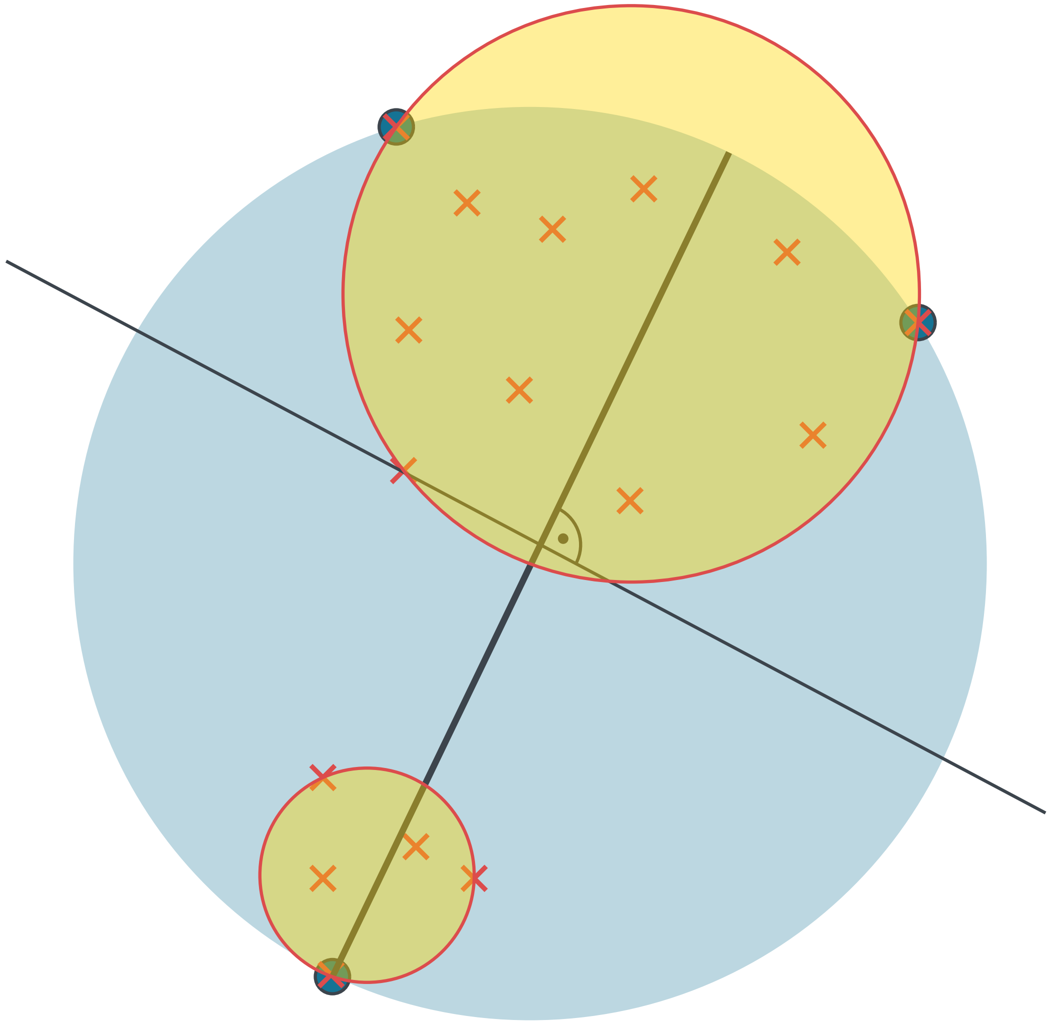
Running time:

$O(n) + 3 \cdot n \cdot O(\log^2 n)$

* We omit $\log \log$ factors here

Algorithm

$k = 2$



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Running time:

$$O(n) \; + \; 3 \cdot n \cdot O(\log^2 n)$$
$$=$$
$$O(n \cdot \log^2 n)$$

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Algorithm

$$k = 3$$

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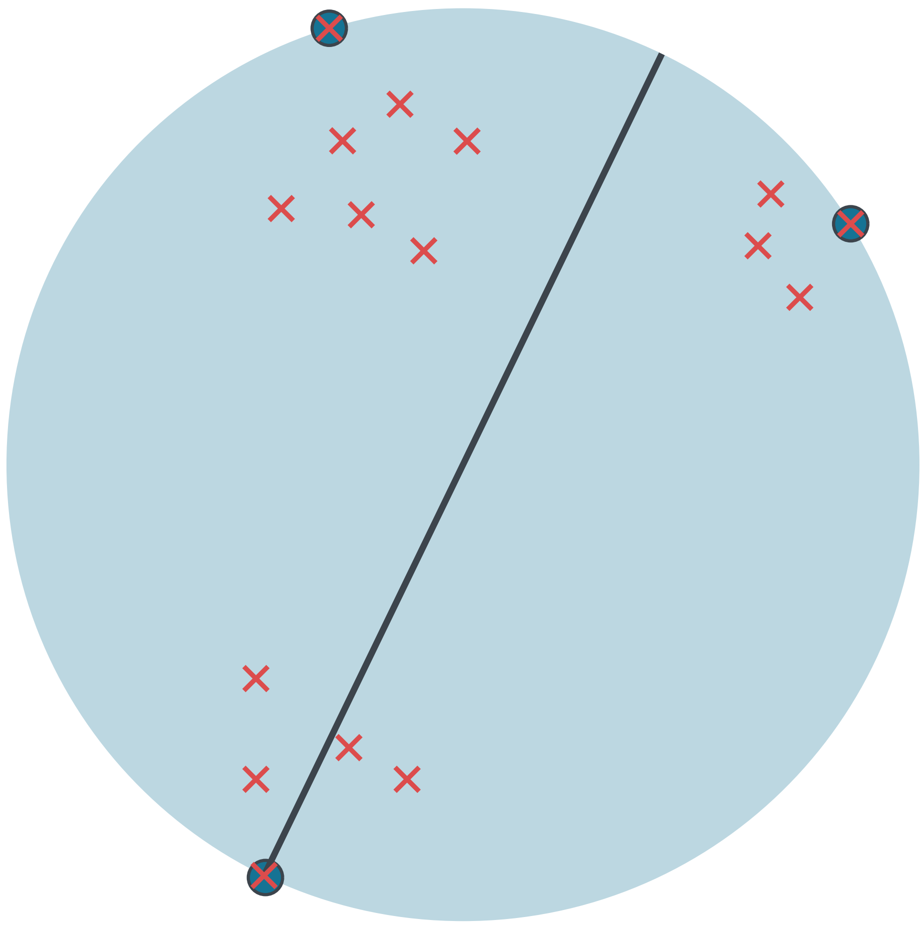

Running time:

$O(n) + 3 \cdot n \cdot$

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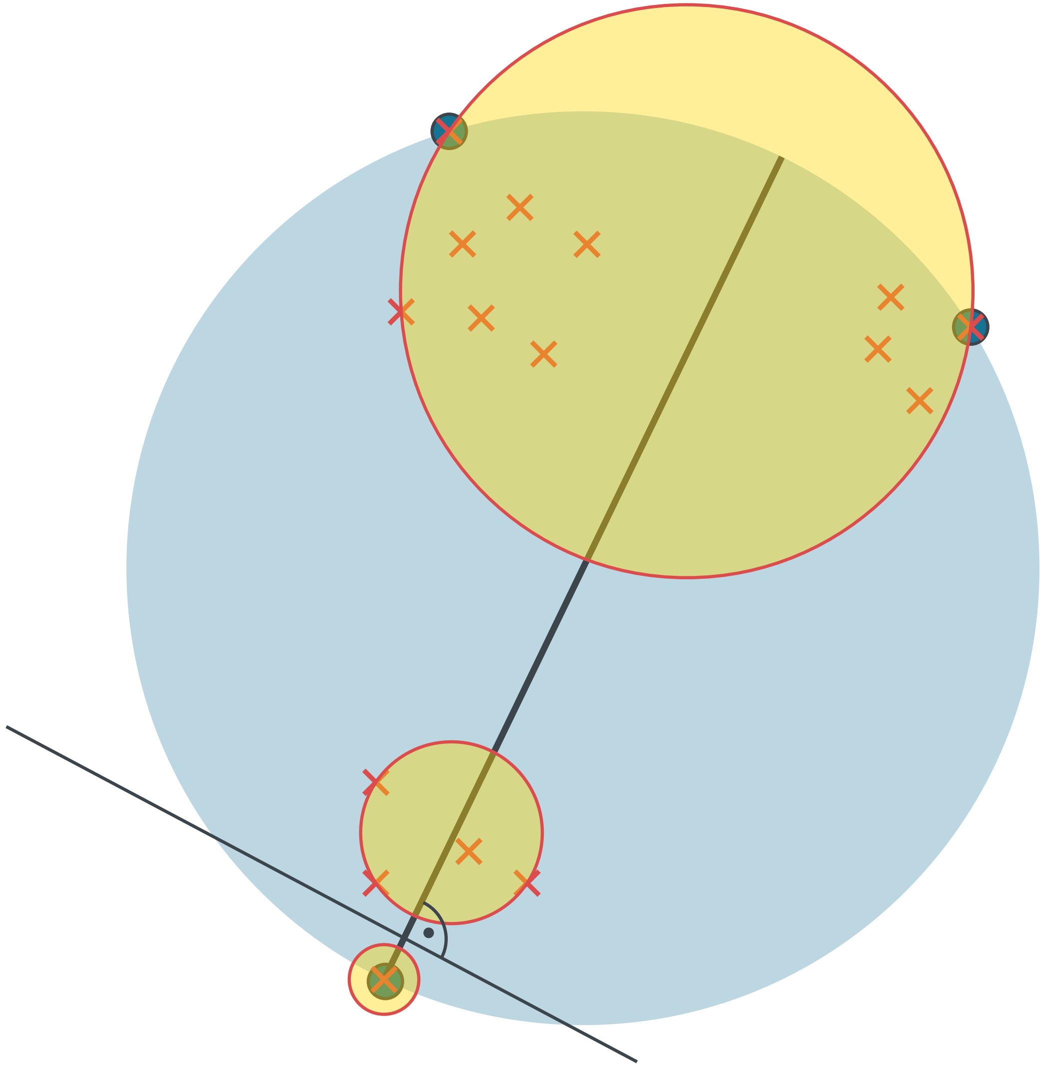

*

Running time:

$$O(n) \ + \ 3 \cdot n \cdot (O(n) + O(n \log^2 n))$$

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Algorithm

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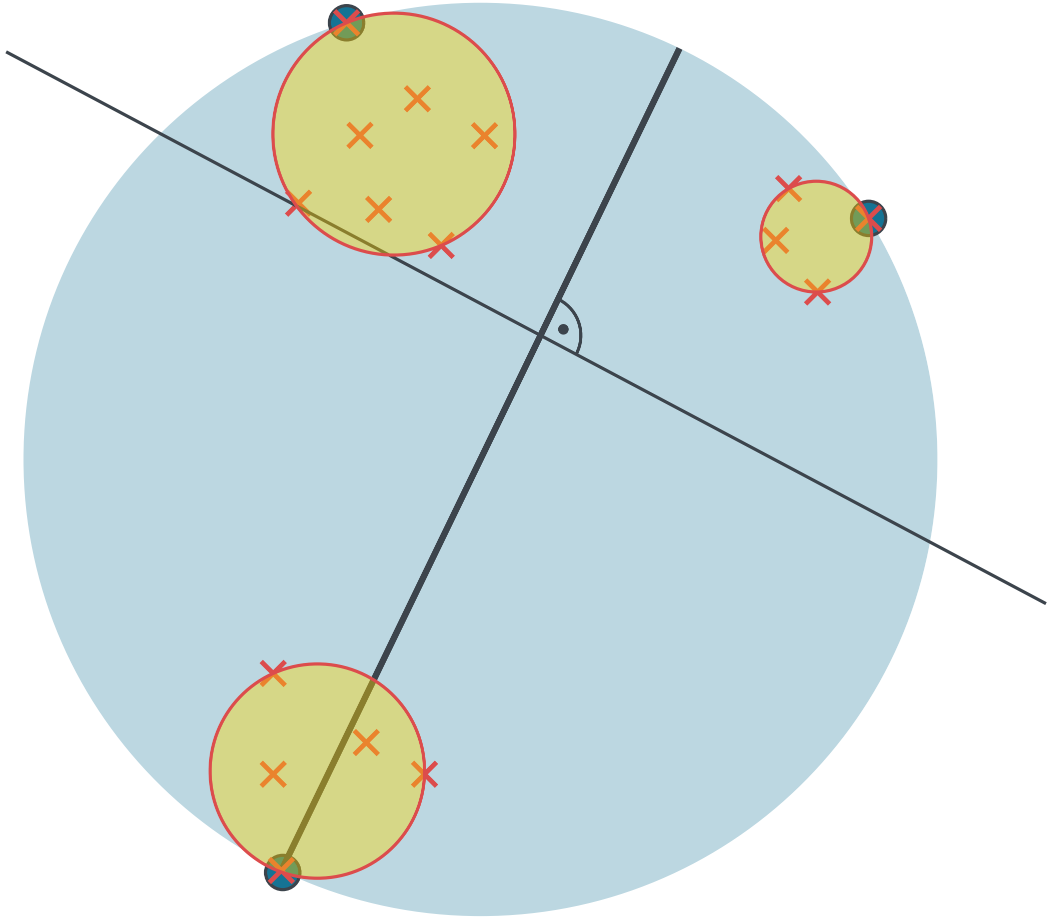

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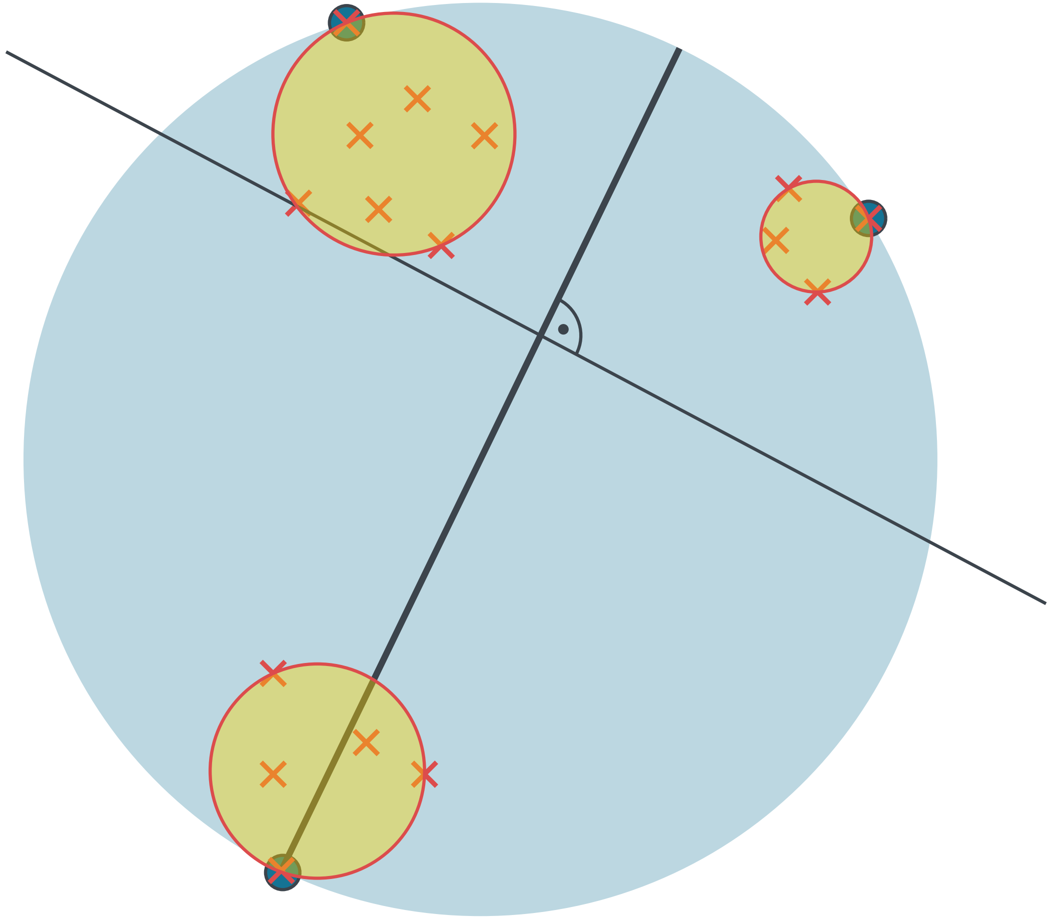

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$$=$$
$$O(n^2 \cdot \log^2 n)$$

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Thank You!

Questions?

Unofficial Paper Soundtrack:

Full Circle
by **Half Moon Run**