Extending simple monotone drawings

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Arrangements of pseudosegments

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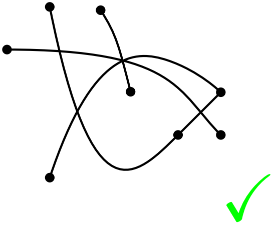
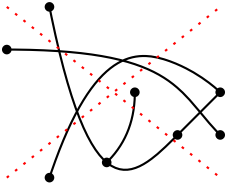
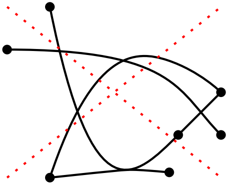
- An arrangement of 1-strings is also called an *arrangement of pseudosegments*, and each curve in the arrangement is called a *pseudosegment*.







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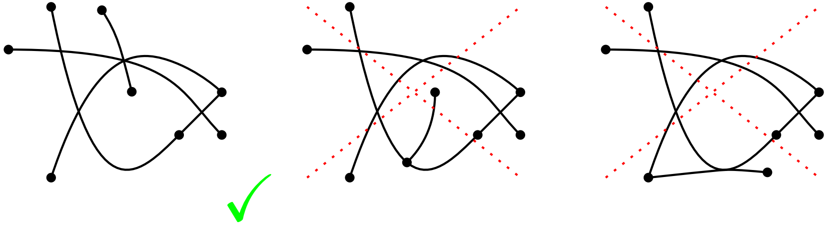


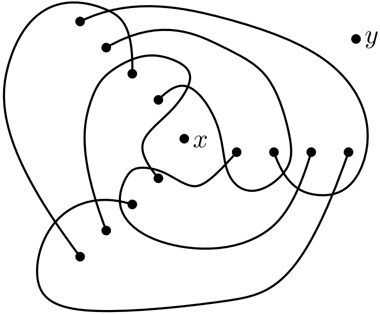
- A simple curve γ in the plane is *x-monotone*, shortly *monotone*, if γ intersects every line parallel to the *y*-axis at most once.

Extendability

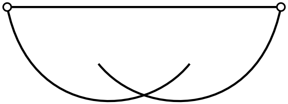
Question

Is every arrangement of monotone pseudosegments extendable? I.e. can we add a new monotone pseudosegment between any (reasonable) pair of points into it?

- Generals arrangements are not extendable



- Arrangements of pseudolines are extendable (Levi's Lemma)
- But pseudosegments in an arrangement of pseudosegments cannot be in general "lengthened" into pseudolines.



Our results

Theorem

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Every arrangement of monotone pseudosegments in the plane is extendable.

- The proof can be easily turned into an algorithm.

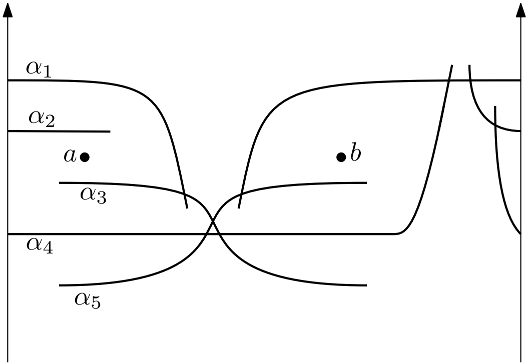
- A drawing of a graph in the plane is *simple* if every pair of edges has at most one common point, either a common endpoint or a proper crossing.
- A drawing of a graph is *monotone* if every edge is drawn as a monotone curve and no two vertices share the same x-coordinate.

Corollary

Every simple monotone drawing of a graph in the plane can be extended to a simple monotone drawing of the complete graph with the same set of vertices.

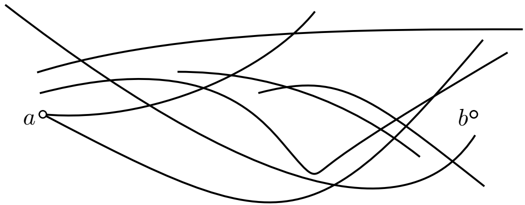
Monotone arrangements on a cyllinder

- In the full version of this article we also study the extendability problem for cylindrically monotone arrangements. We show that extending an arrangement of cylindrically monotone pseudosegments is not always possible and, in fact, the corresponding decision problem is NP-hard.



Main ideas from the proof

- We want to add new monotone psedosegment between two points *a* and *b*.
- We can restrict ourselves to the vertical strip between *a* and *b*.



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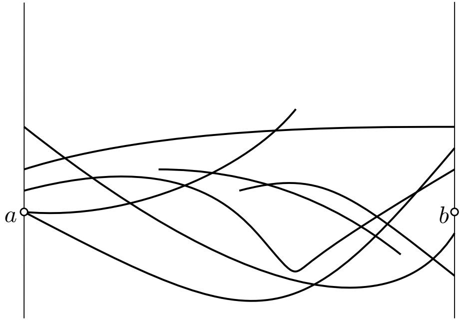

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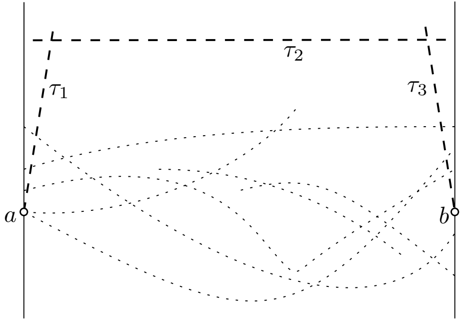
- We proceed inductively and maintain a connected curve connecting *a* and *b* that is formed by a lower envelope of a subset $\mathcal{U}$ of pseudosegments.
- We start with a lower envelope of auxiliary pseudosegments that connect *a* and *b* from above.
- In each step we select a pseudosegment γ intersecting our lower envelope twice and add it to $\mathcal{U}$. In other words we reroute the lower envelope along γ, so that γ is not below the lower envelope.
- At the end we draw the new pseudosegment along the lower envelope.



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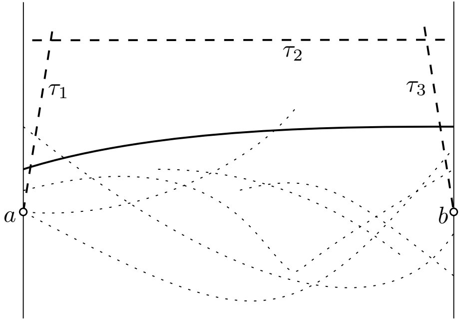

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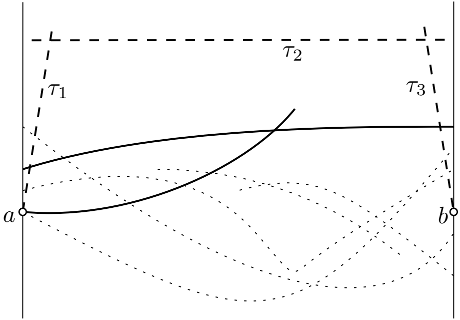

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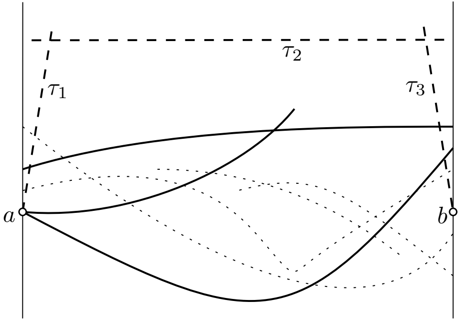

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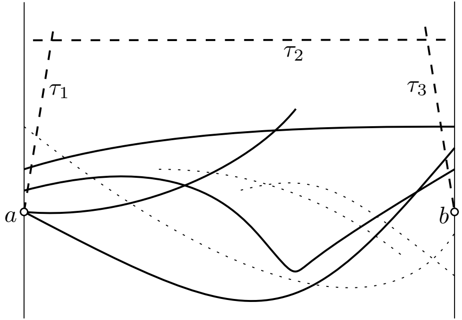
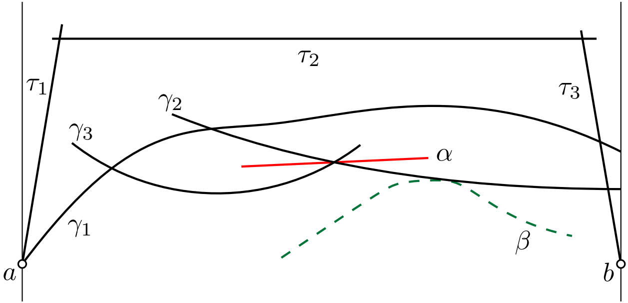


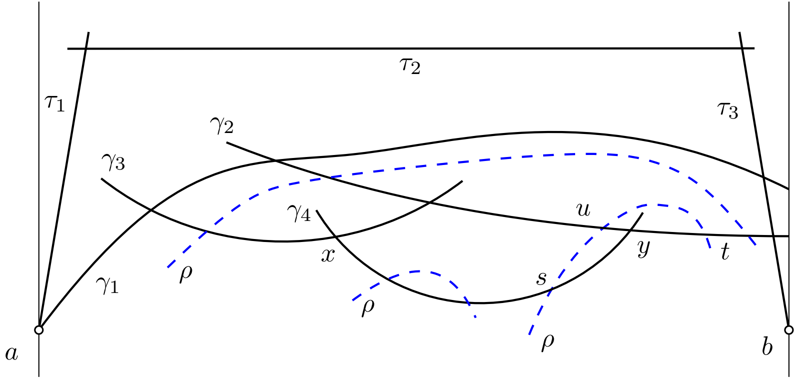
What can go wrong during the procedure?

- Some remaining pseudosegment touches the lower envelope from below with its inner point. $\rightarrow$ By definition it is not possible.



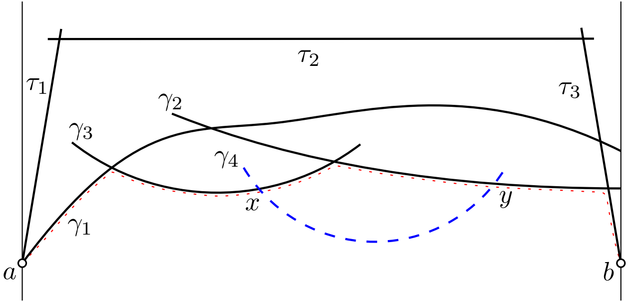
What can go wrong during the procedure?

- Some remaining pseudosegment intersects the lower envelope three or more times. → This case reduces to the next one.
- Some remaining pseudosegment intersect the lower envelope first from below and then from above. → By induction it is not possible.



So nothing can go wrong

- In all other cases after each step the lower envelope remains a connected curve connecting *a* with *b* and not containing any other endpoints of pseudosegments even after this addition.



Thank you for your attention!