The k-Transmitter Watchman Route Problem is NP-Complete Even in Histograms and Star-Shaped Polygons

Anna Brötzner, Bengt J. Nilsson, Christiane Schmidt

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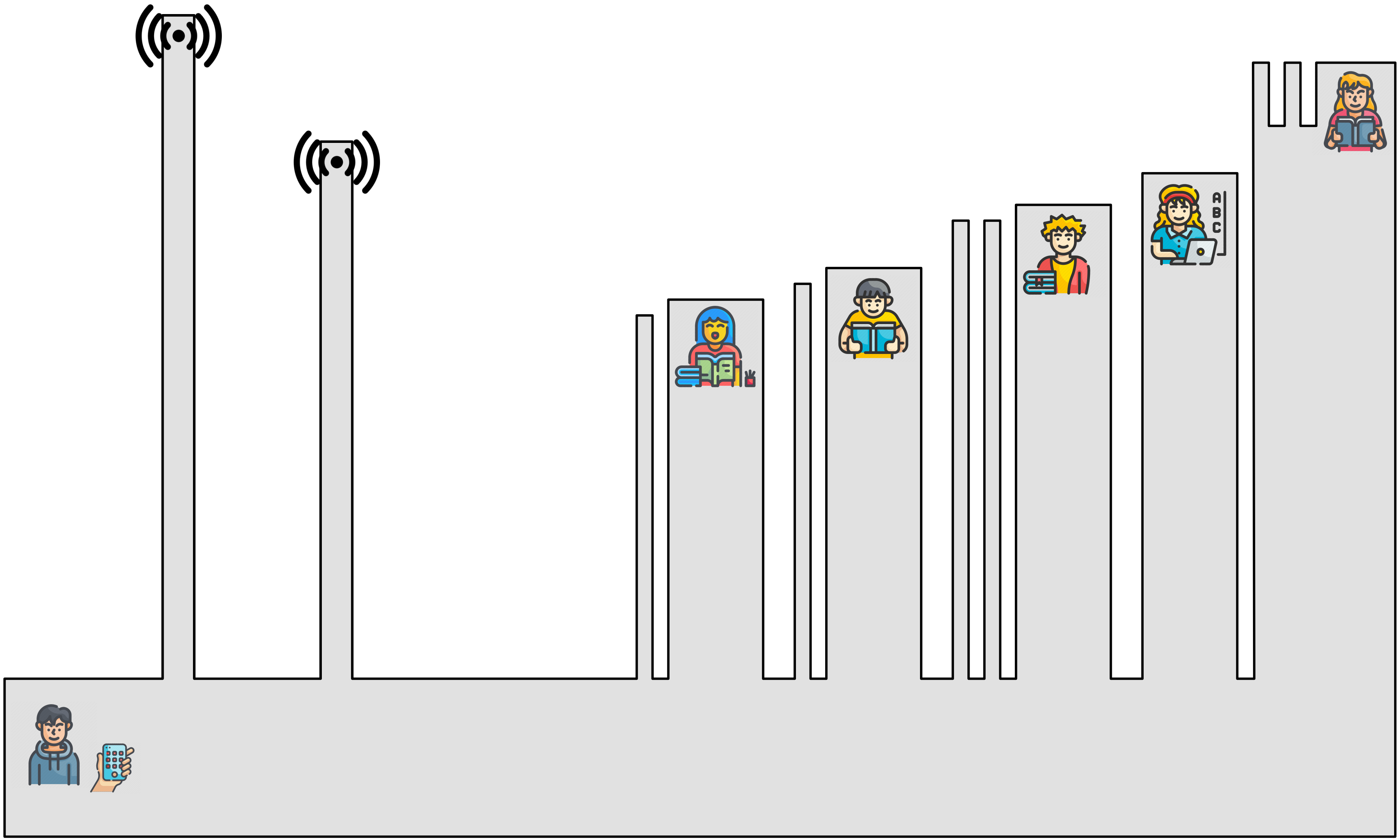
How to Send Party Invitations



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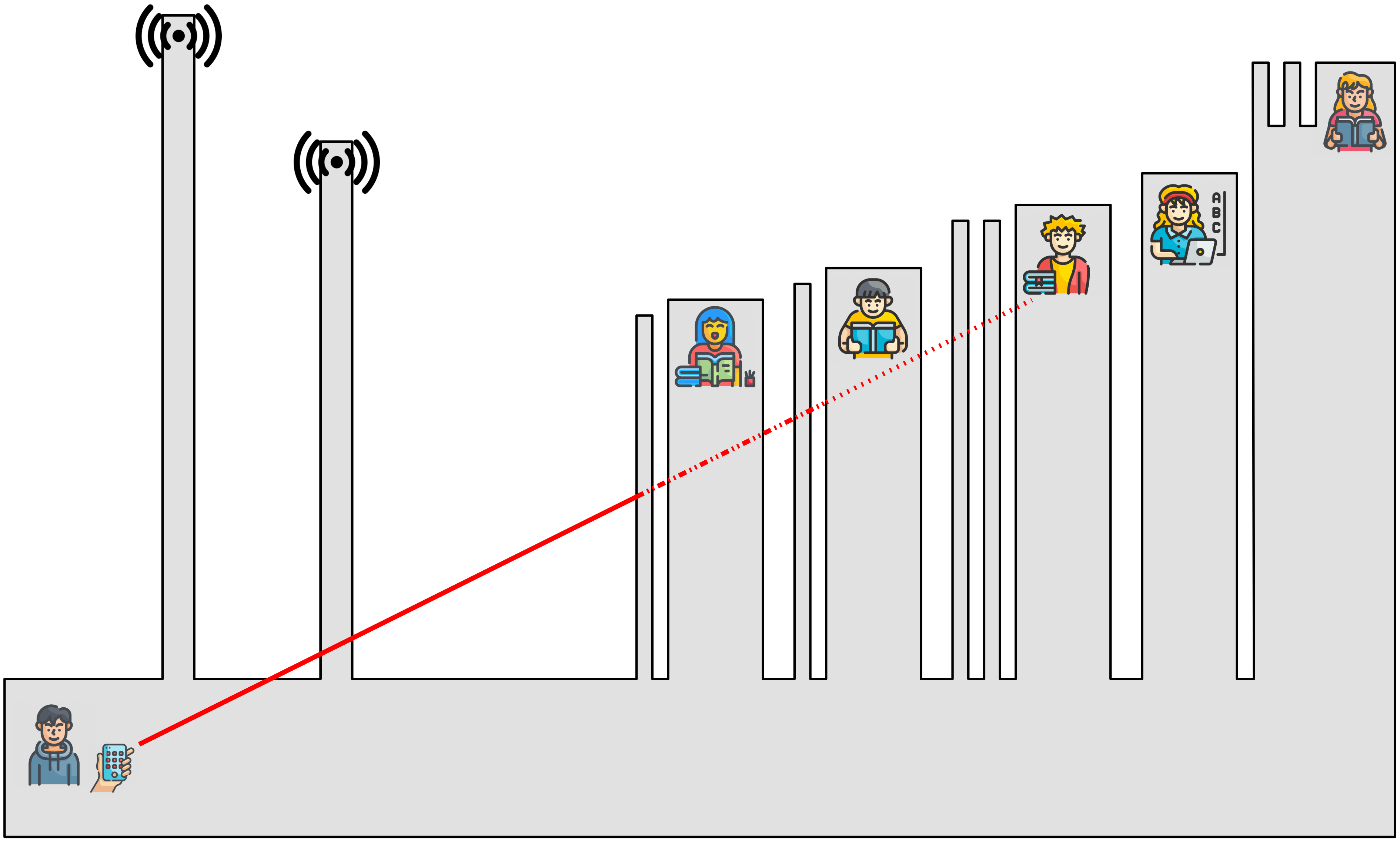

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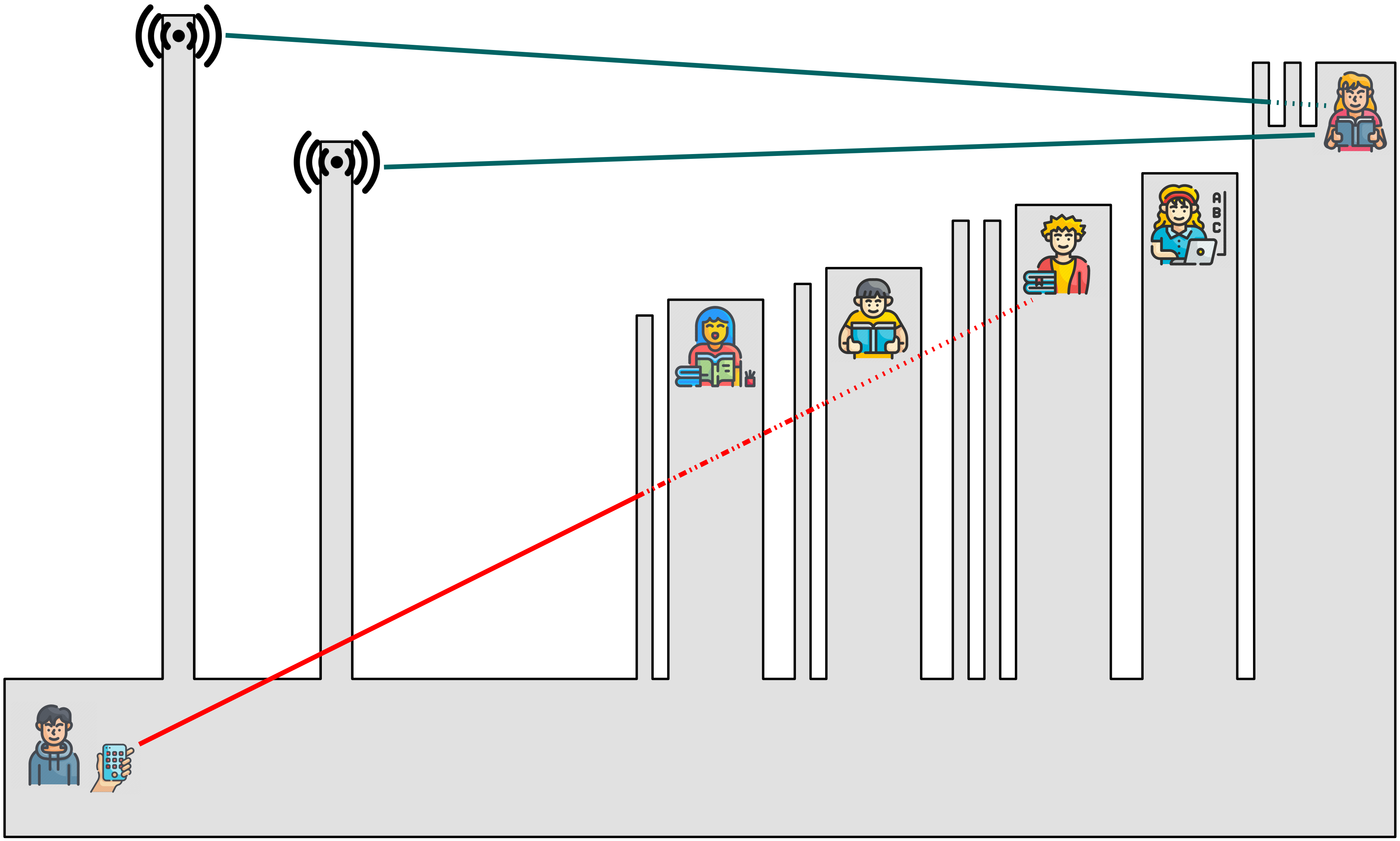

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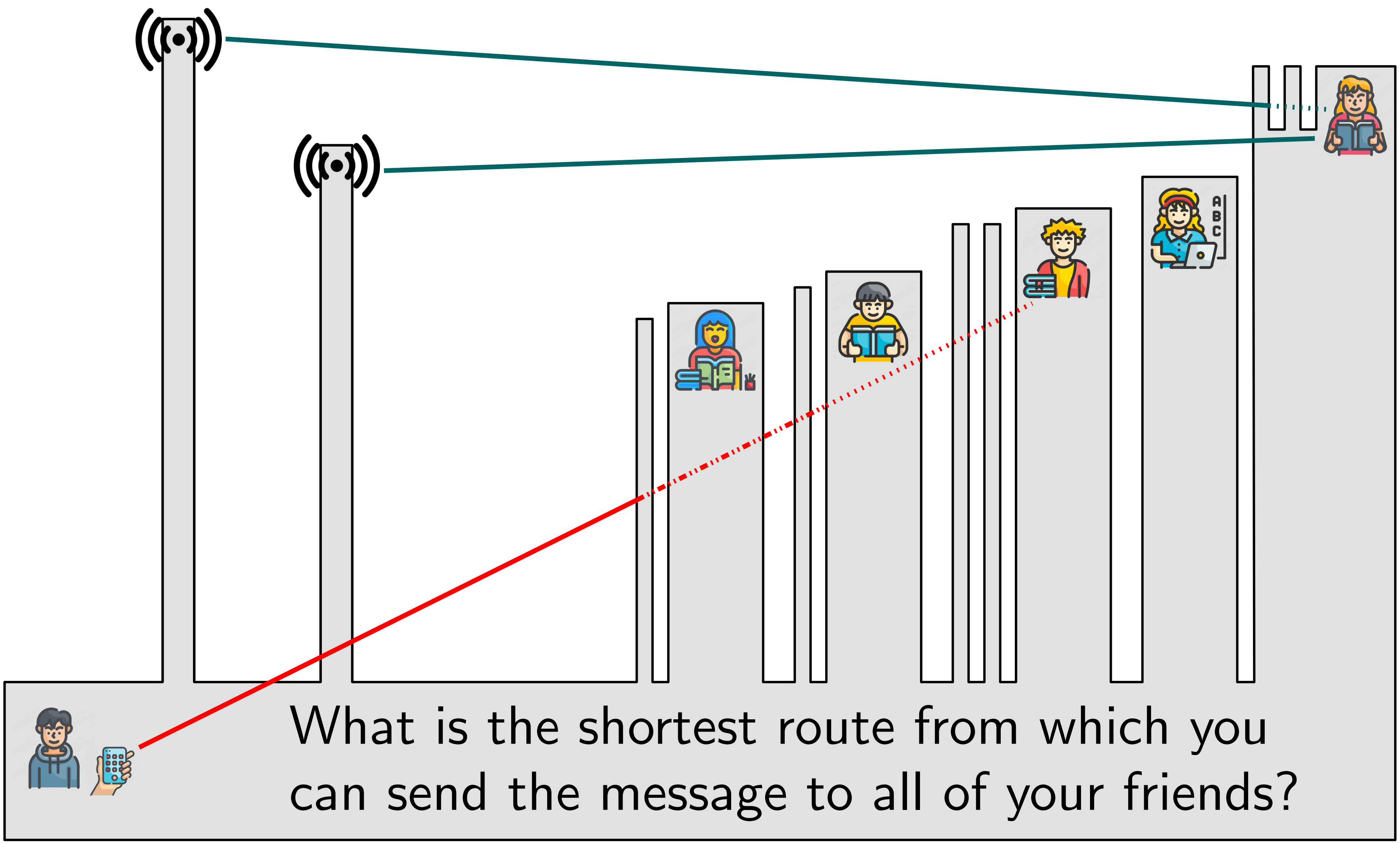

What is the shortest route from which you can send the message to all of your friends?

Problem Setup

k-**Transmitter** q: sees a point p in the polygon if $\overline{pq}$ intersects at most k boundary edges.

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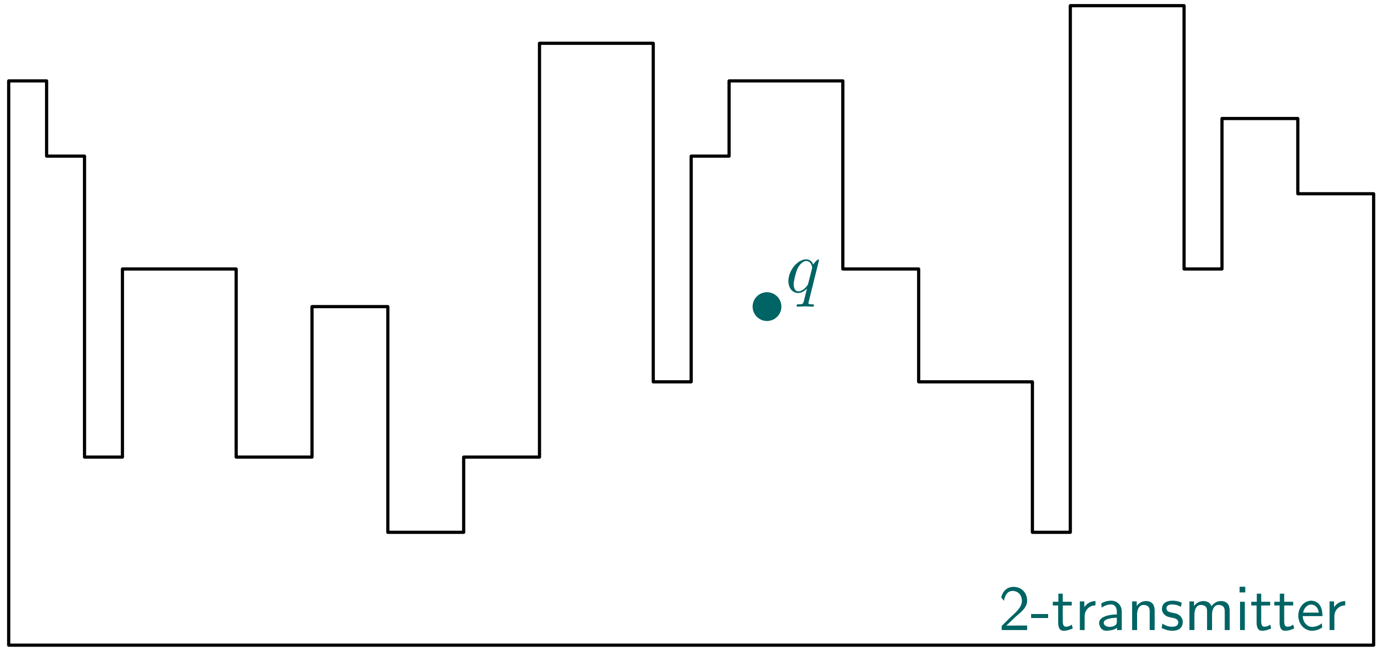

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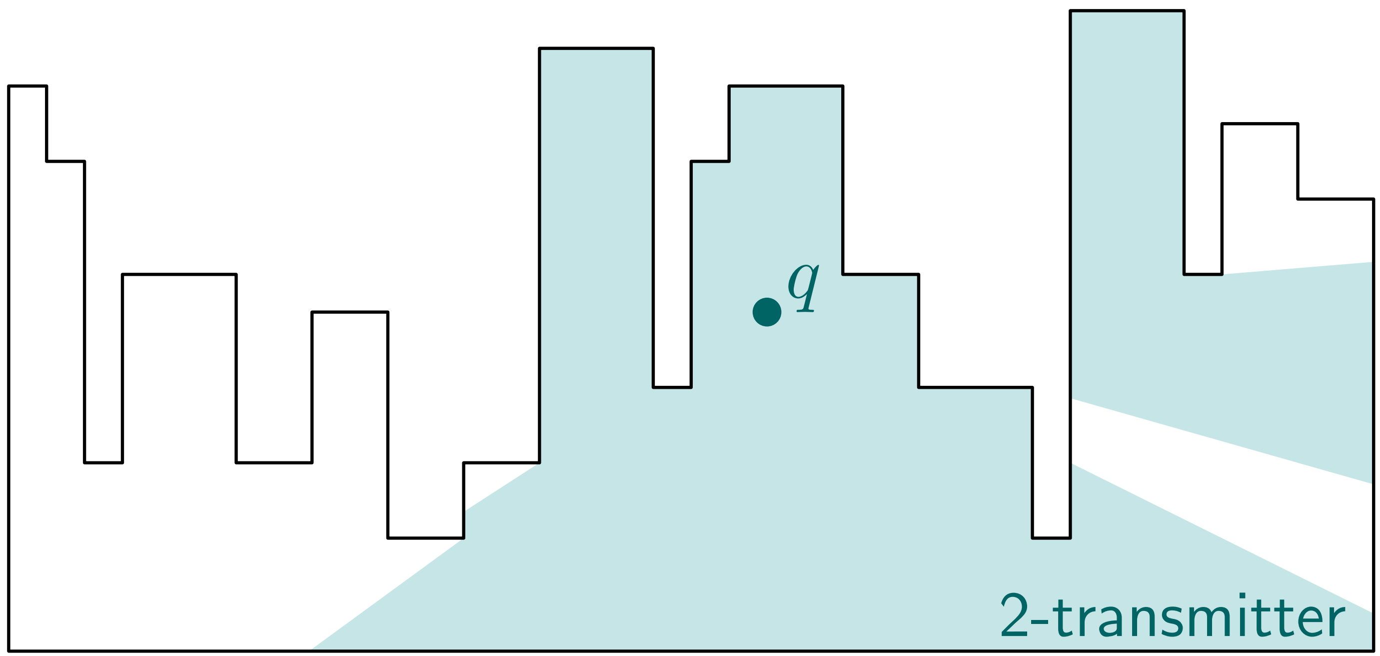

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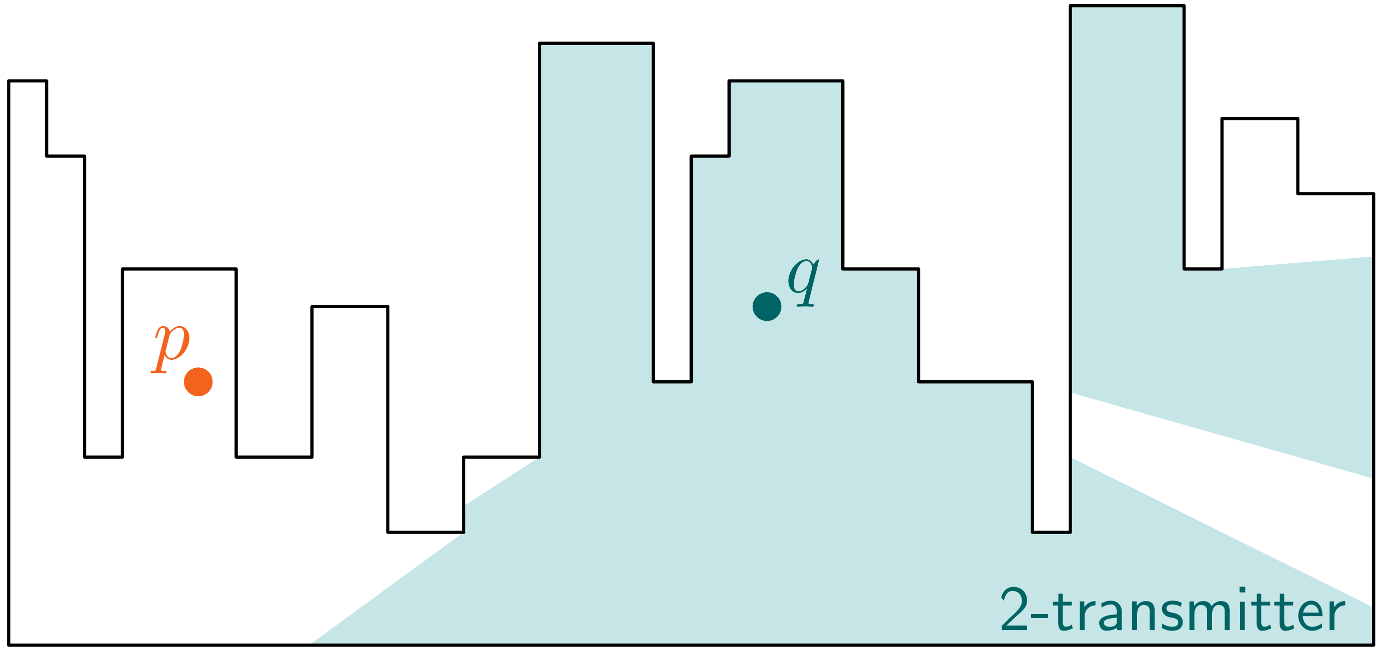

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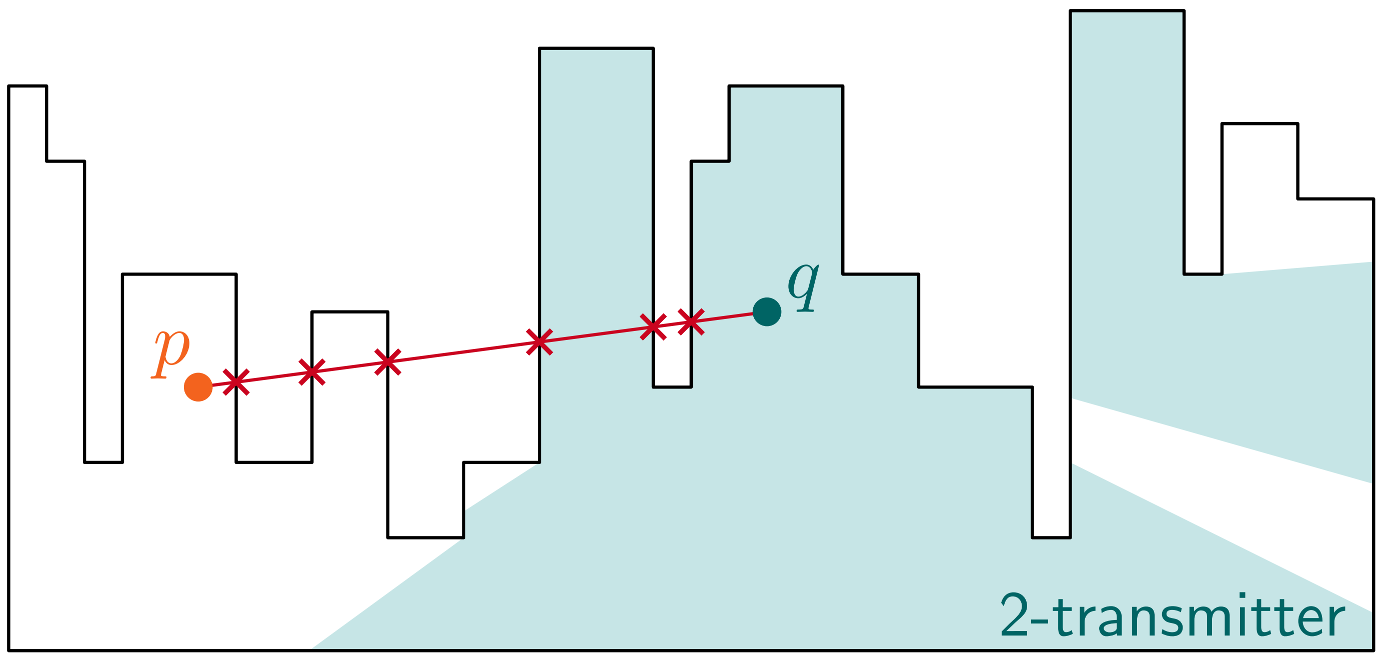

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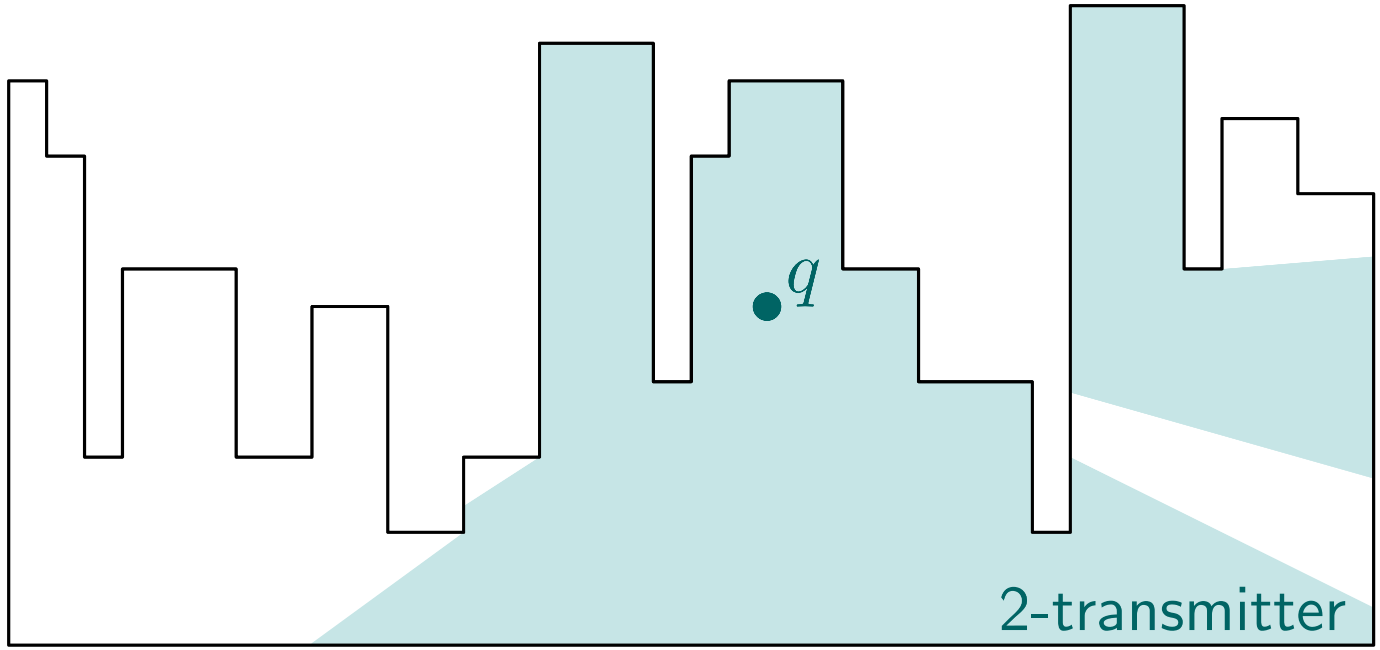

Watchman: mobile transmitter walking along a route

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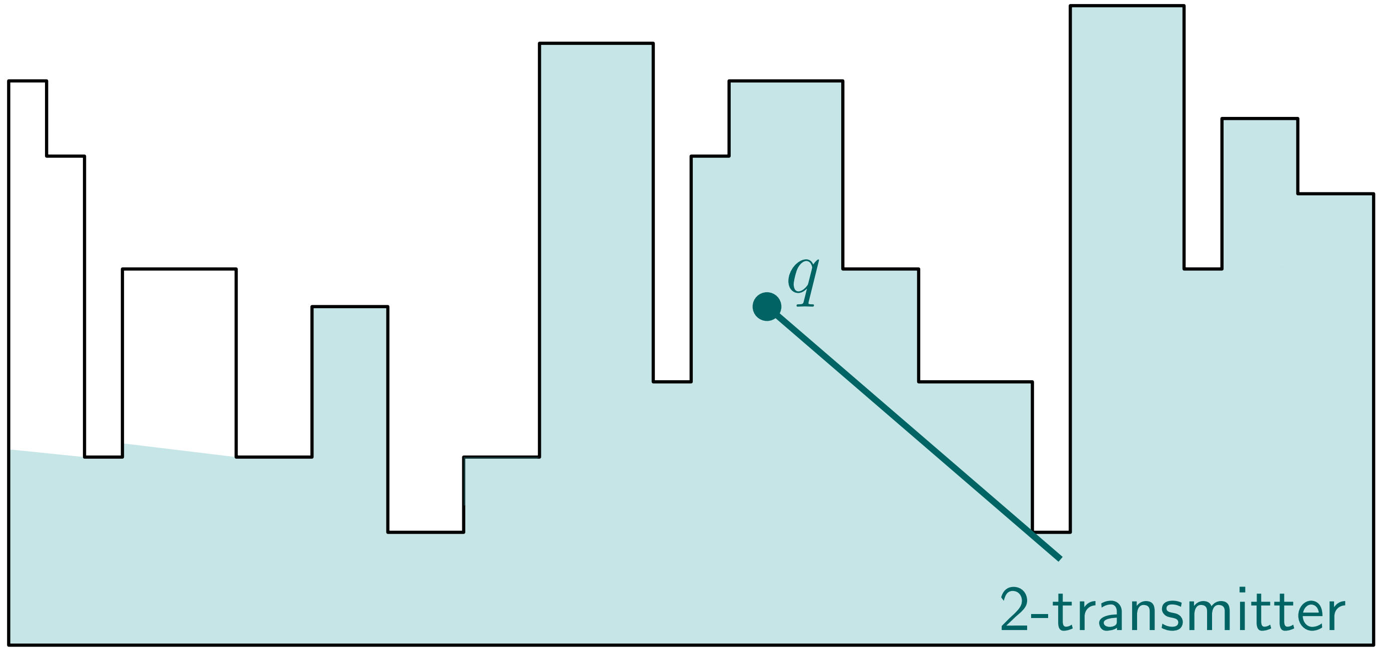

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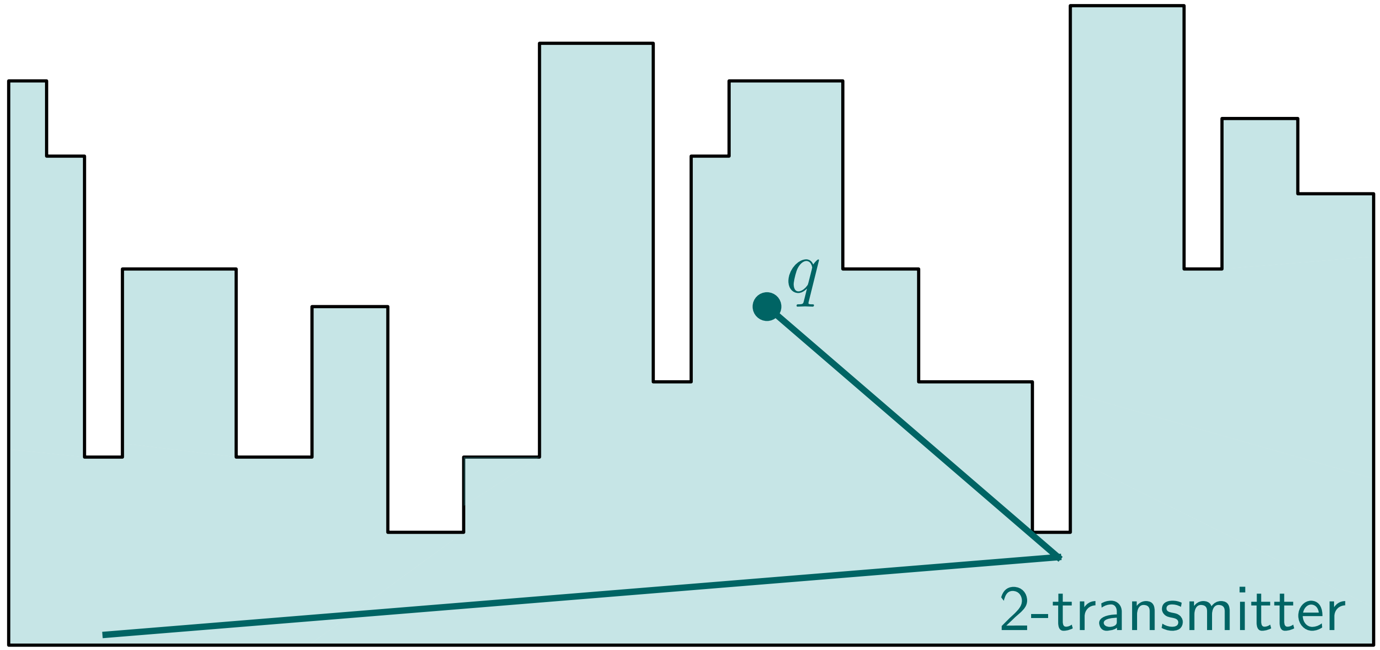

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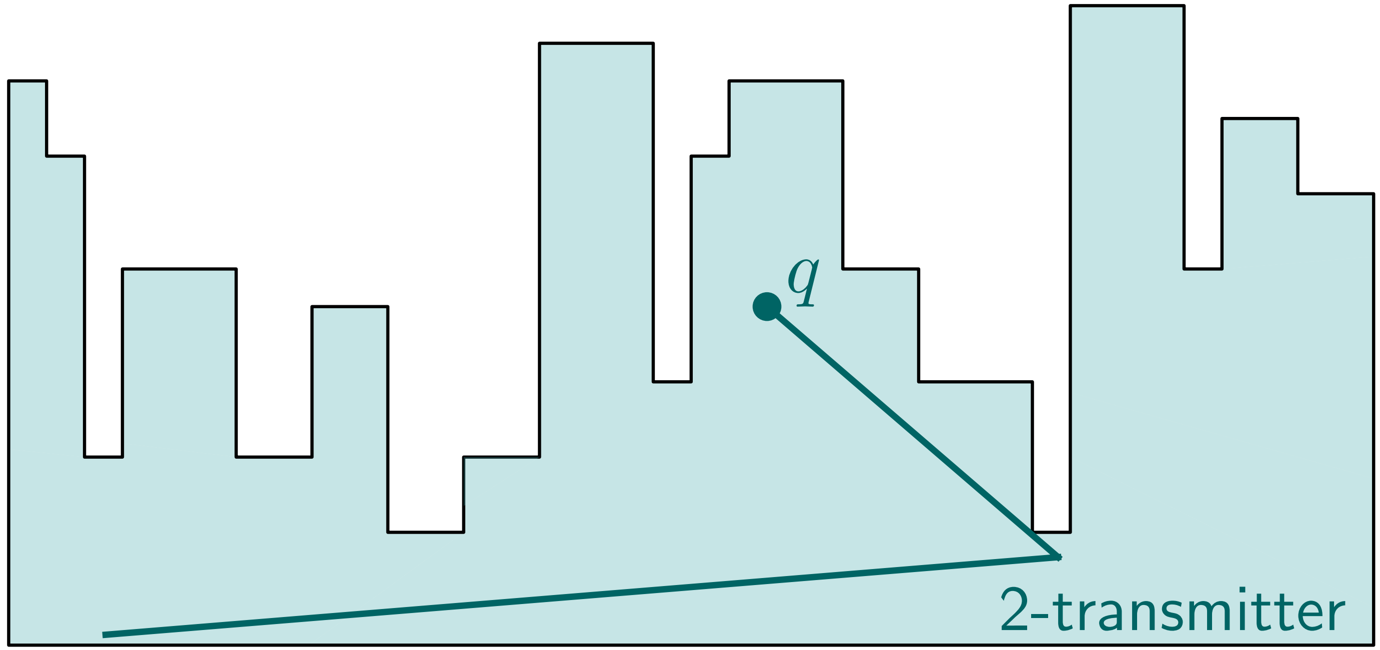

Watchman: mobile transmitter walking along a route

Our goal: see a set S of points inside a polygon P

k-Transmitter WRP

Watchman Route Problem with Starting Point:
Given a polygon P with n vertices, a starting point s in P, and a set of interior points S in P, find a minimum length watchman route that starts at s and lies within P such that all points in S are visible from the route.

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Theorem. *The k-Transmitter Watchman Route Problem for a given discrete set of points to be guarded is* NP*-hard both with and without a fixed starting point and cannot be approximated to within a logarithmic factor.* [N.,S. 2022]

k-Transmitter WRP

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Histogram

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Uni-monotone polygons
(aka monotone mountains aka Alps)

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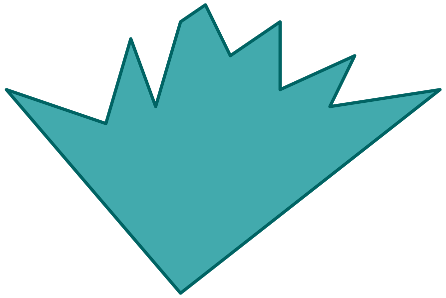

Star-shaped polygons

k-Transmitter WRP

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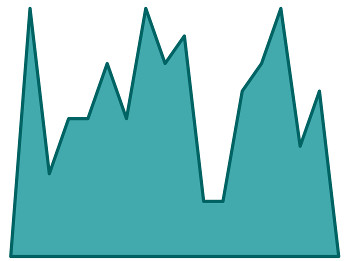

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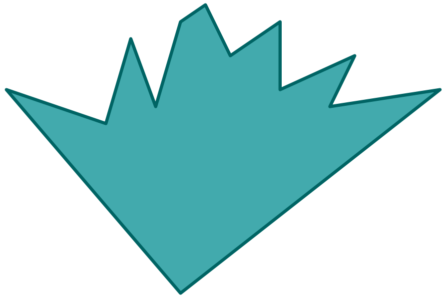

Star-shaped polygons

Theorem. *For any* $k \geq 2$, *k-TrWRP*(S, P, s) *is* NP*-hard for histograms, uni-monotone polygons, and star-shaped polygons, and cannot be approximated within a logarithmic factor* $c \log n$, *for any* $c > 0$.

NP-Hardness for Histograms

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Reduction from Set Cover

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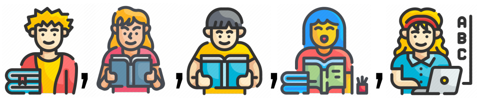
Reduction from Set Cover

Set Cover: Given a universe $\mathcal{U} = \{$  $\}$ and a family $\mathcal{R}$ of subsets of $\mathcal{U}$, find a subfamily $\mathcal{C} \subseteq \mathcal{R}$ that contains all elements of $\mathcal{U}$ and is of minimum cardinality.

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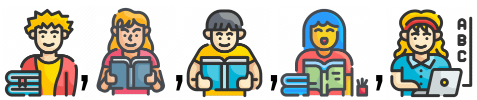
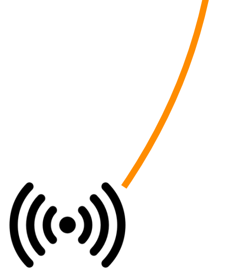
Set Cover: Given a universe $\mathcal{U} = \{$$\}$ and a family $\mathcal{R}$ of subsets of $\mathcal{U}$, find a subfamily $\mathcal{C} \subseteq \mathcal{R}$ that contains all elements of $\mathcal{U}$ and is of minimum cardinality.



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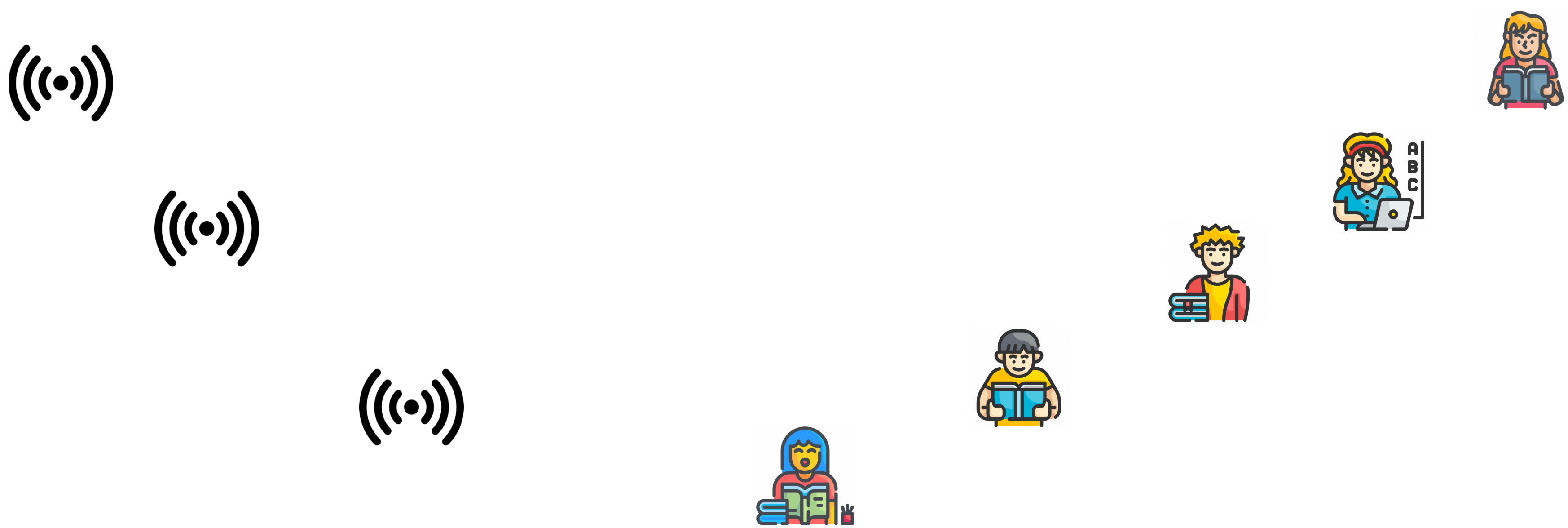

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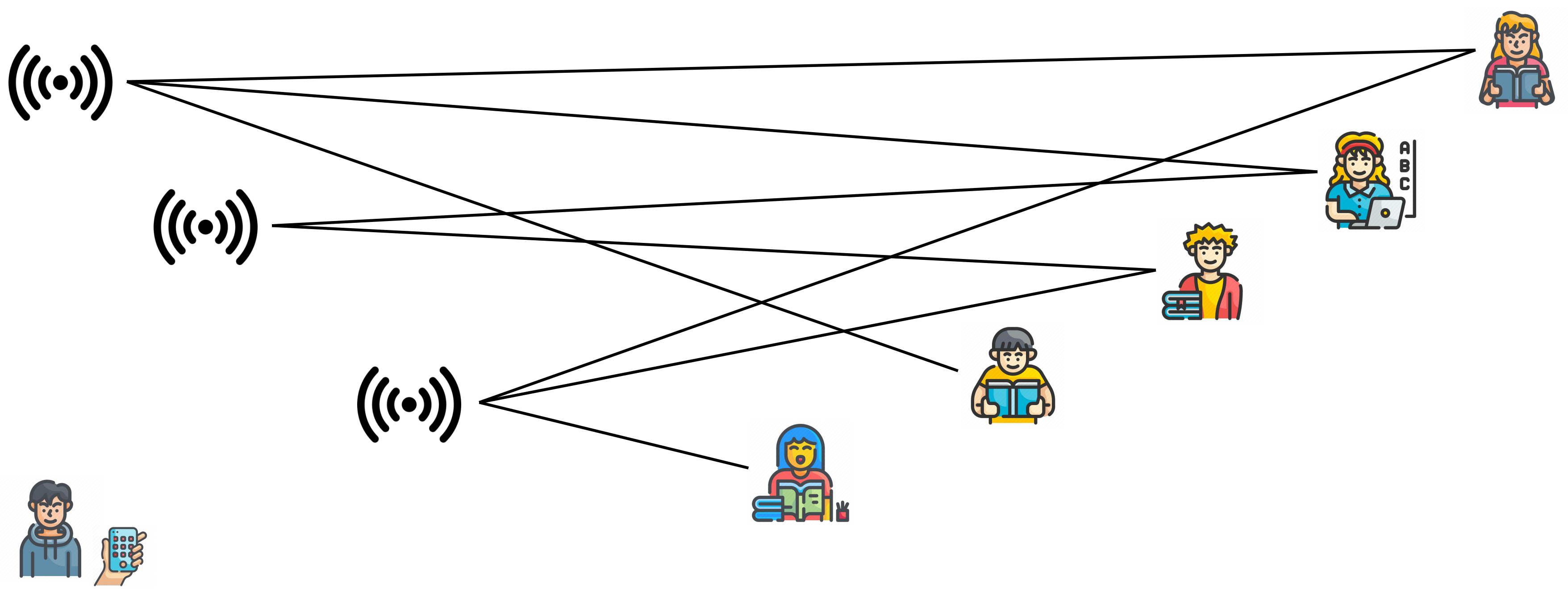

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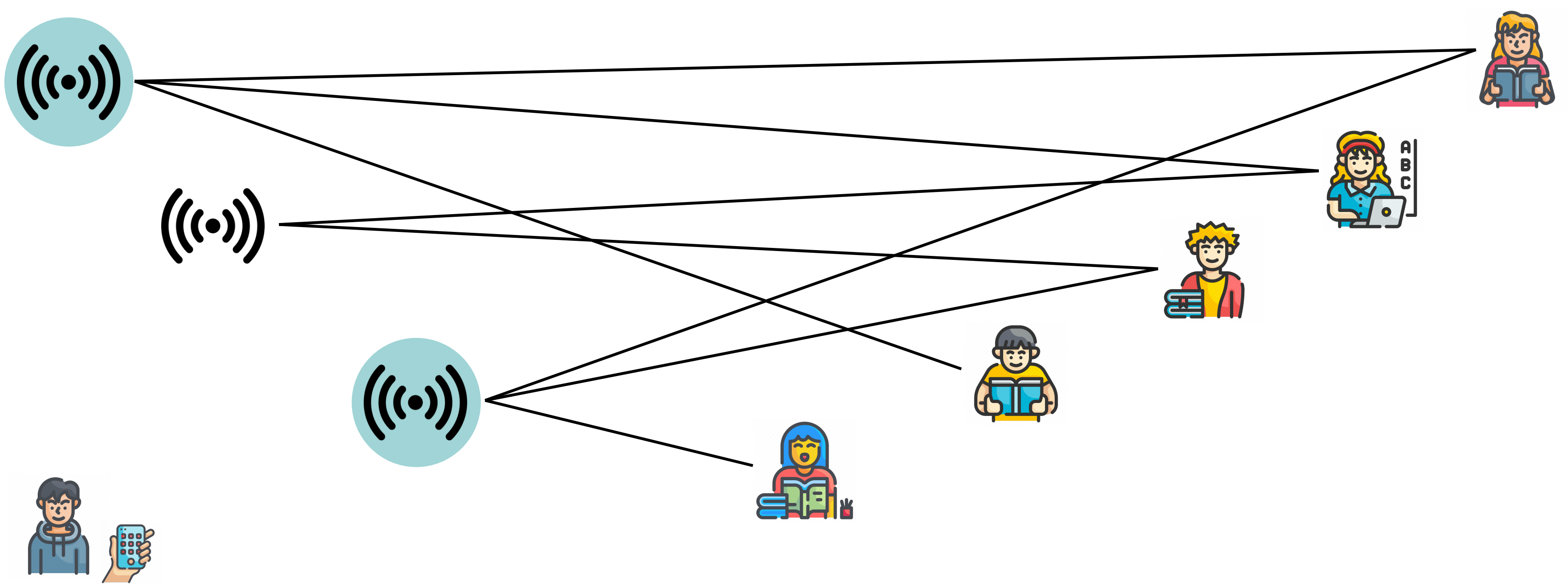

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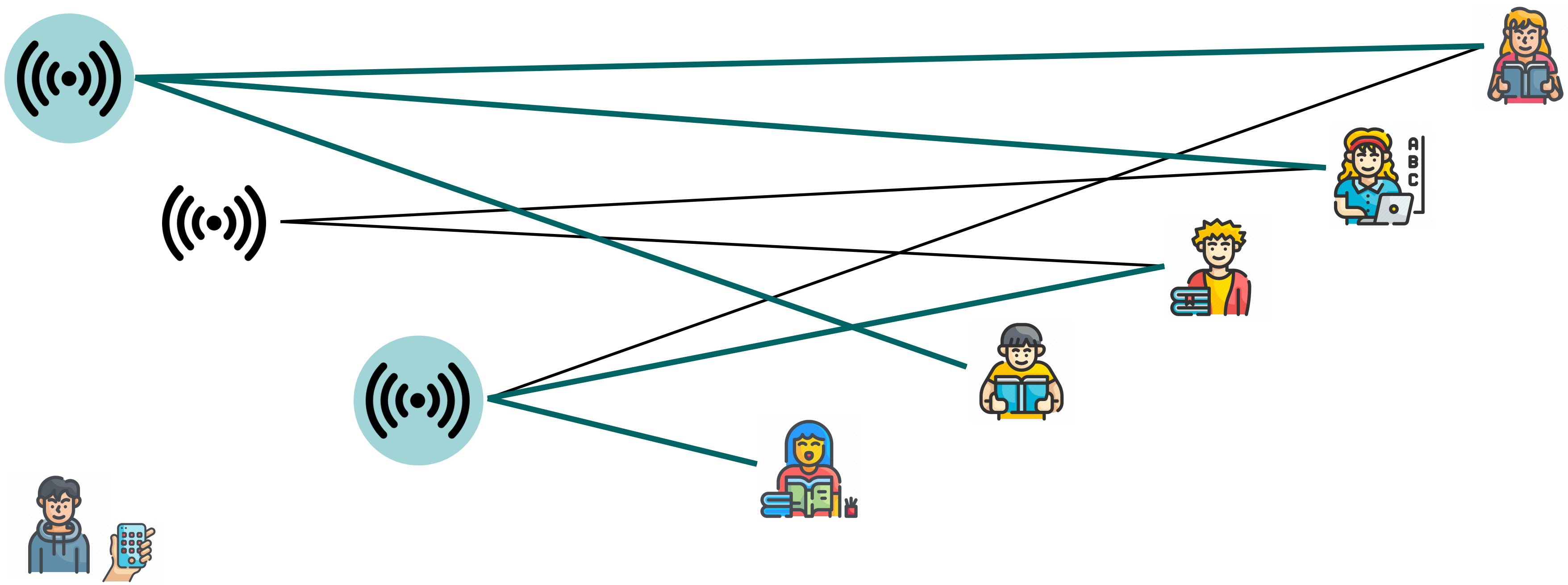

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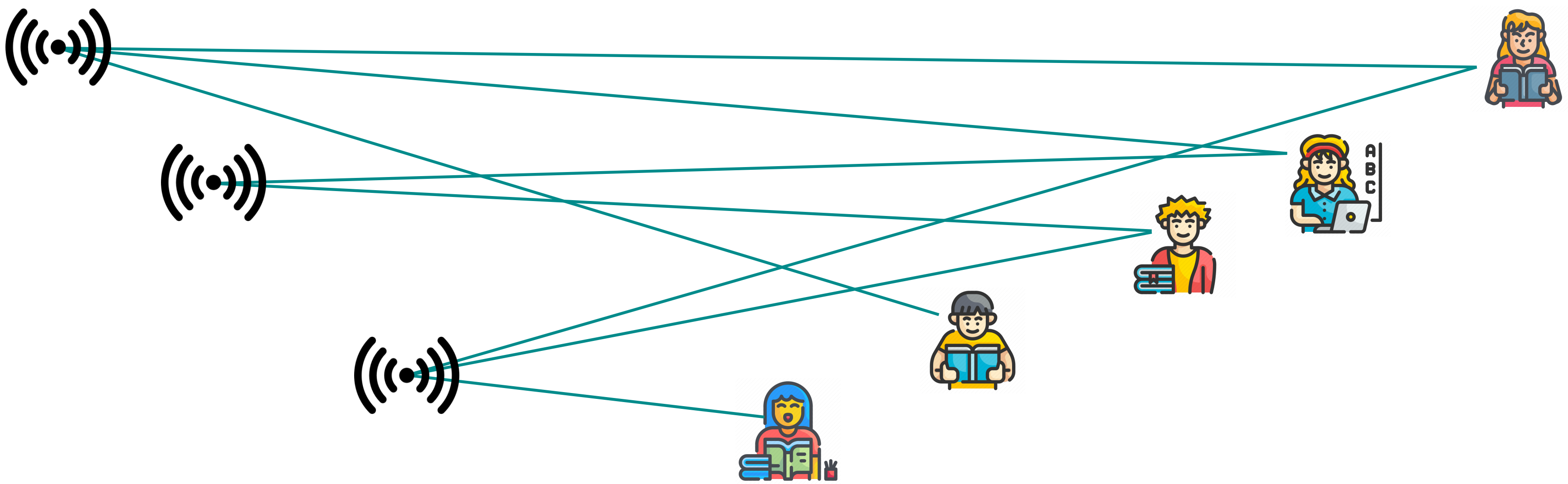

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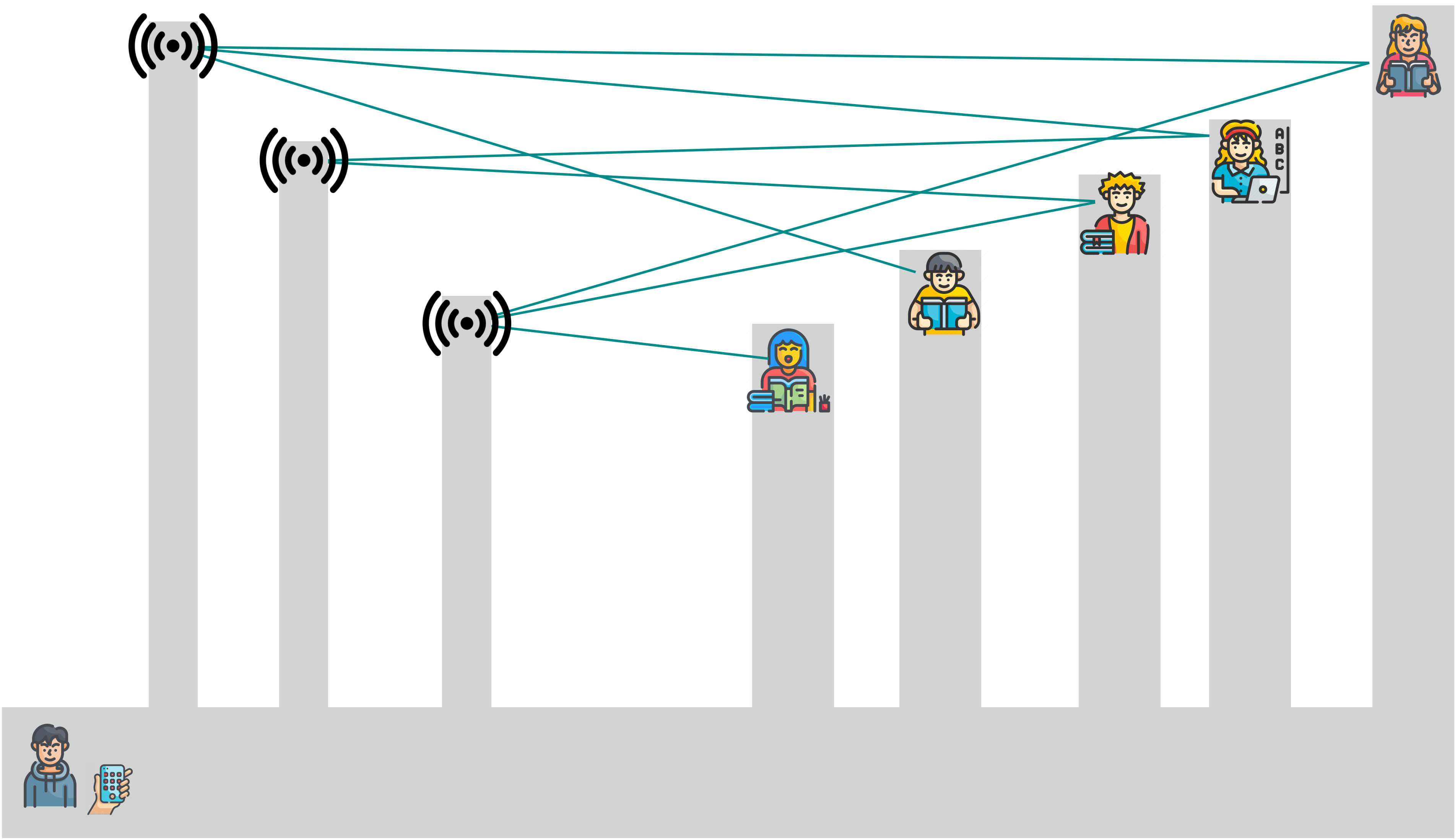

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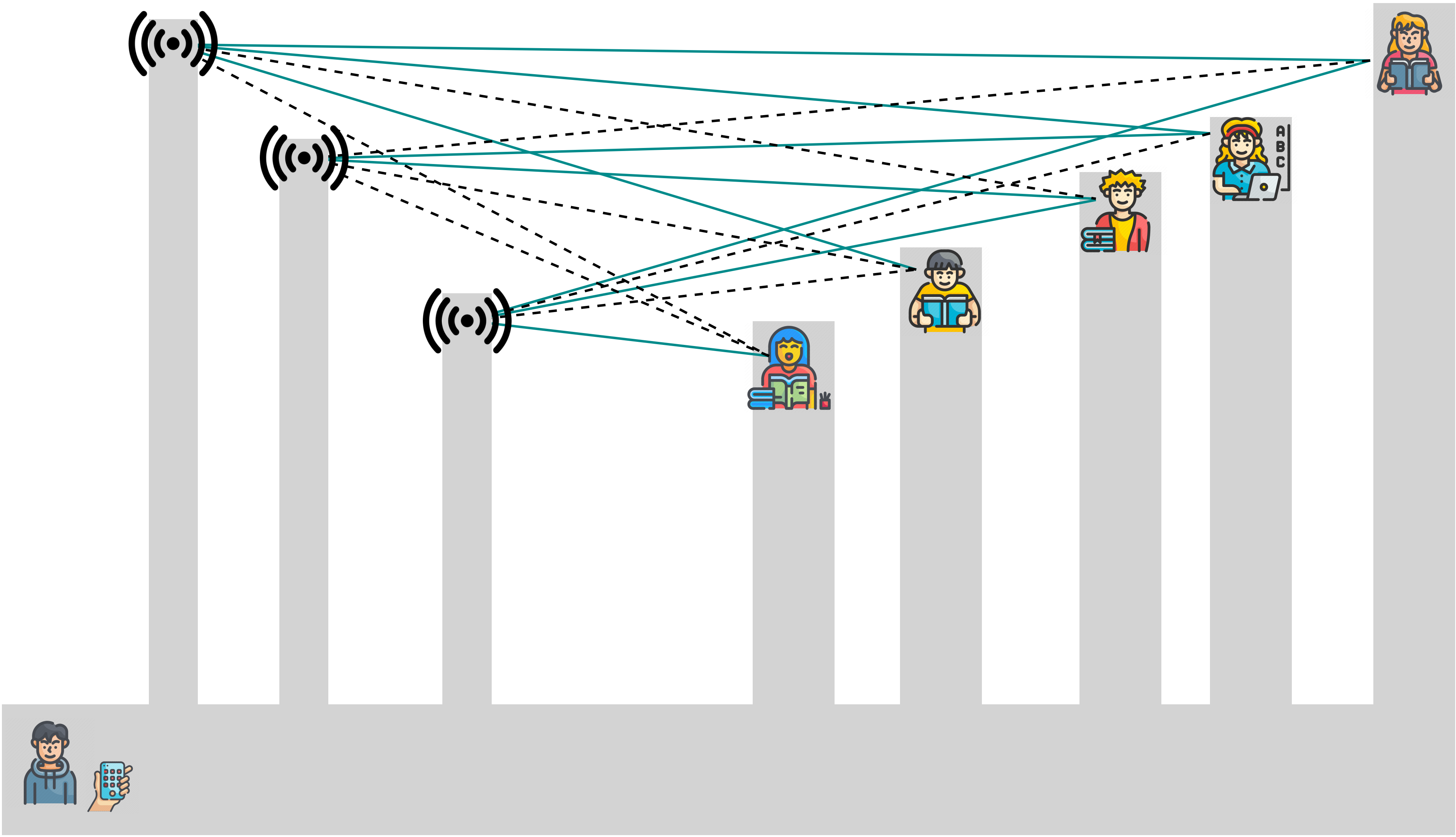

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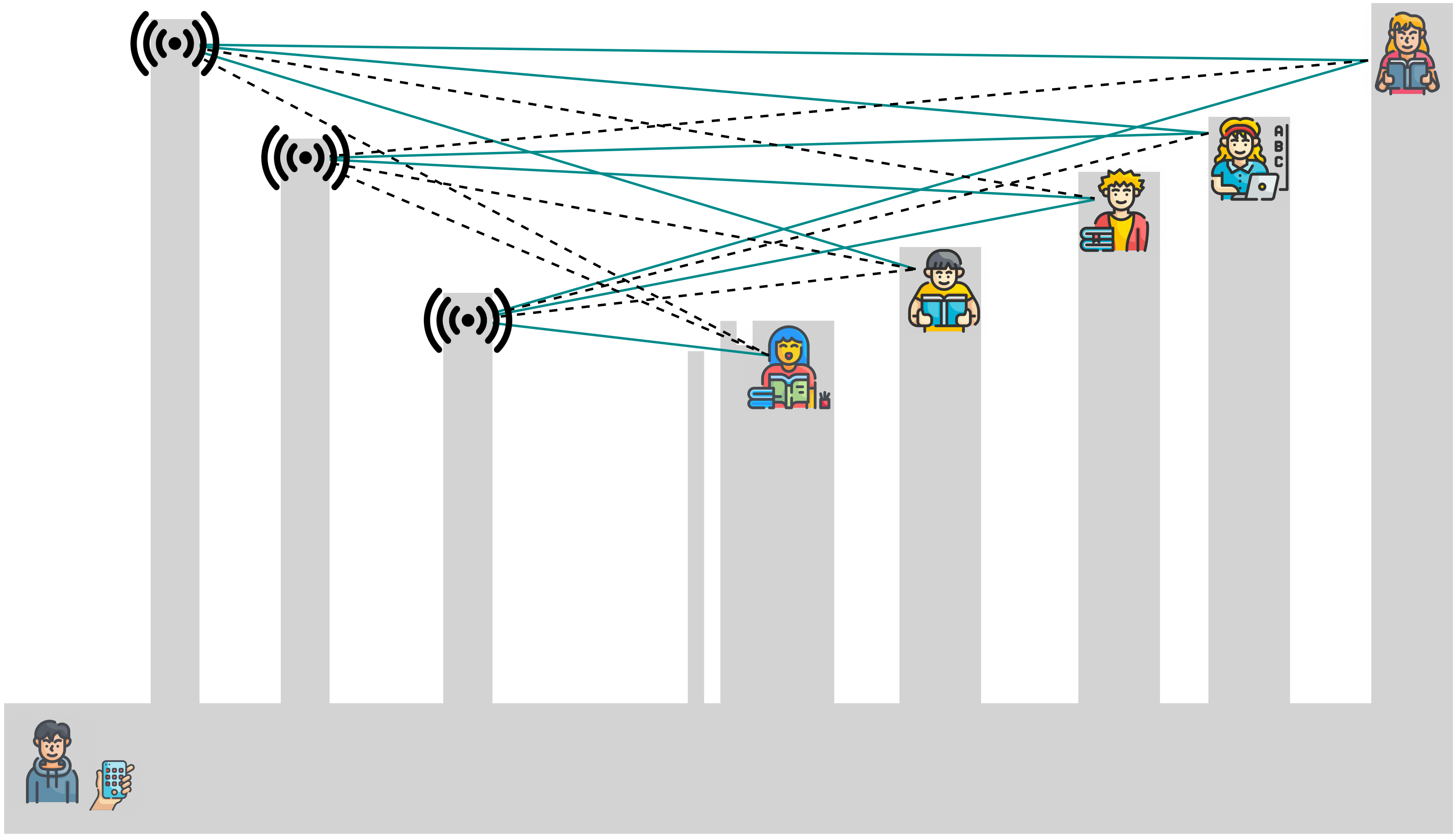

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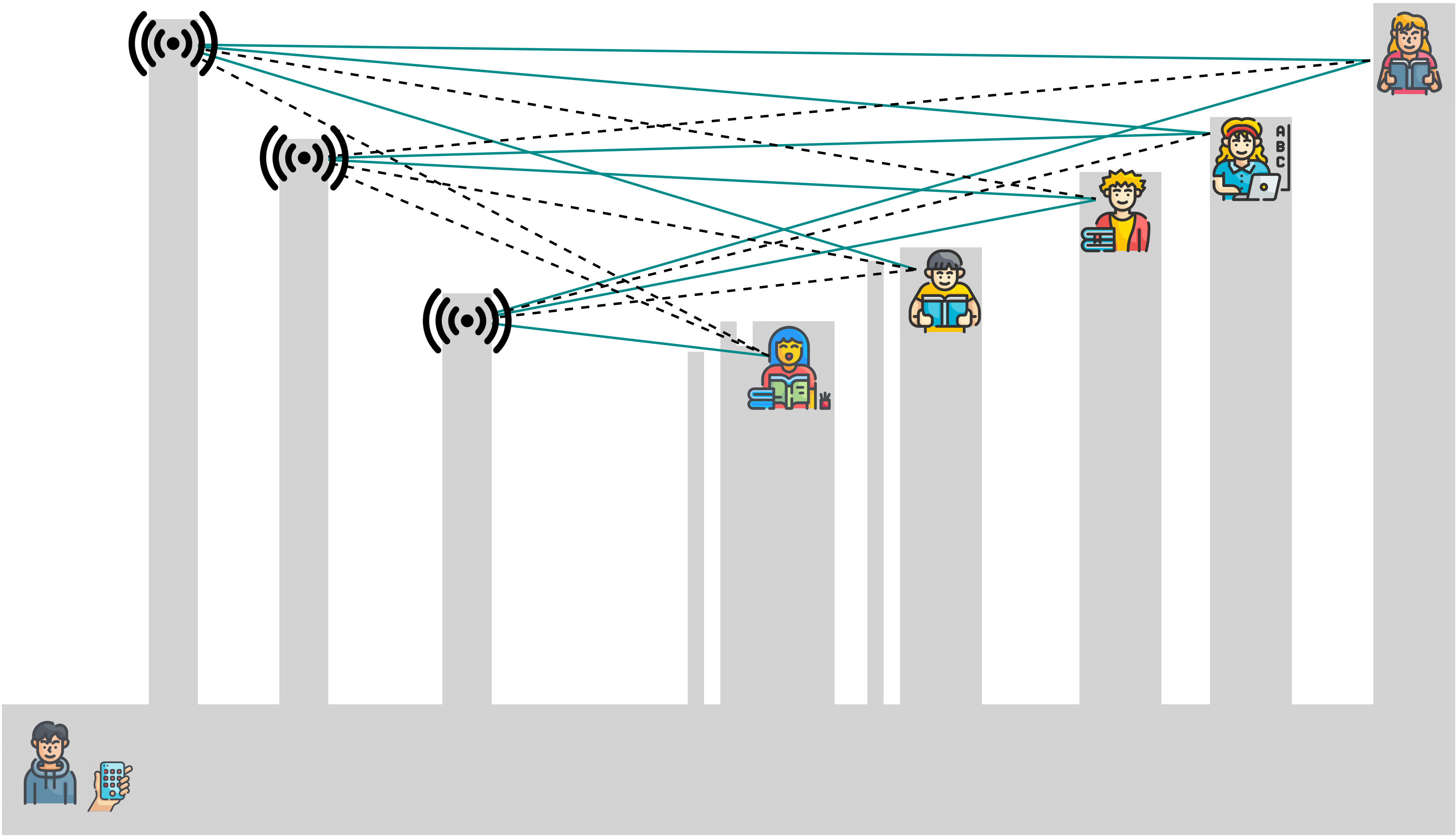

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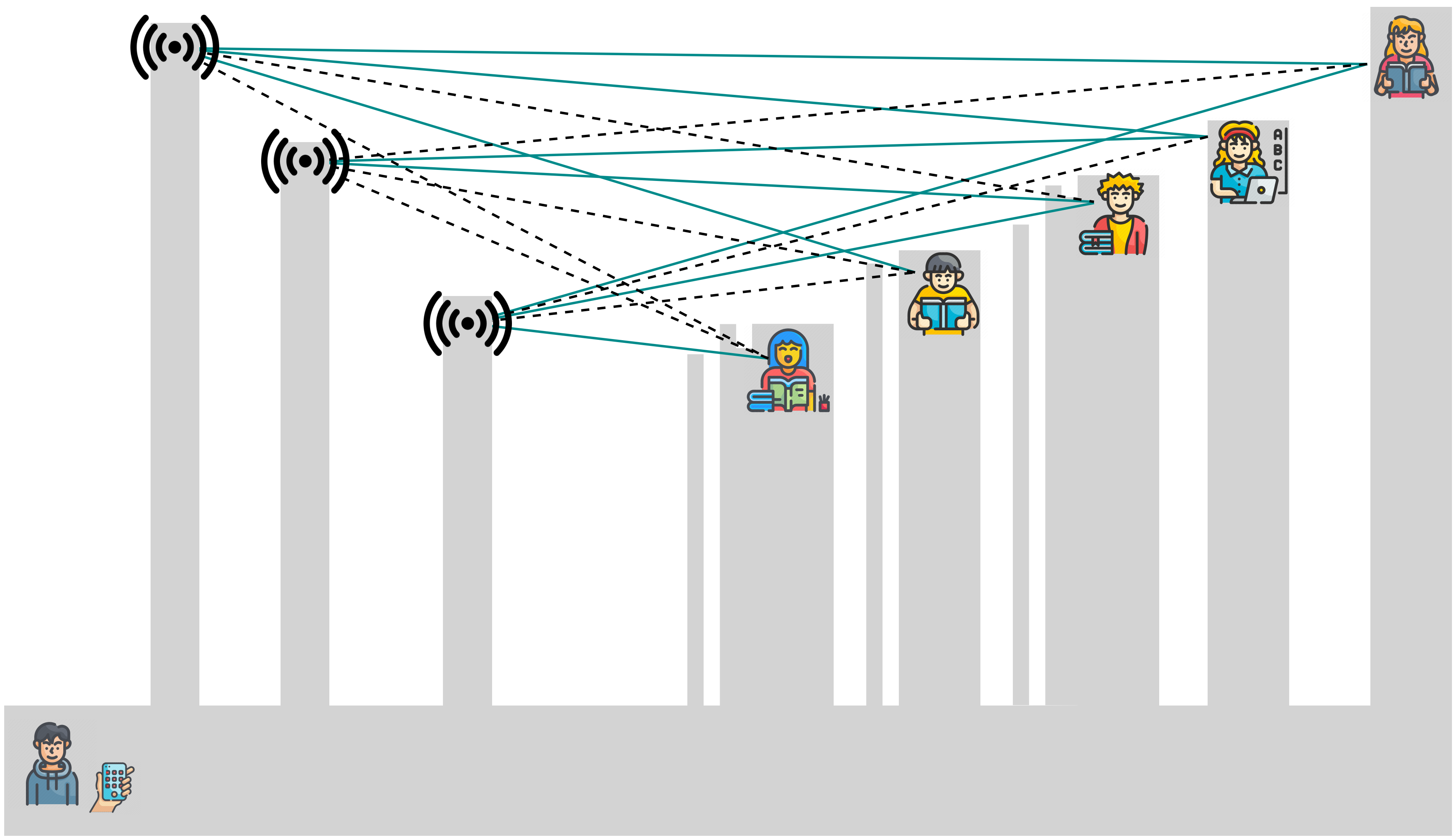

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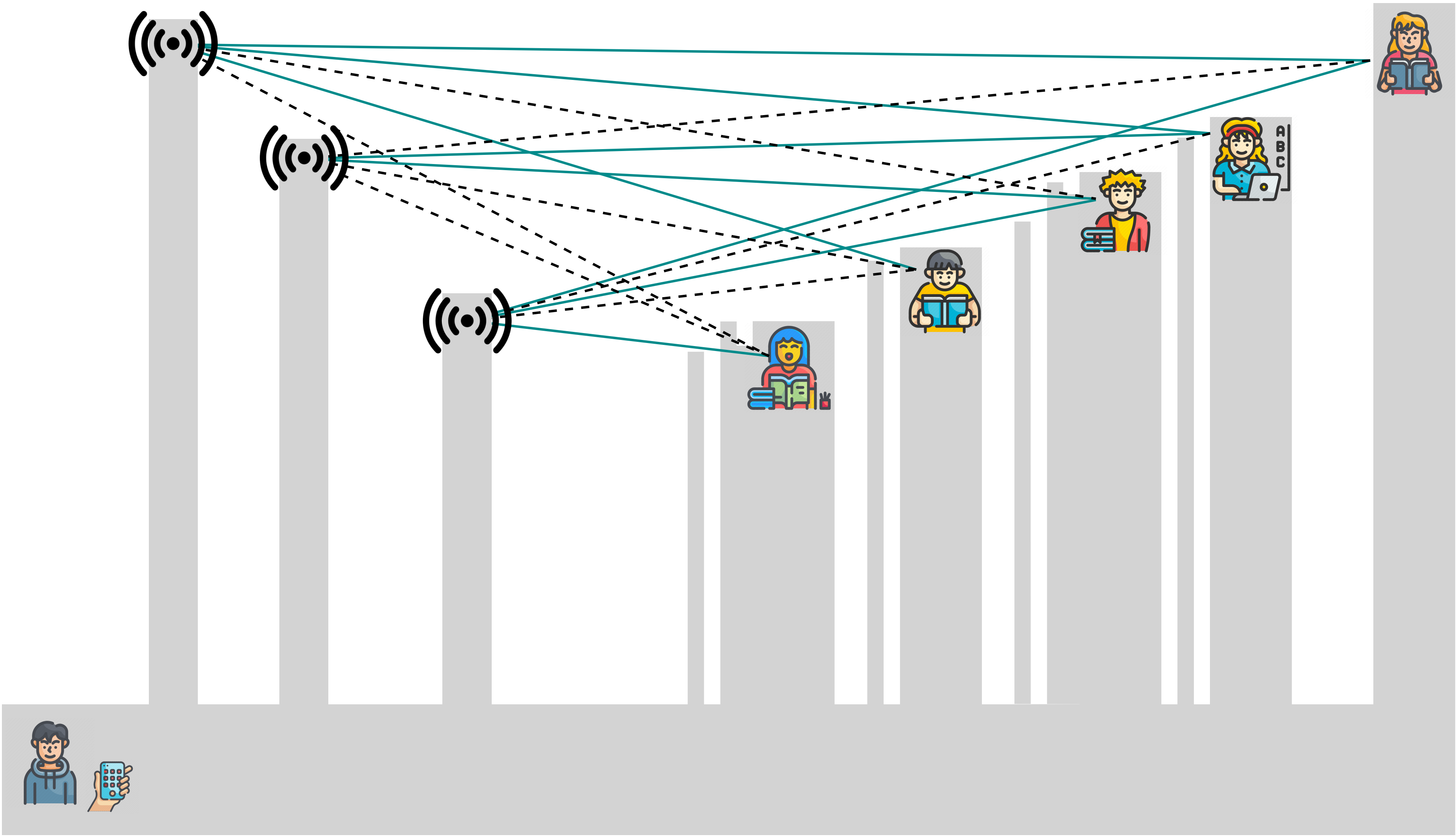

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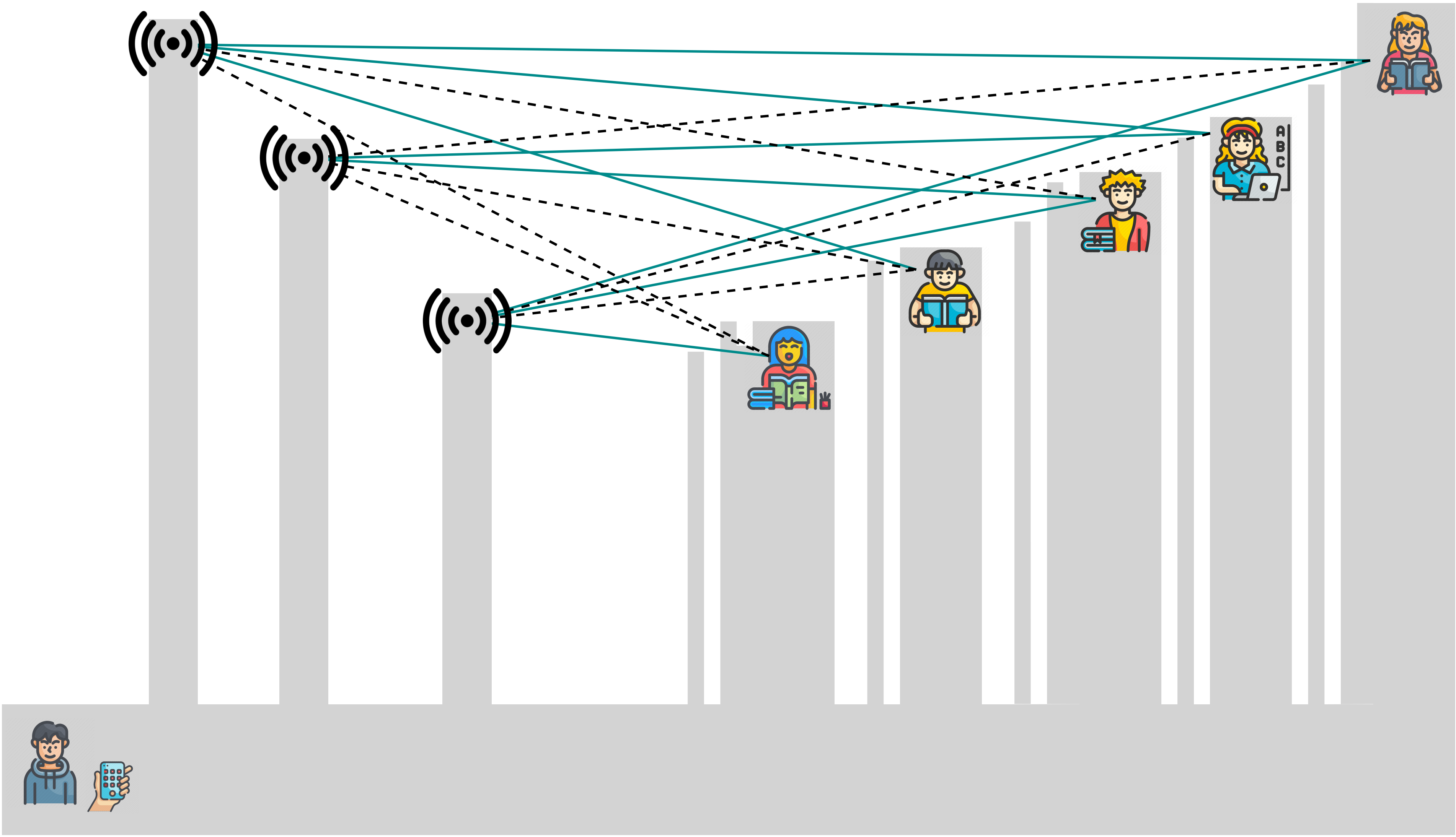

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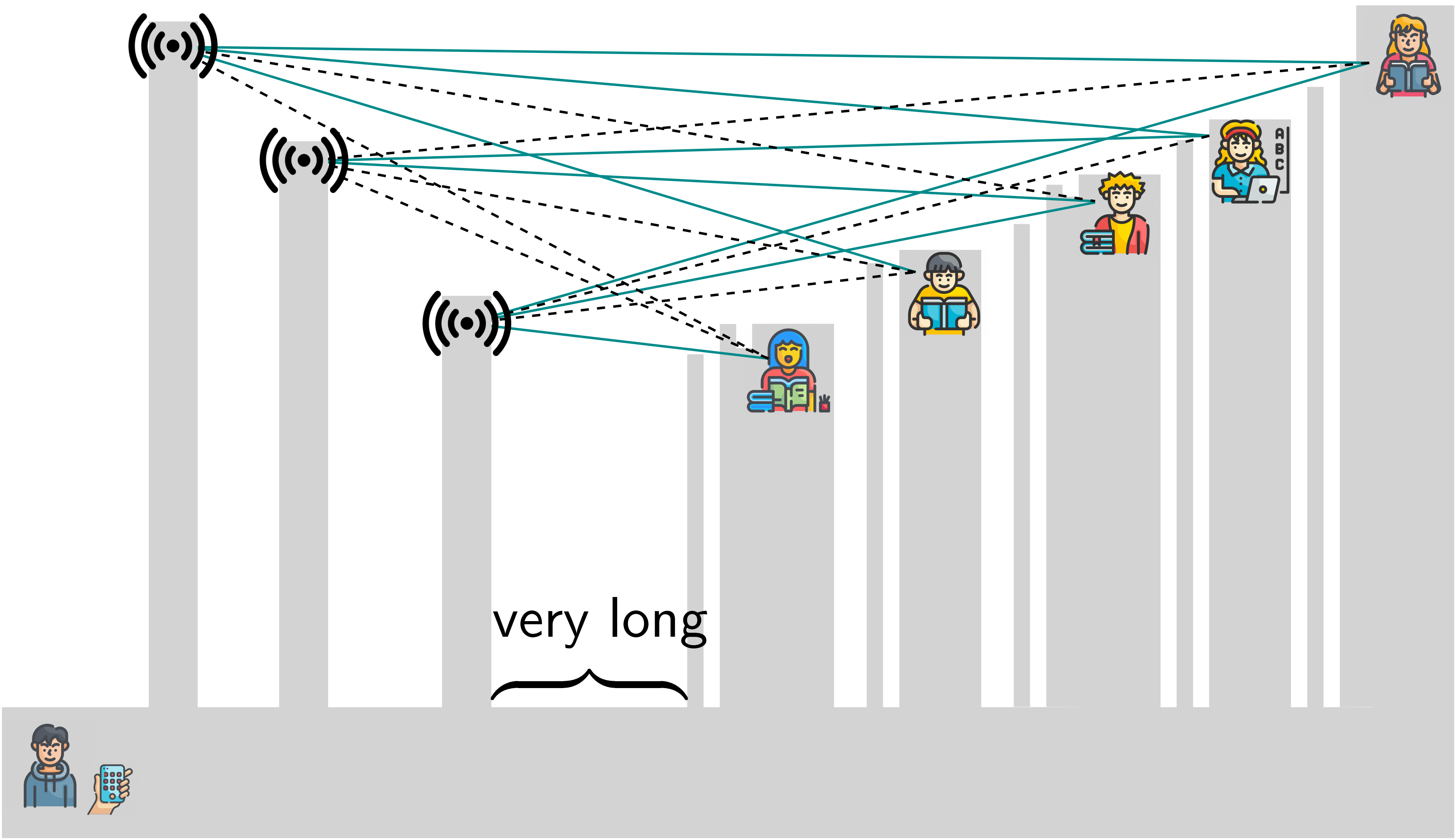

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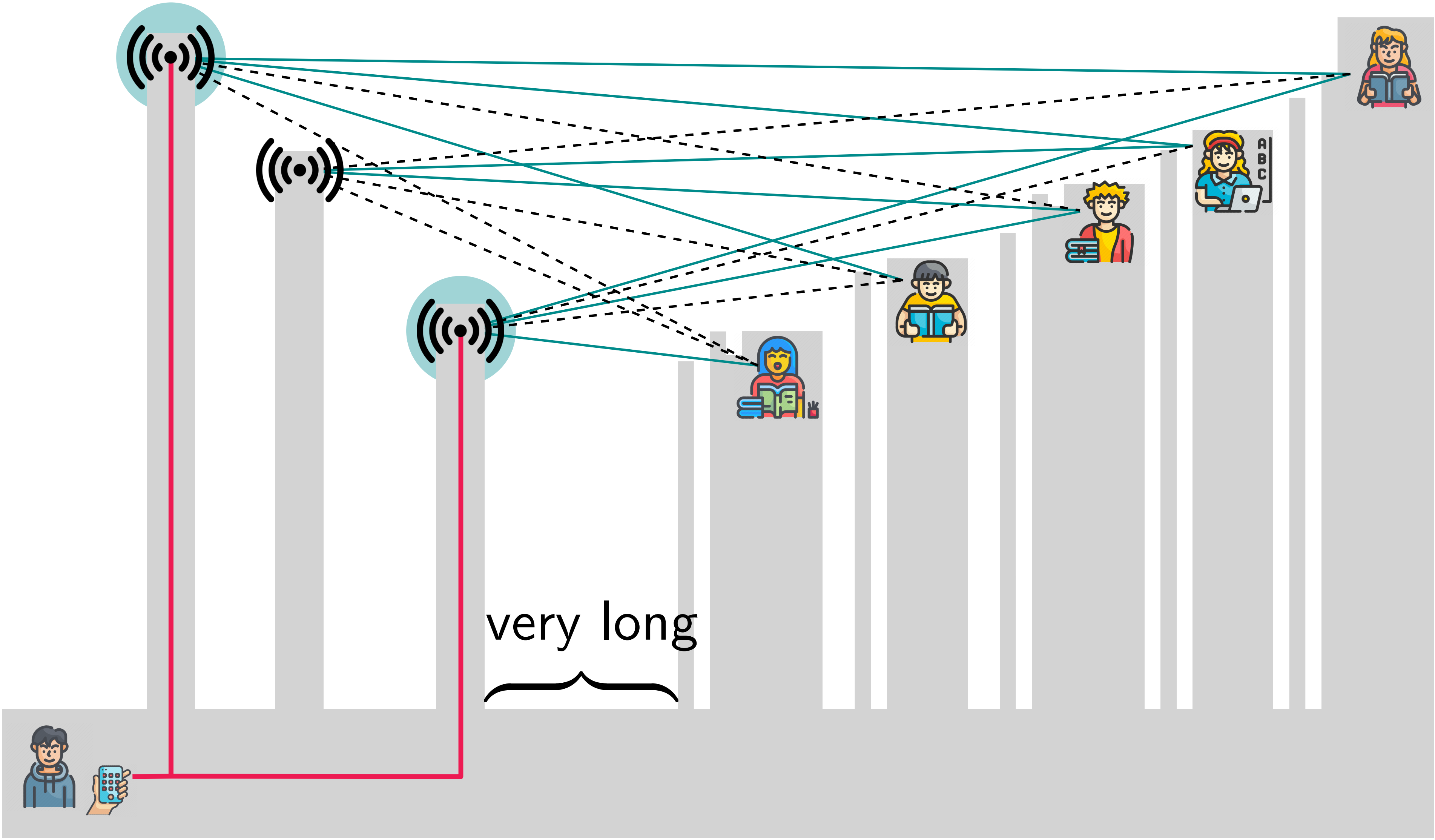

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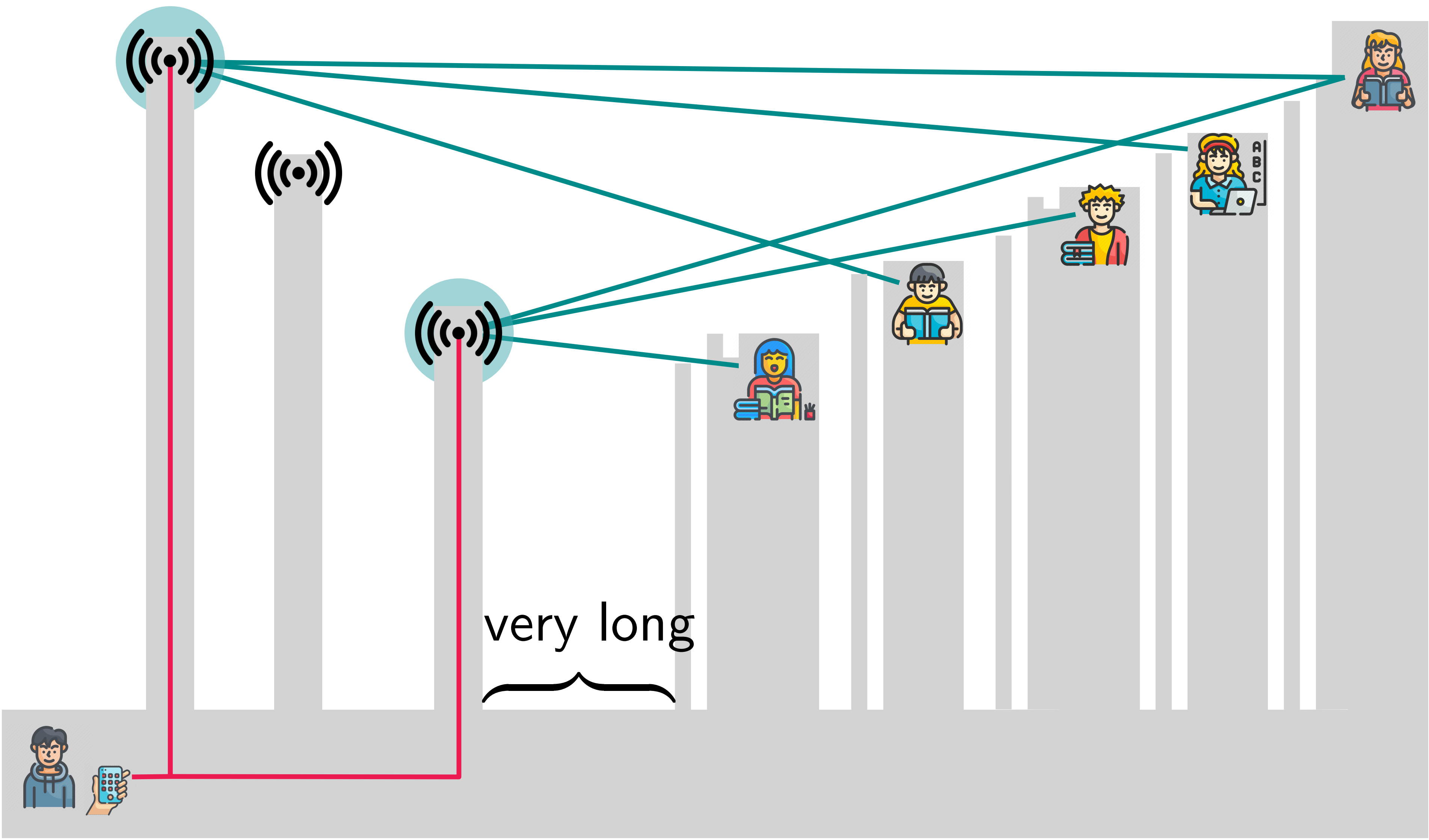

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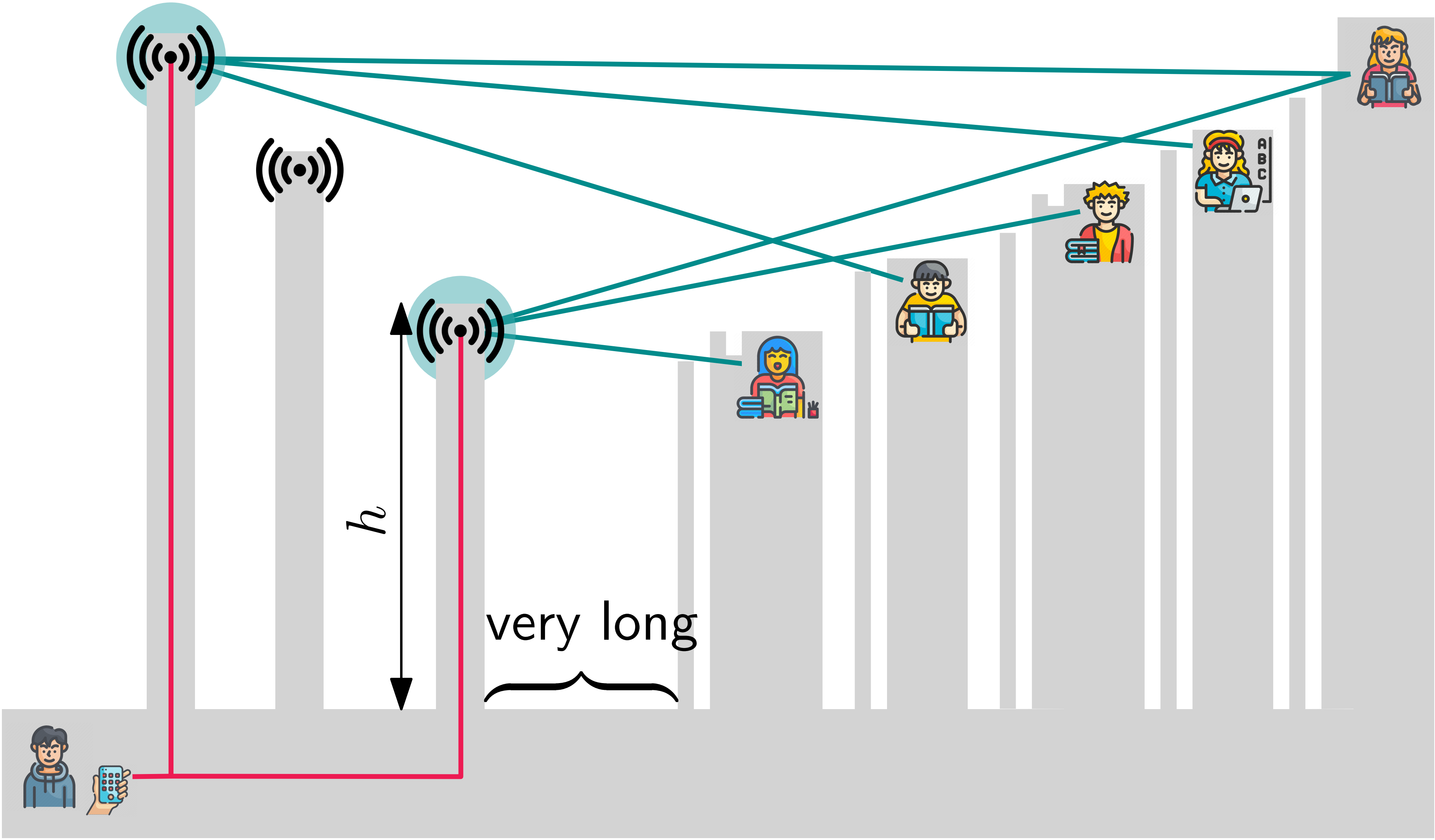

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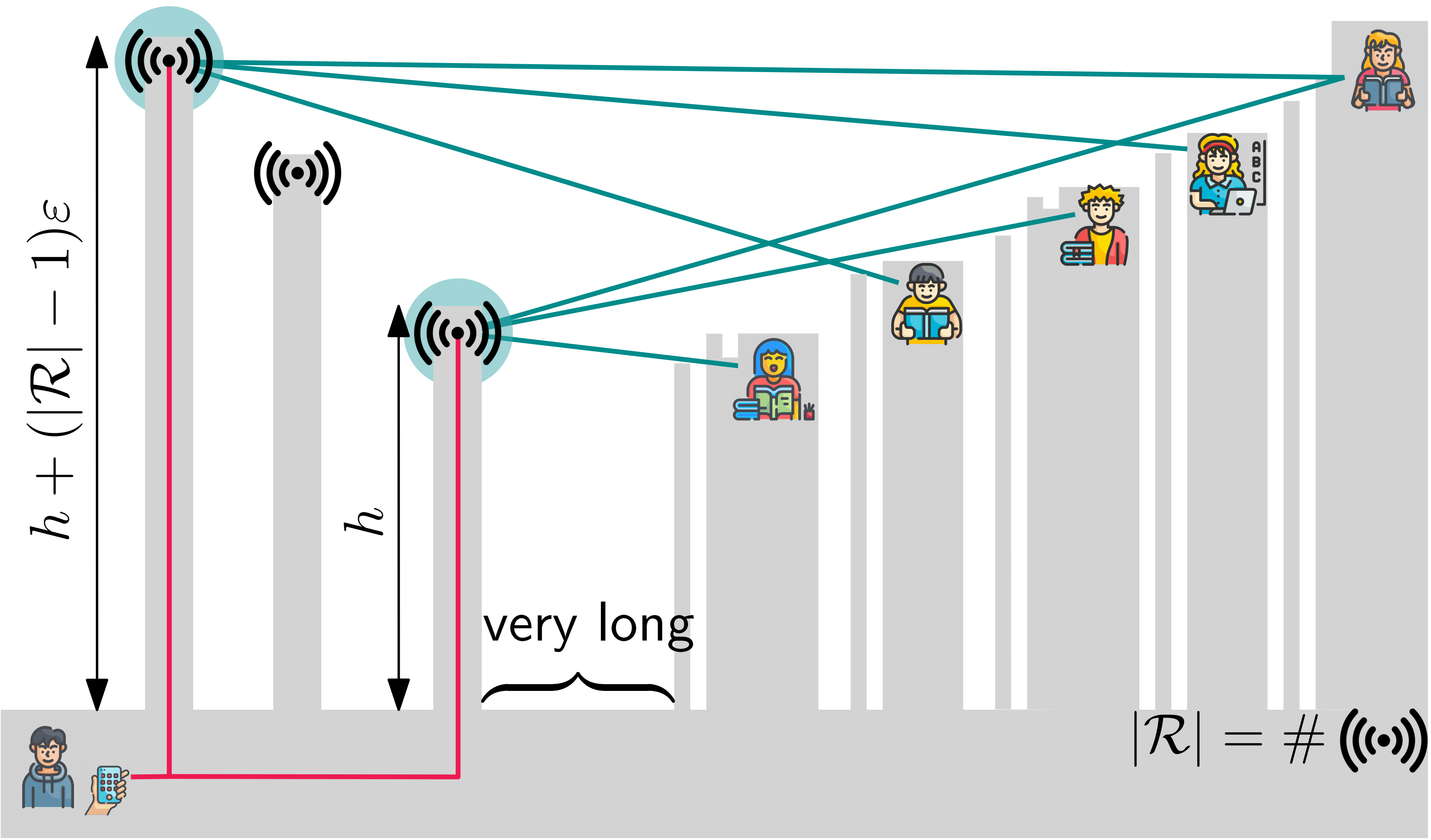

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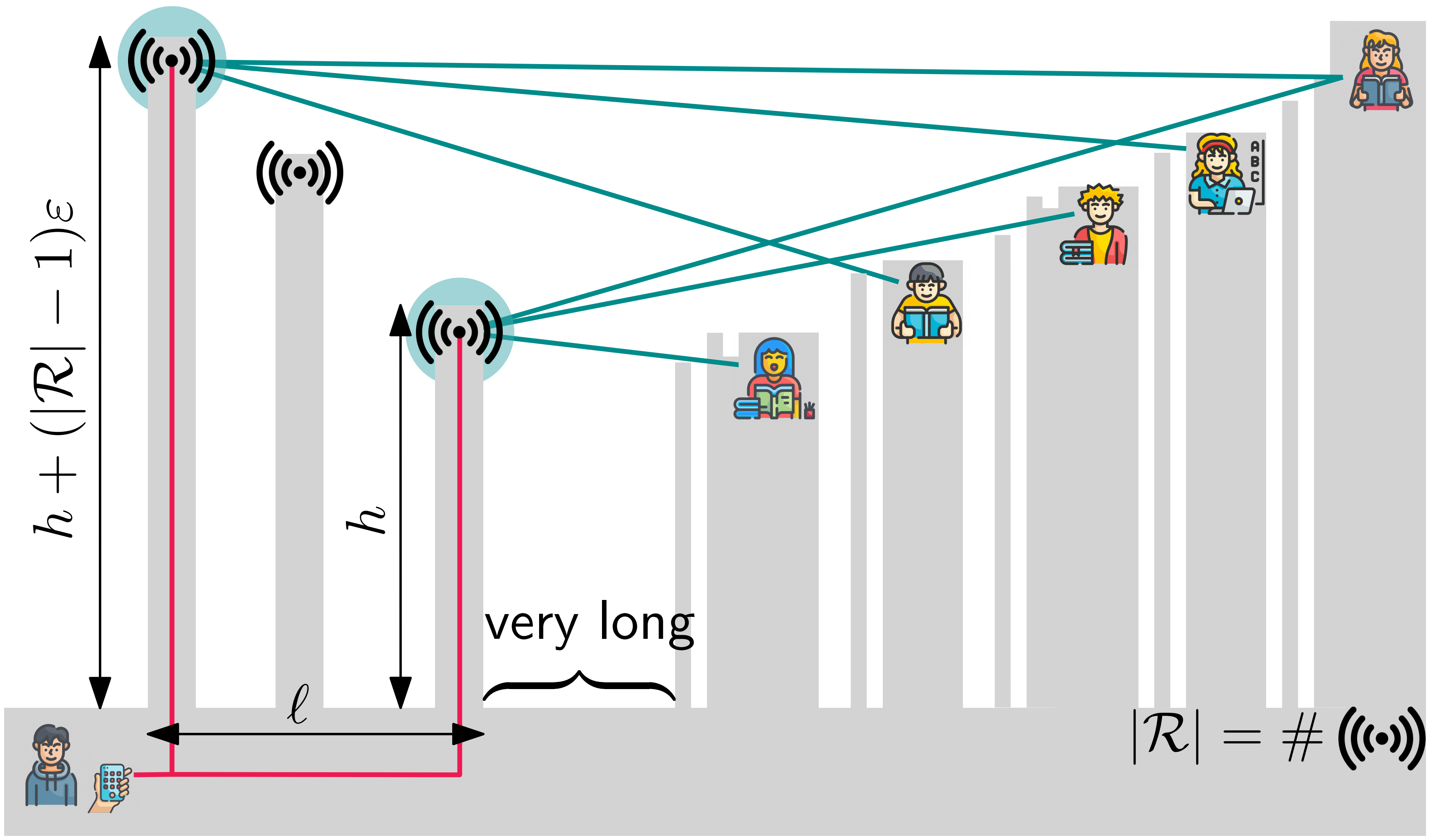

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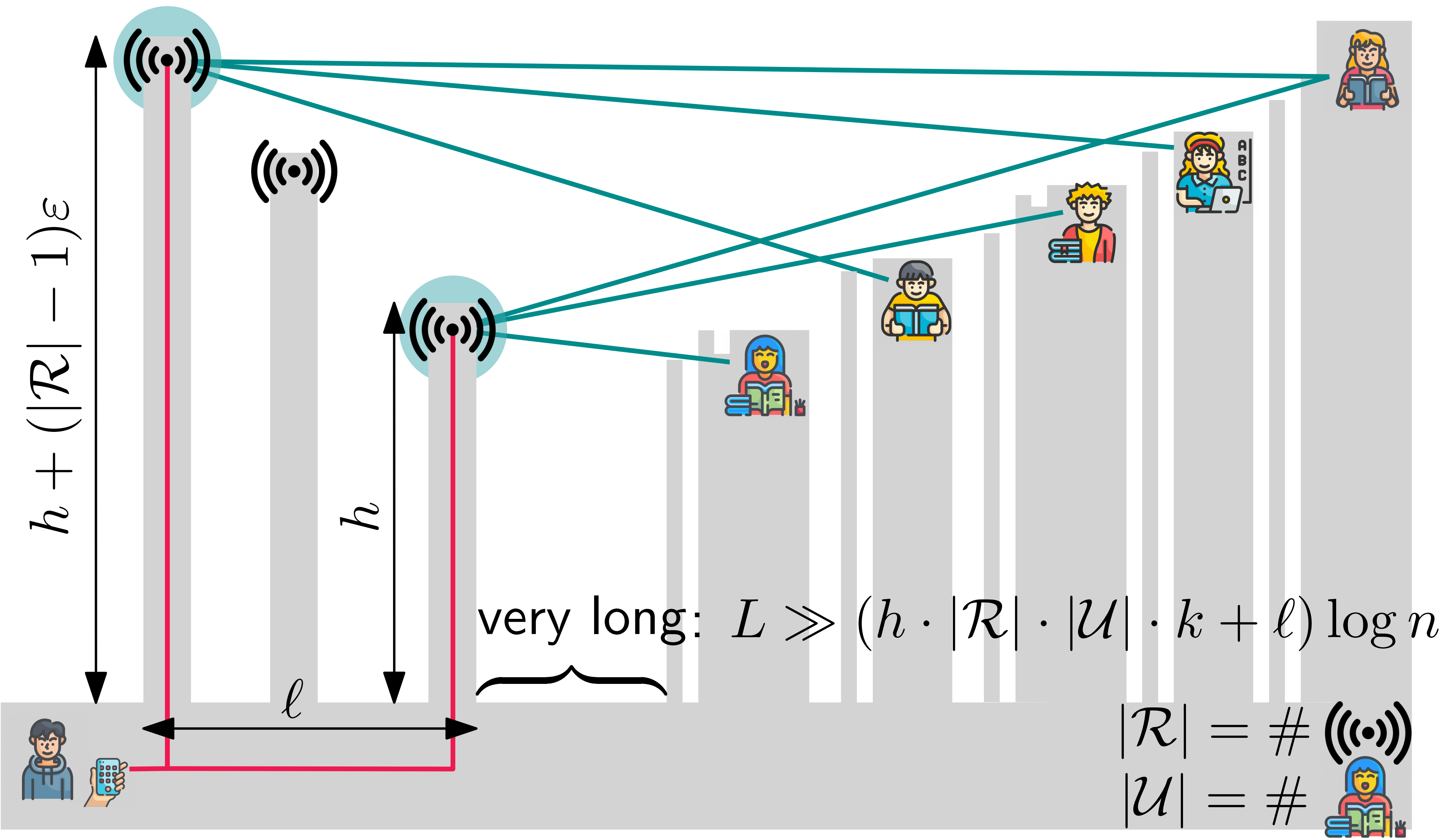
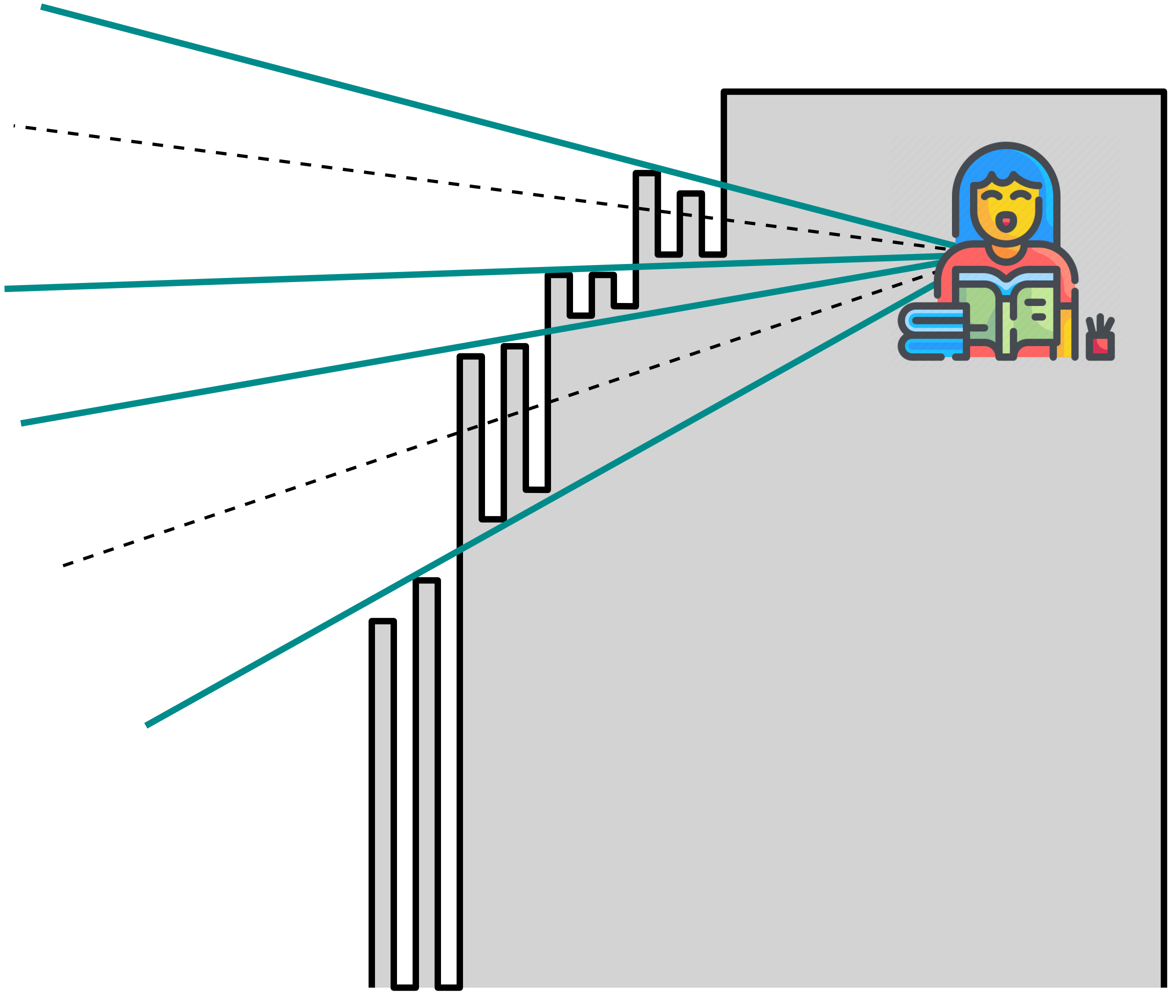


NP-Hardness for Histograms

block visibility



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block visibility

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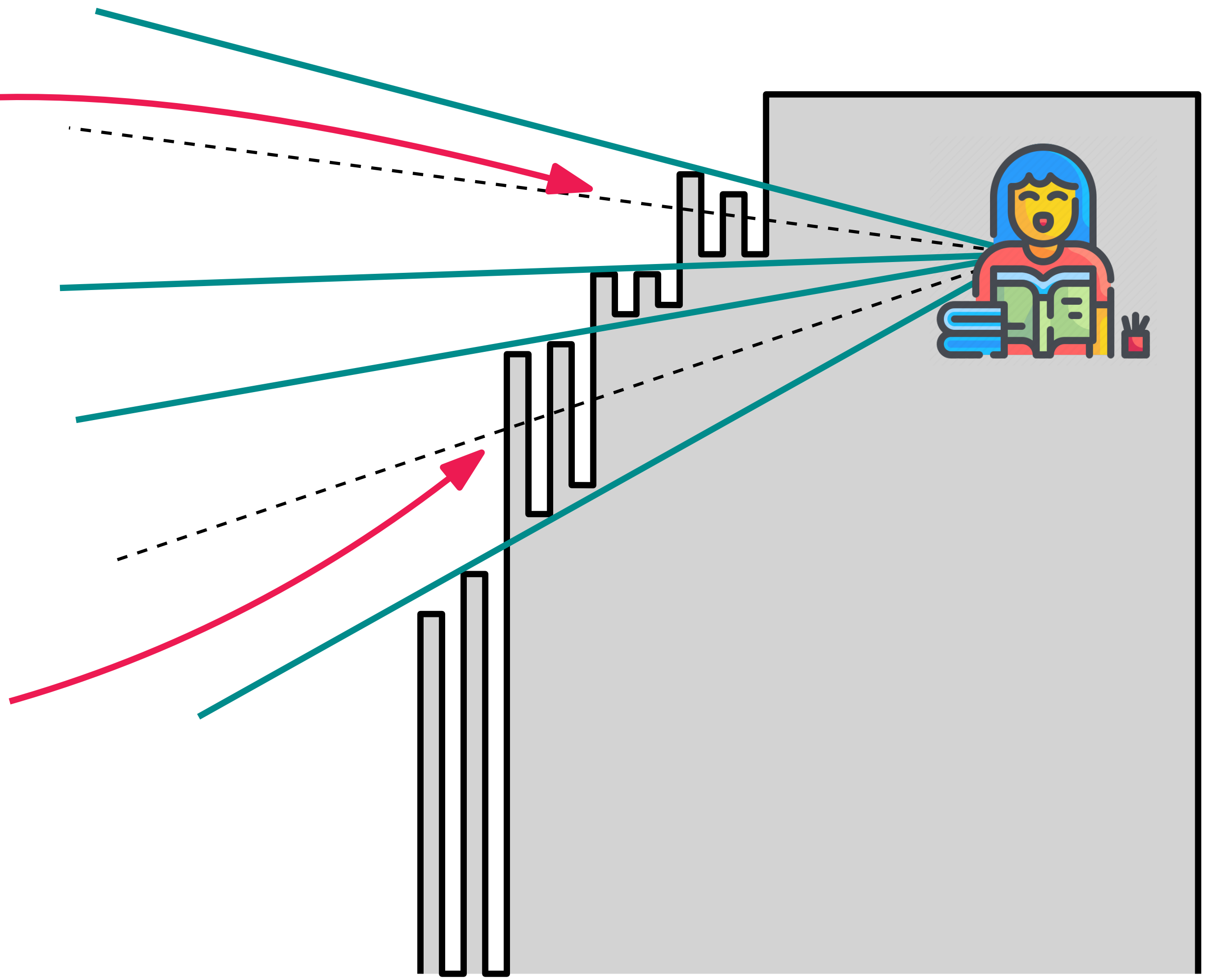

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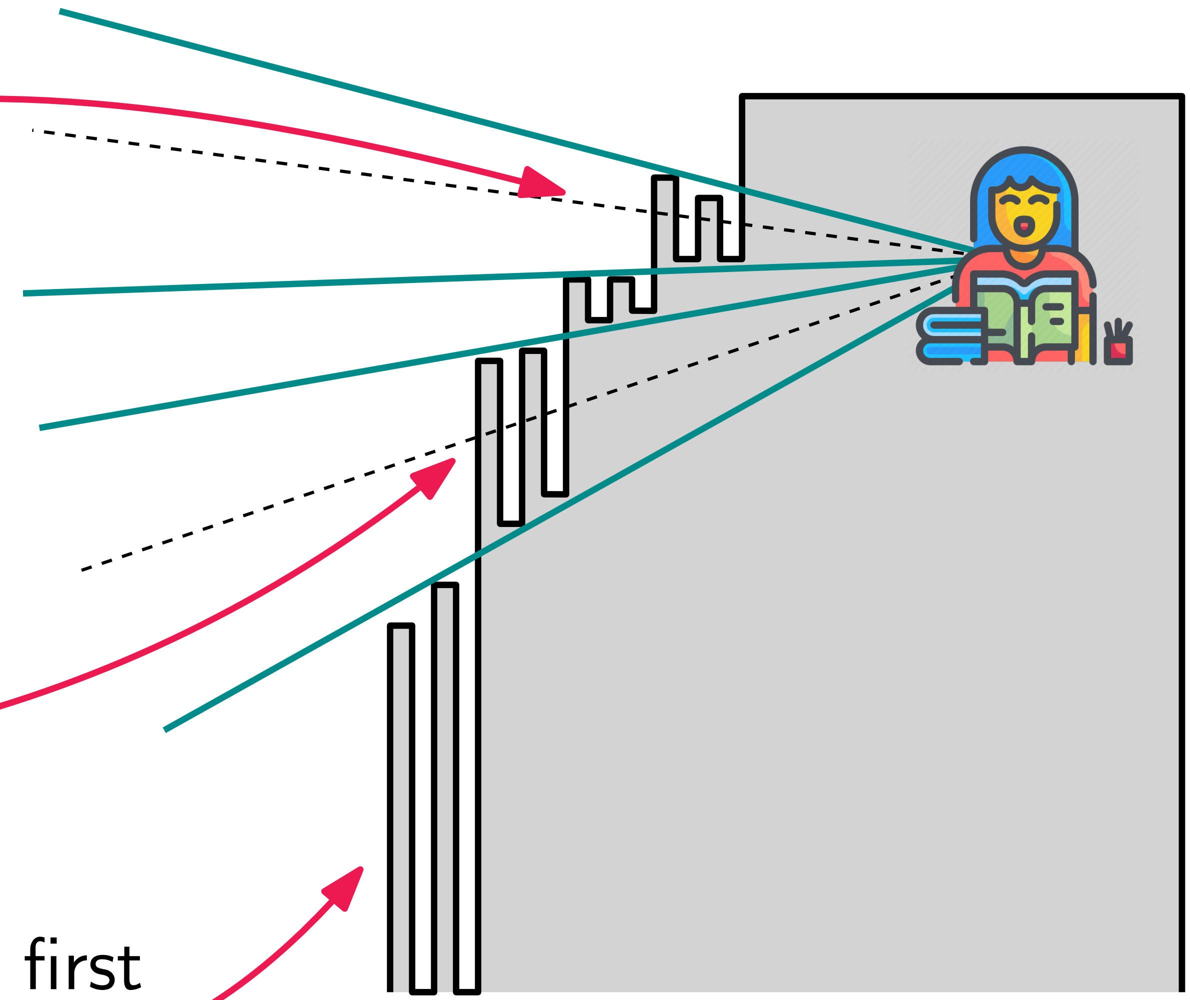

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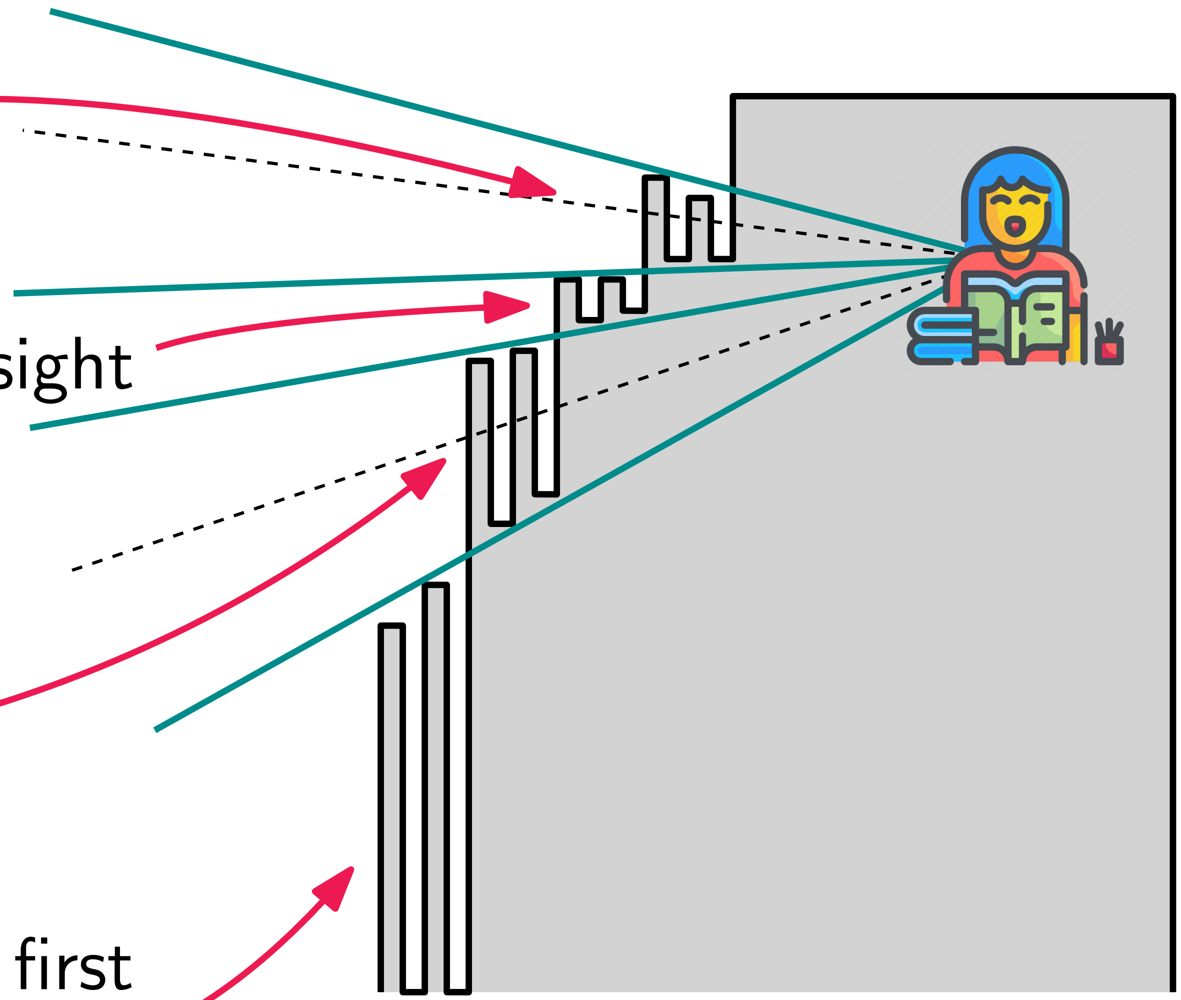
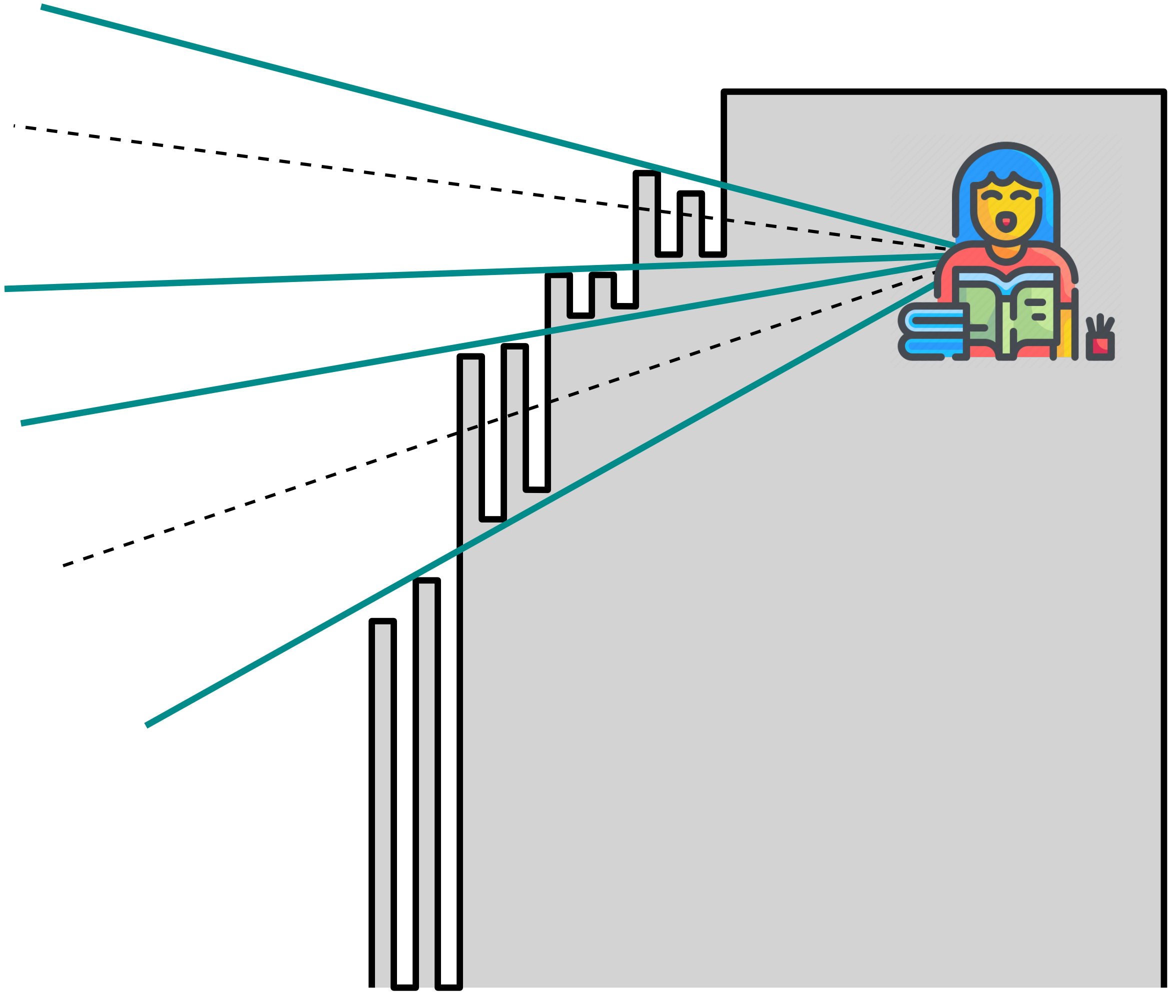


NP-Hardness for Histograms

Integer Coordinates



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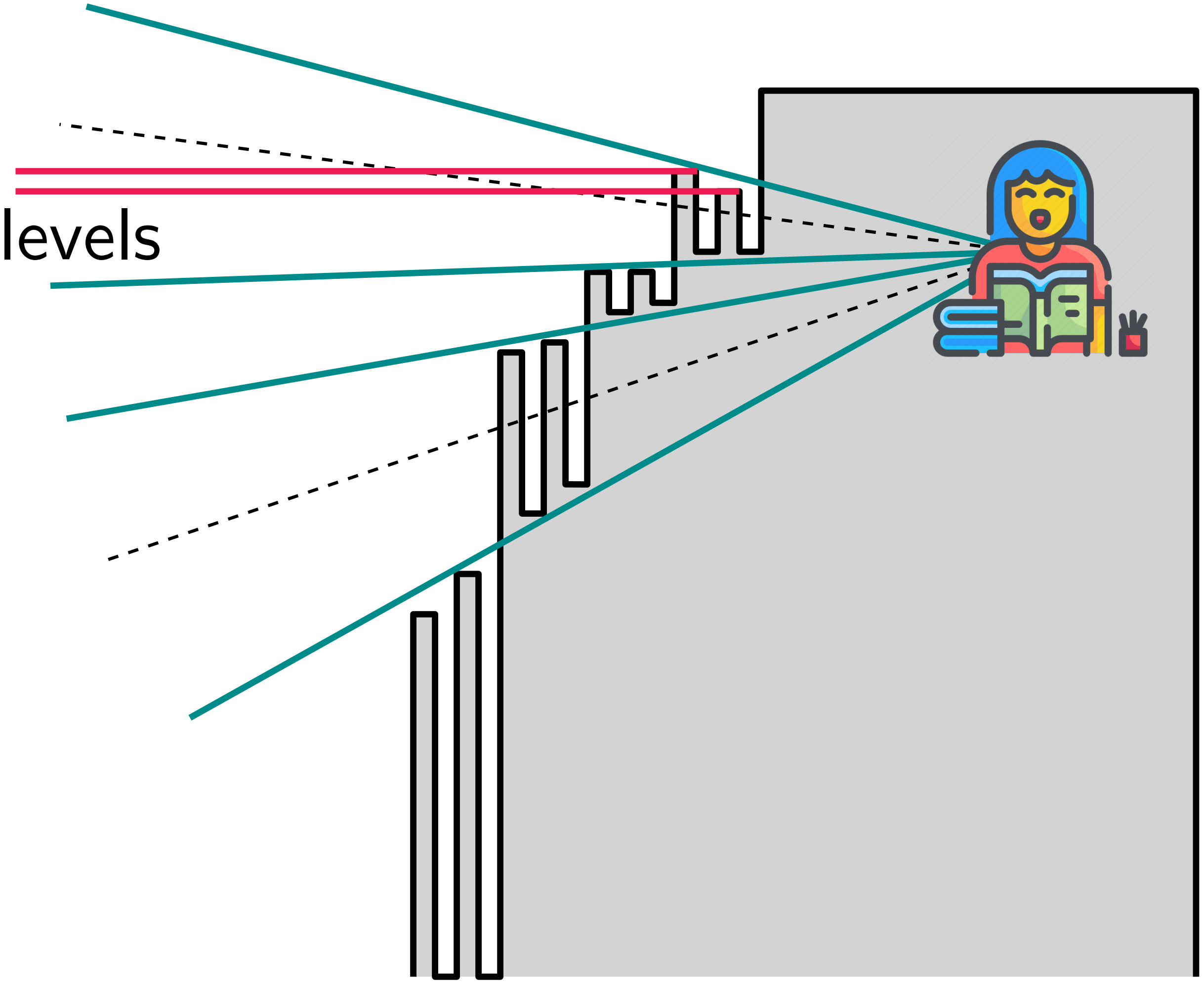
Integer Coordinates

$O(k \cdot |\mathcal{R}|)$
integral y-levels



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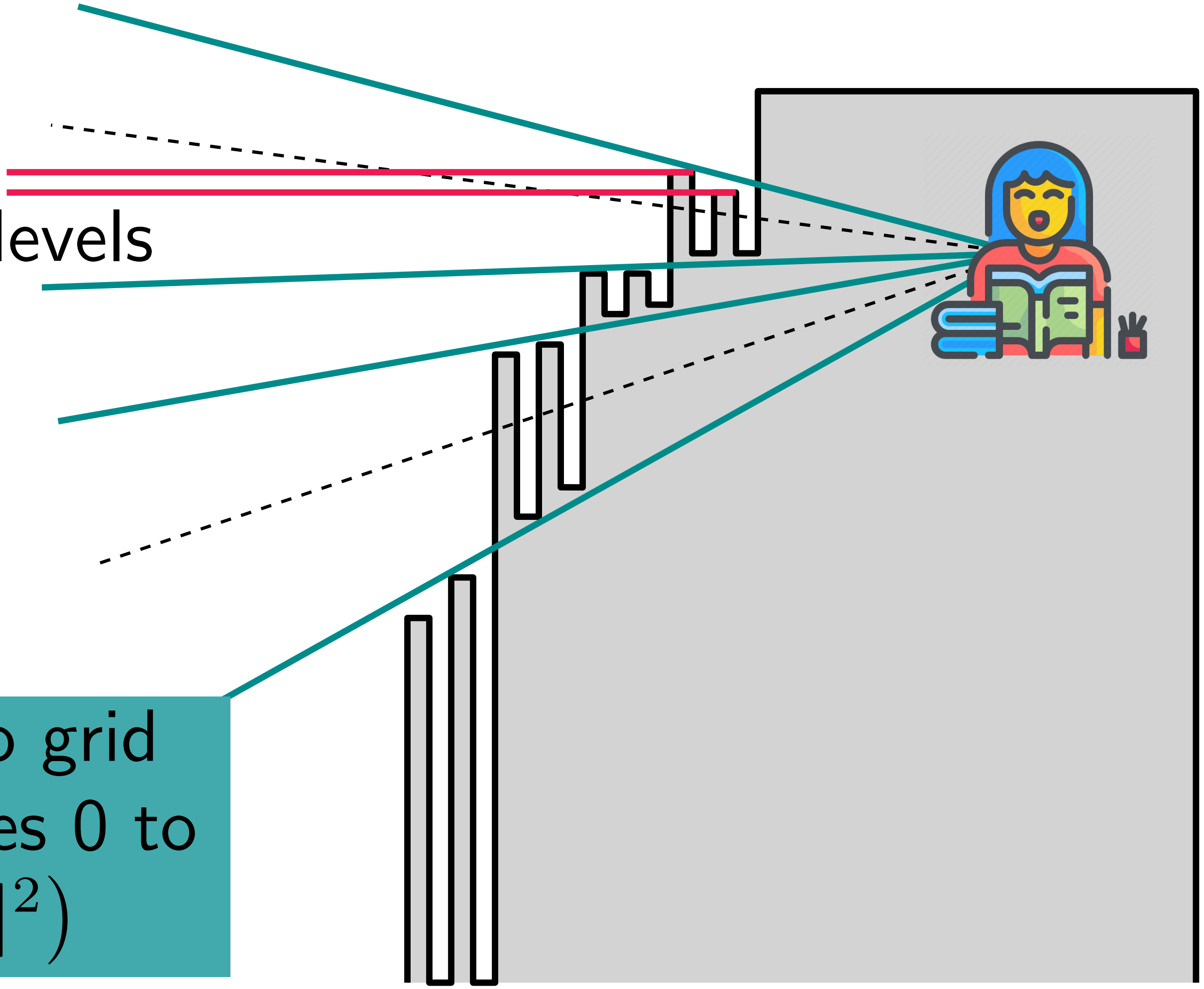
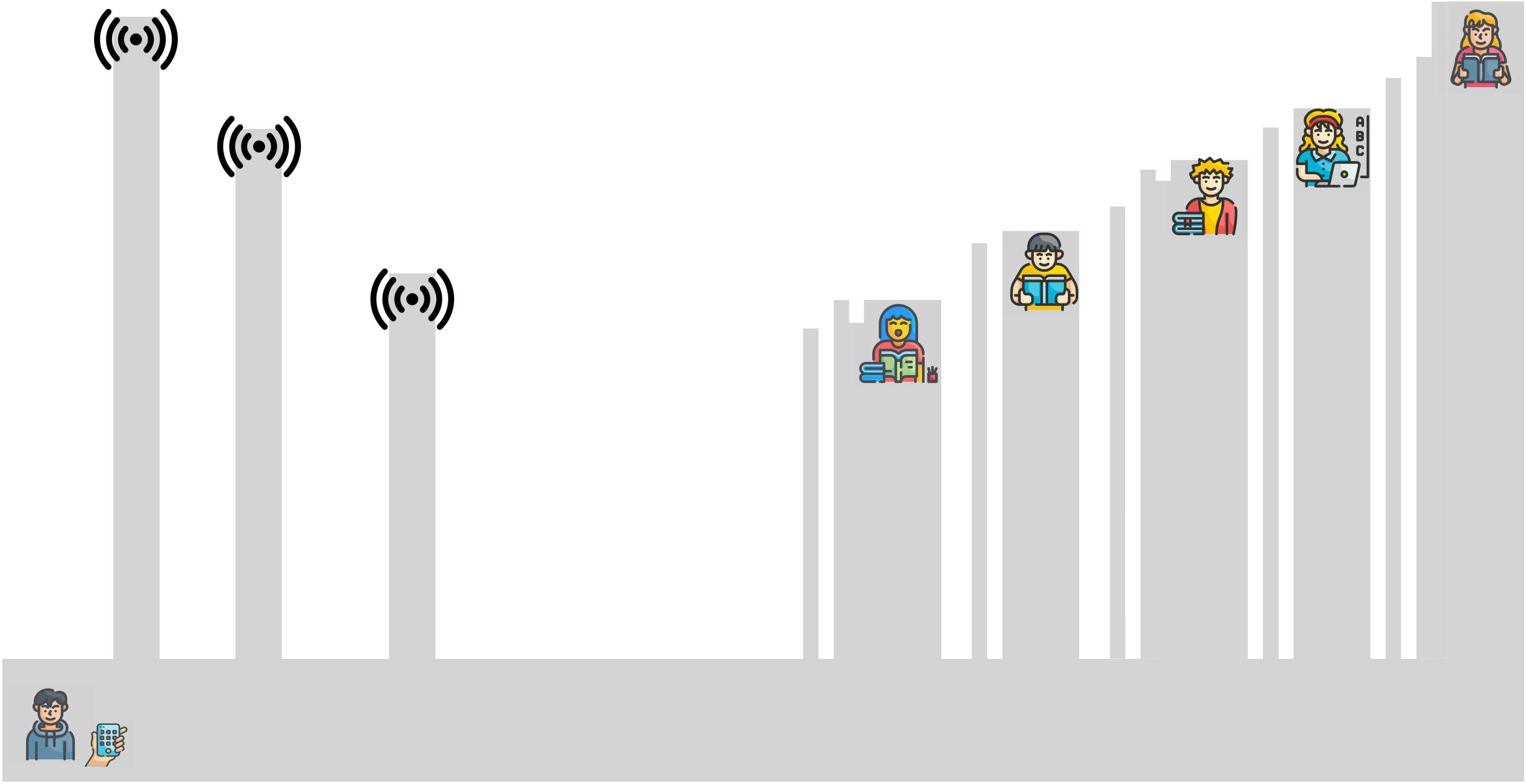


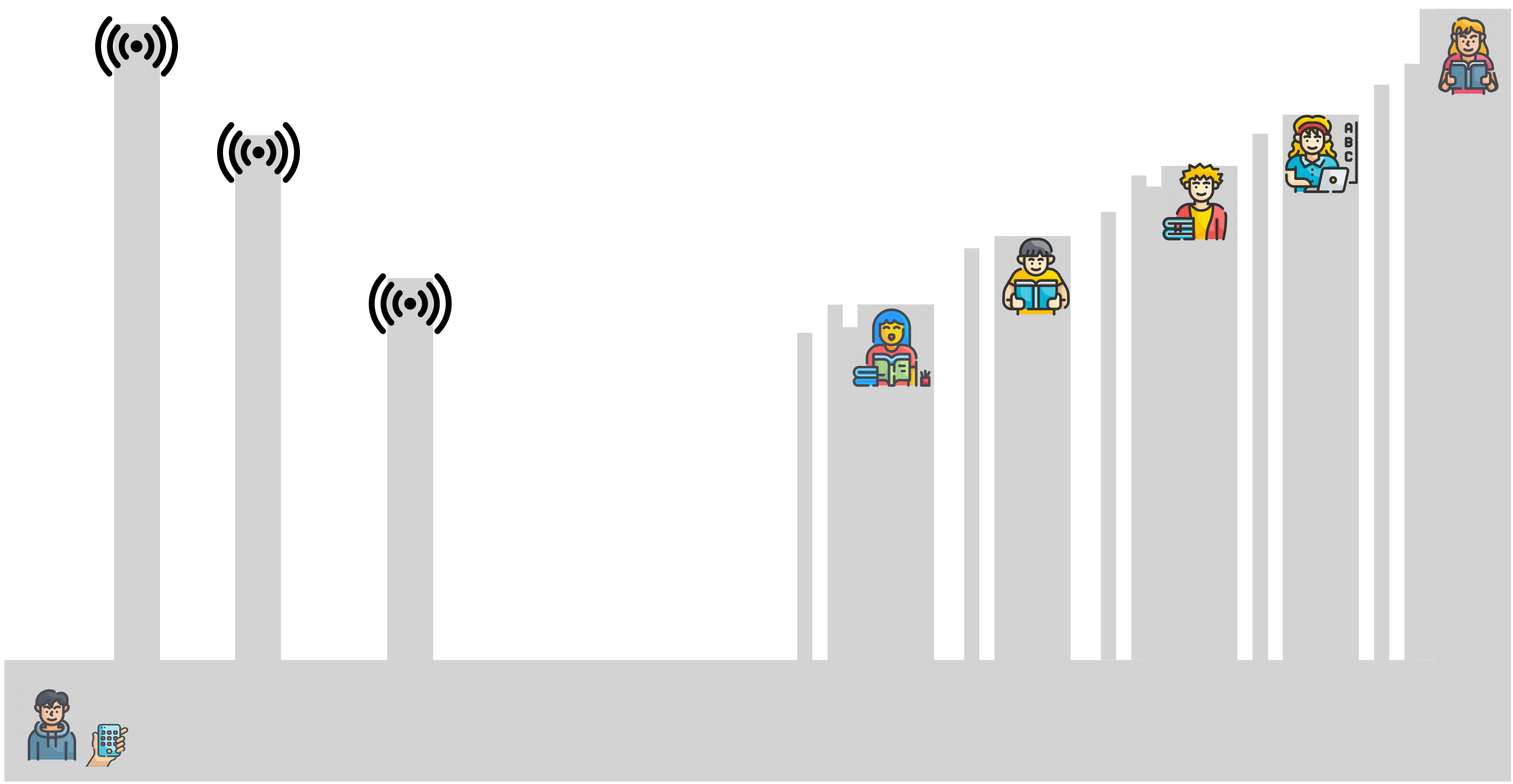
fit polygon into grid with coordinates 0 to $O\left(k^2 \cdot |\mathcal{U}| \cdot |\mathcal{R}|^2\right)$

Inapproximability



Inapproximability

- **Set Cover** cannot be approximated to within a factor $(1 - o(1)) \ln |\mathcal{U}|$ in polynomial time (unless P$=$ NP).



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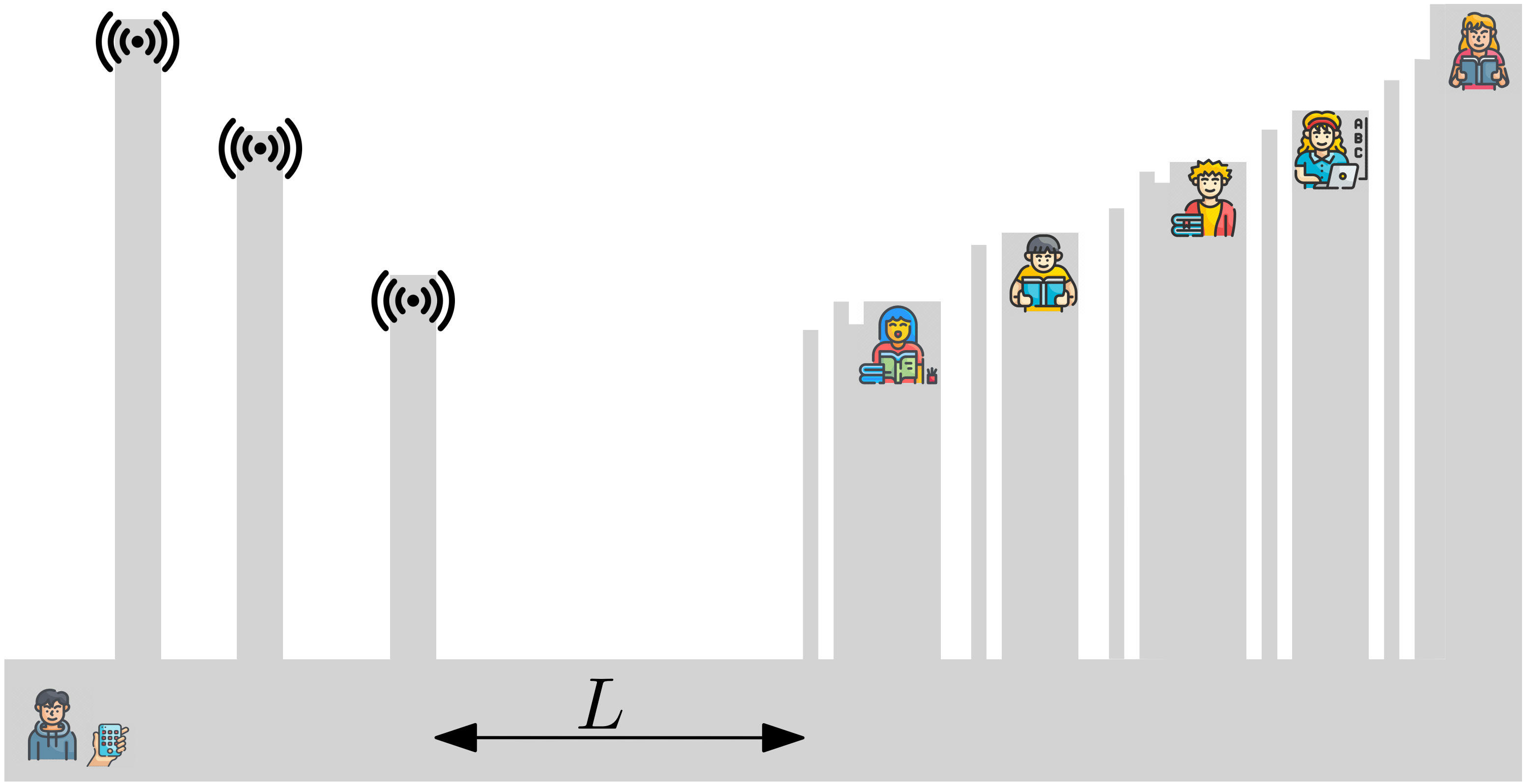
- Length of the corridor $L \gg (h \cdot |\mathcal{R}| \cdot |\mathcal{U}| \cdot k + \ell) \log n$



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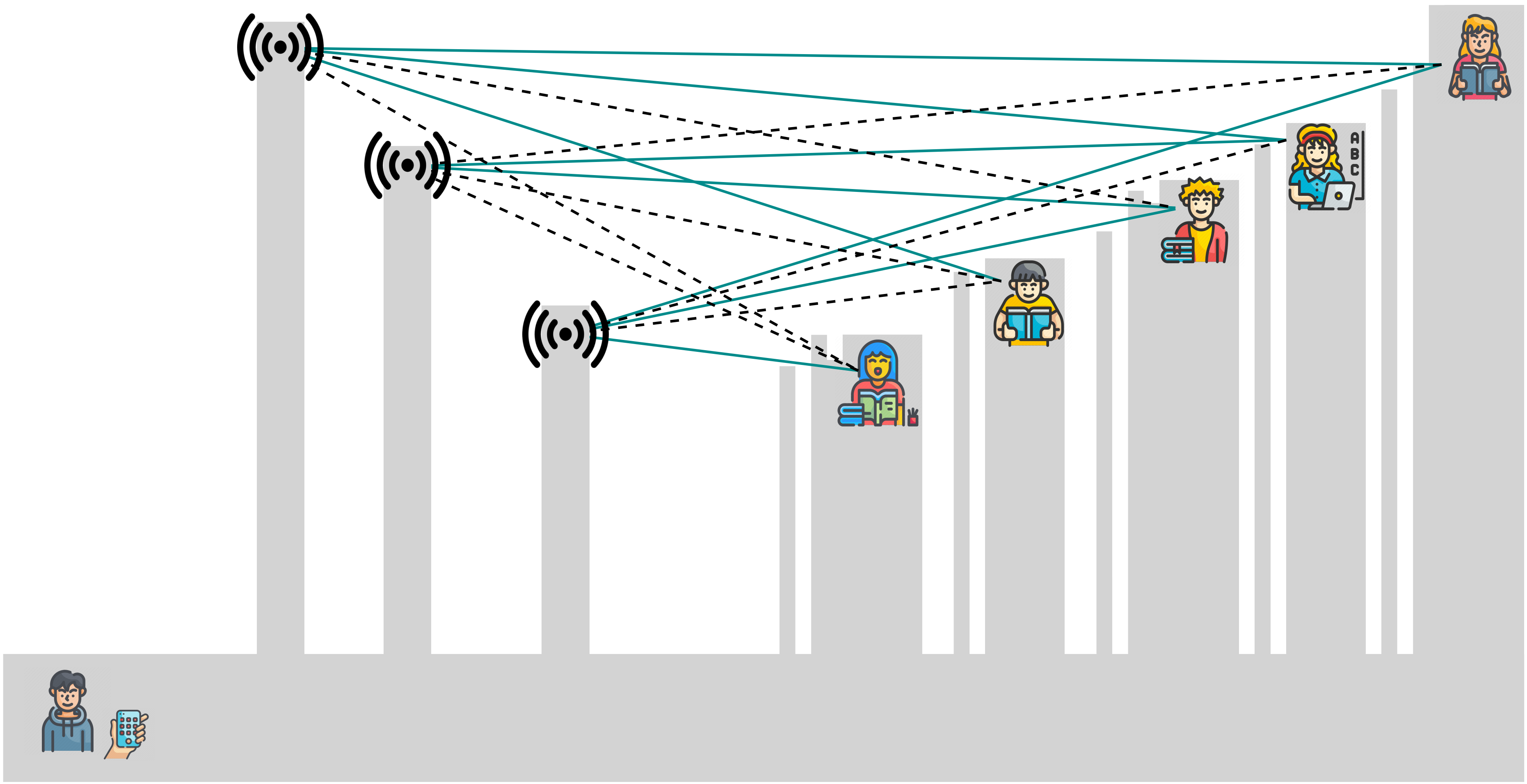

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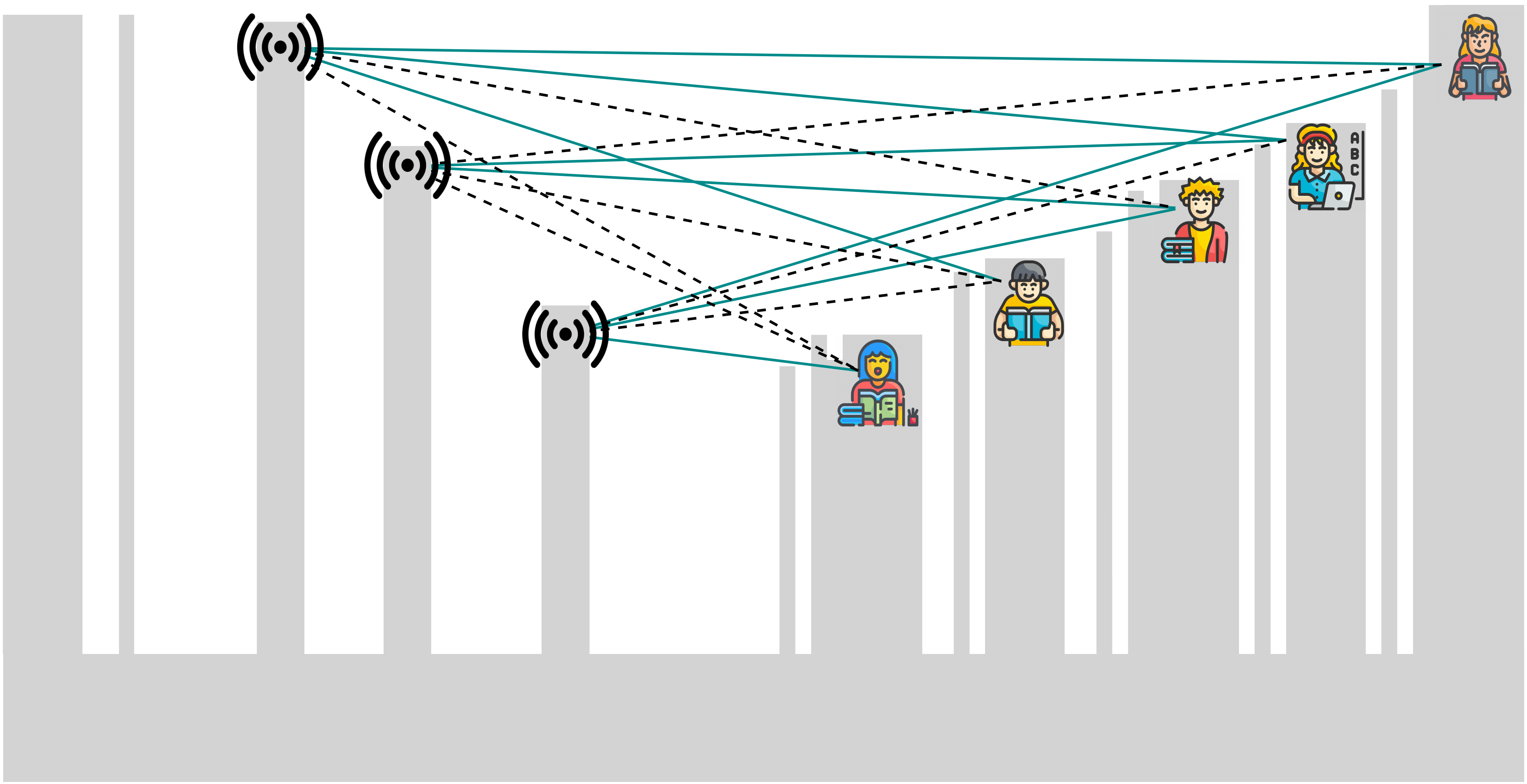

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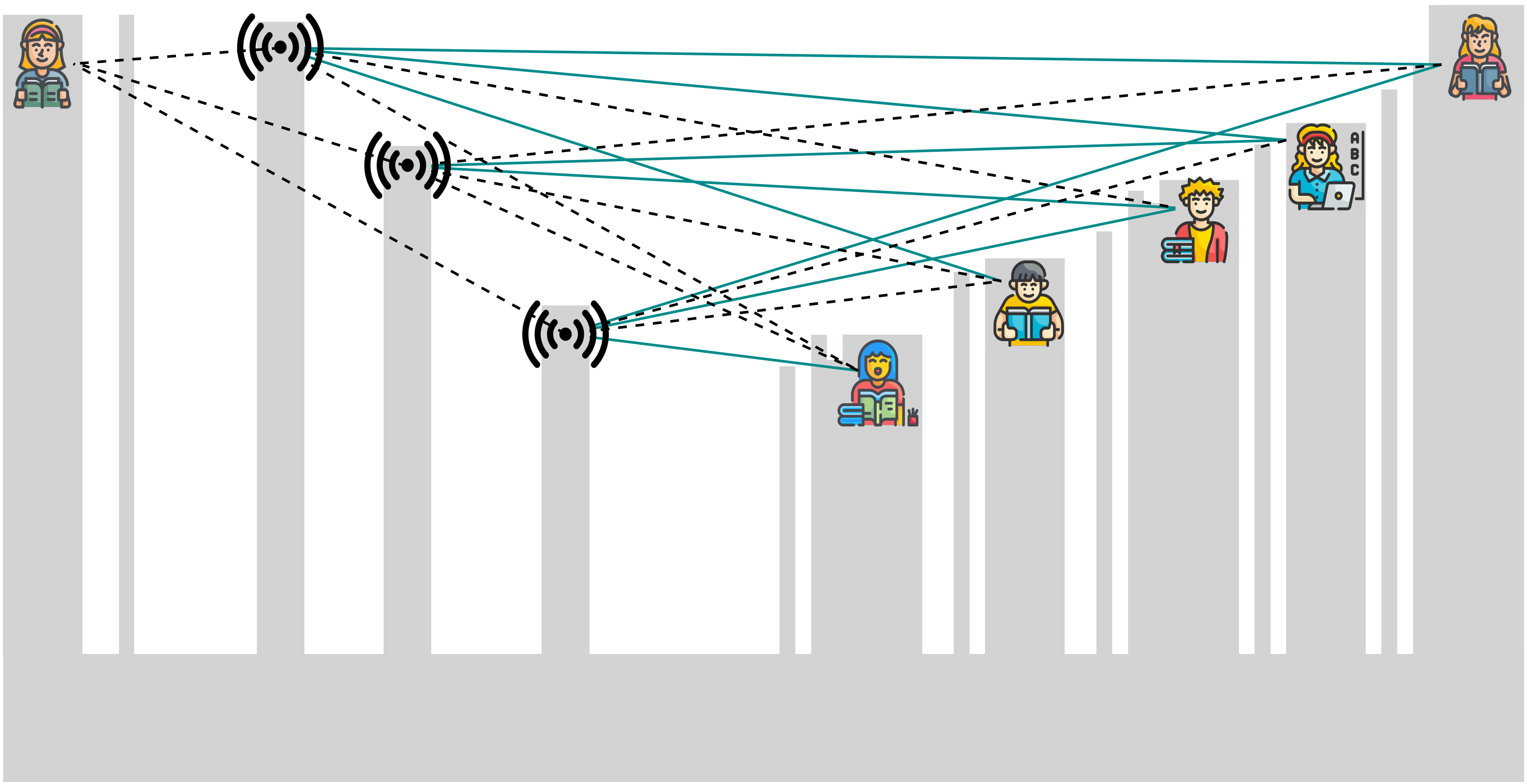

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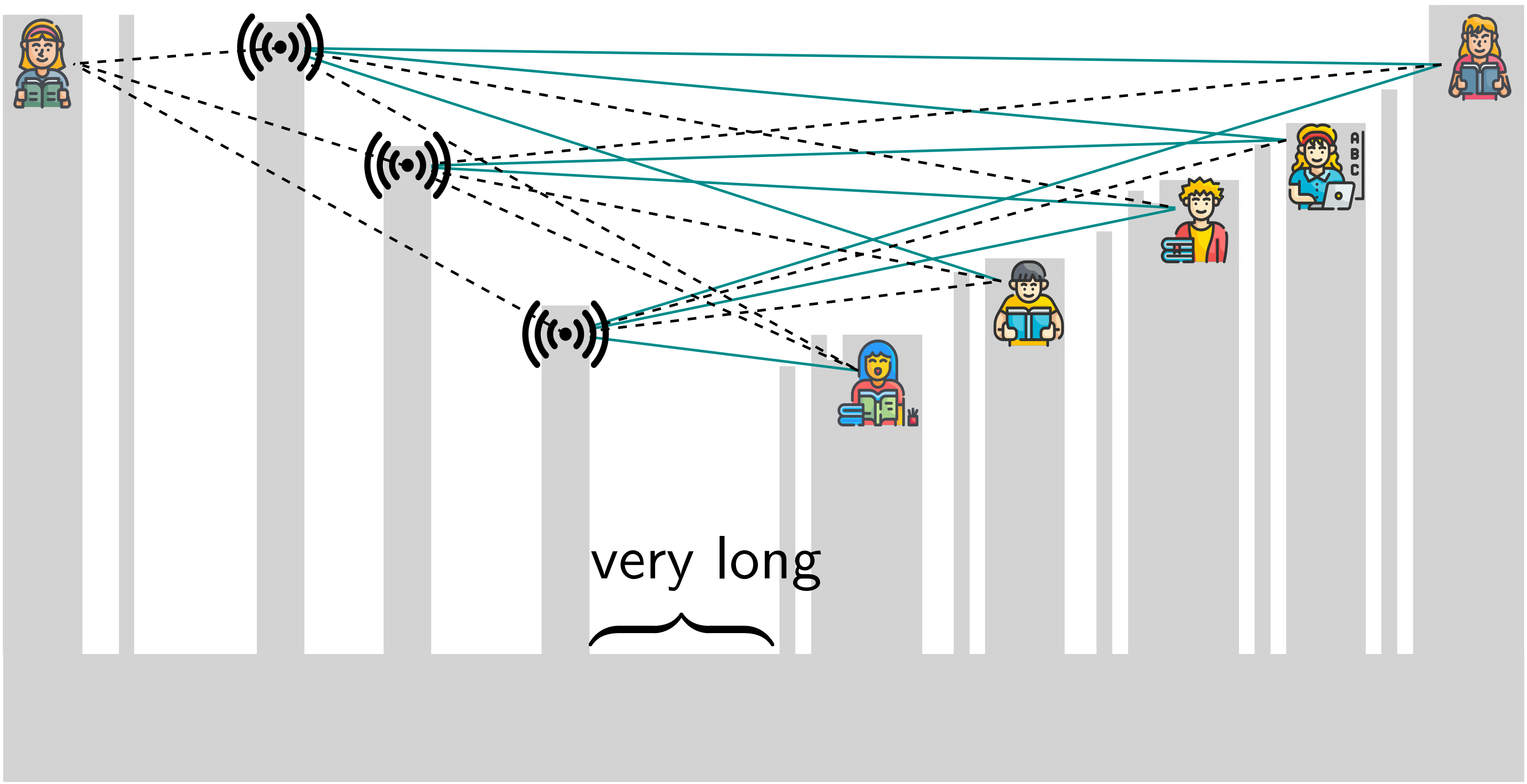

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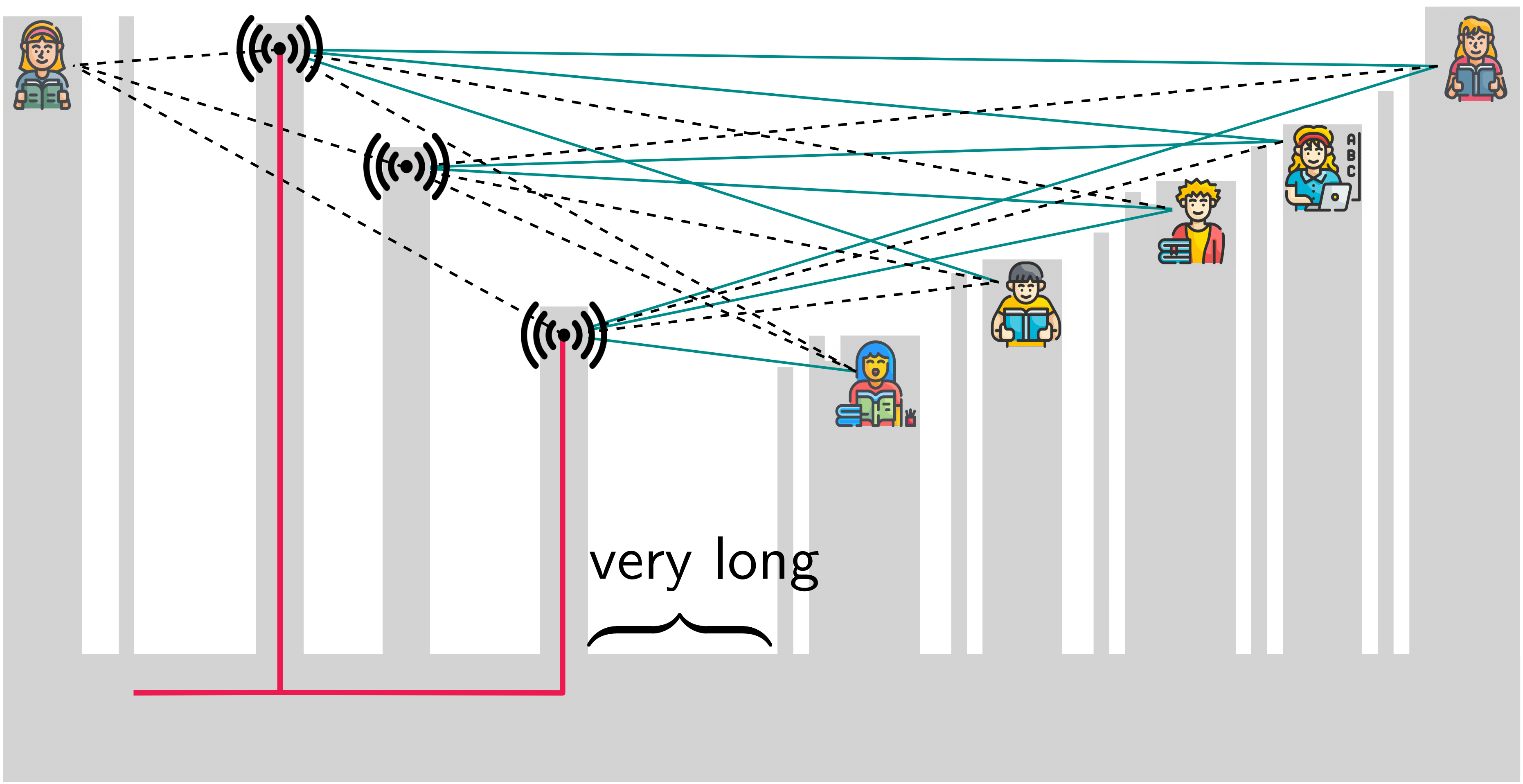

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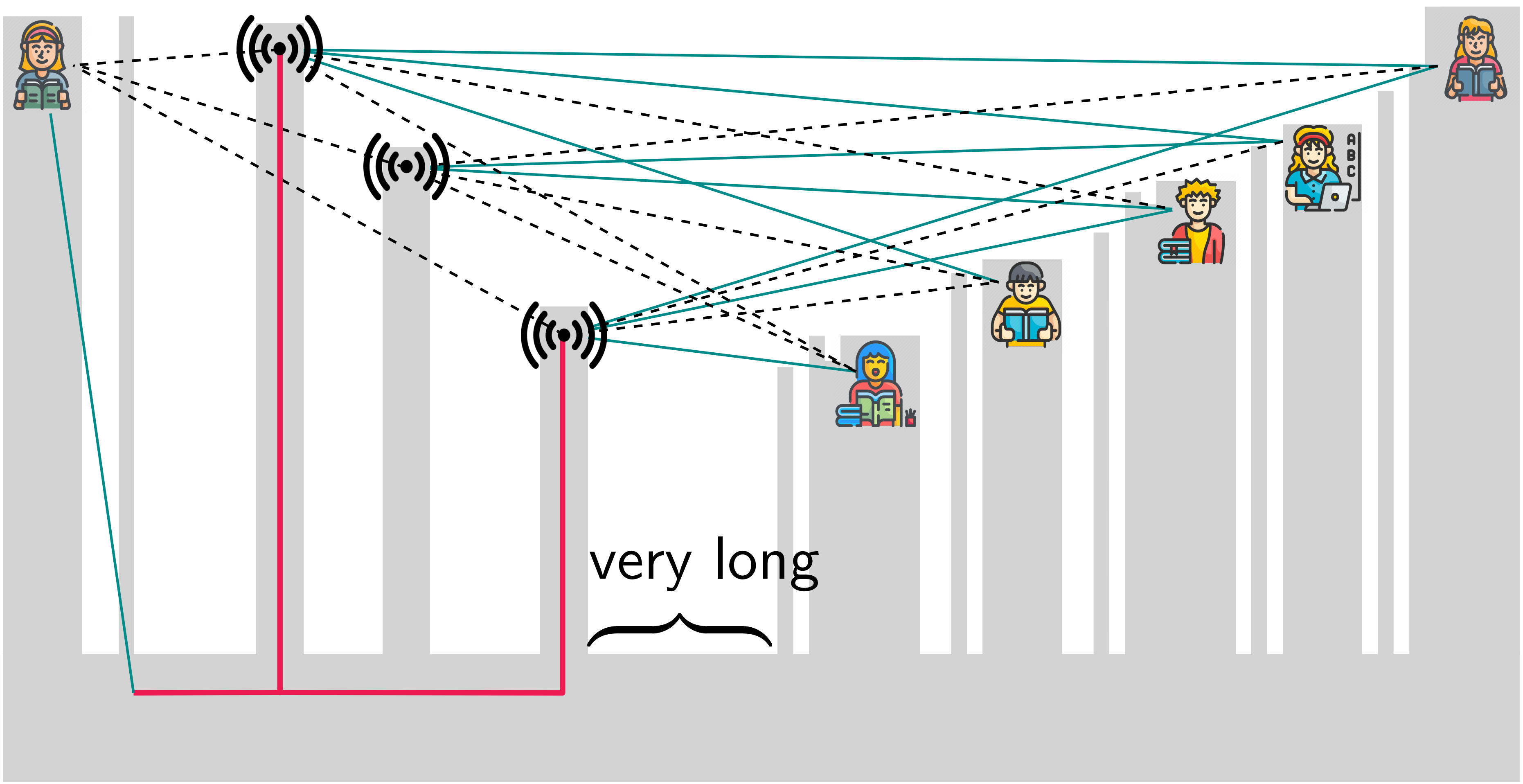

Uni-Monotone Polygons

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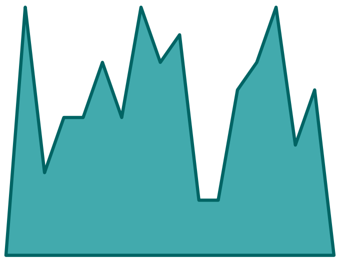
- it is x-*monotone*: any vertical line intersects P in at most one connected component
- either the upper or the lower chain is a horizontal segment



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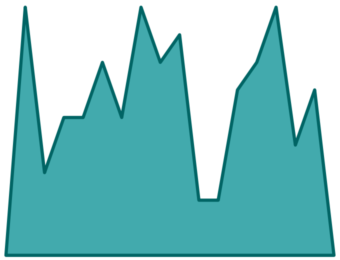
$\Rightarrow$ Histograms are uni-monotone

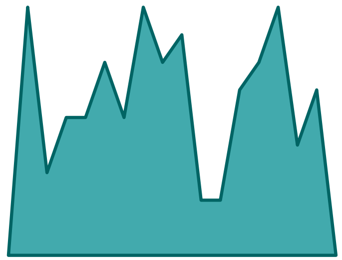


Uni-Monotone Polygons

Polygon P is called *uni-monotone* if

- it is x-*monotone*: any vertical line intersects P in at most one connected component
- either the upper or the lower chain is a horizontal segment

$\Rightarrow$ Histograms are uni-monotone



Corollary. For any $k \geq 2$, k-TrWRP(S, P, s) and k-TrWRP(S, P) are NP-hard for uni-monotone polygons and cannot be approximated within a logarithmic factor $c \log n$, for any $c > 0$.

Star-Shaped Polygons

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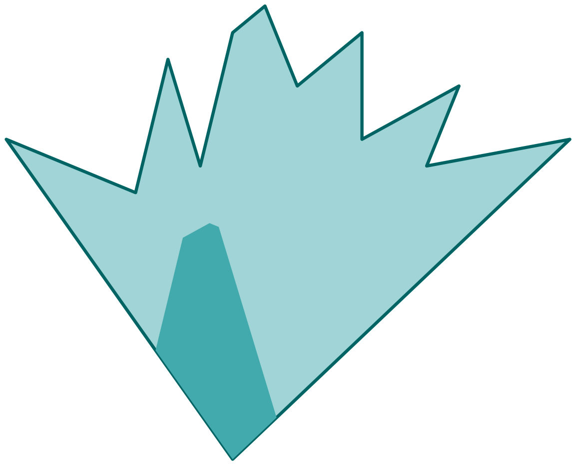
A polygon P is *star-shaped* if it contains a region, called the *kernel*, from which every point in P is 0-seen.



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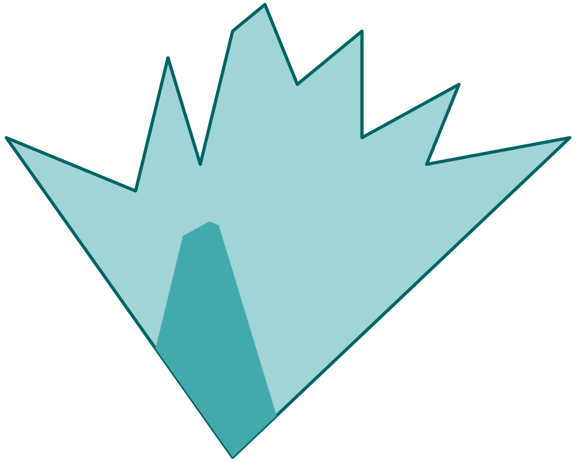
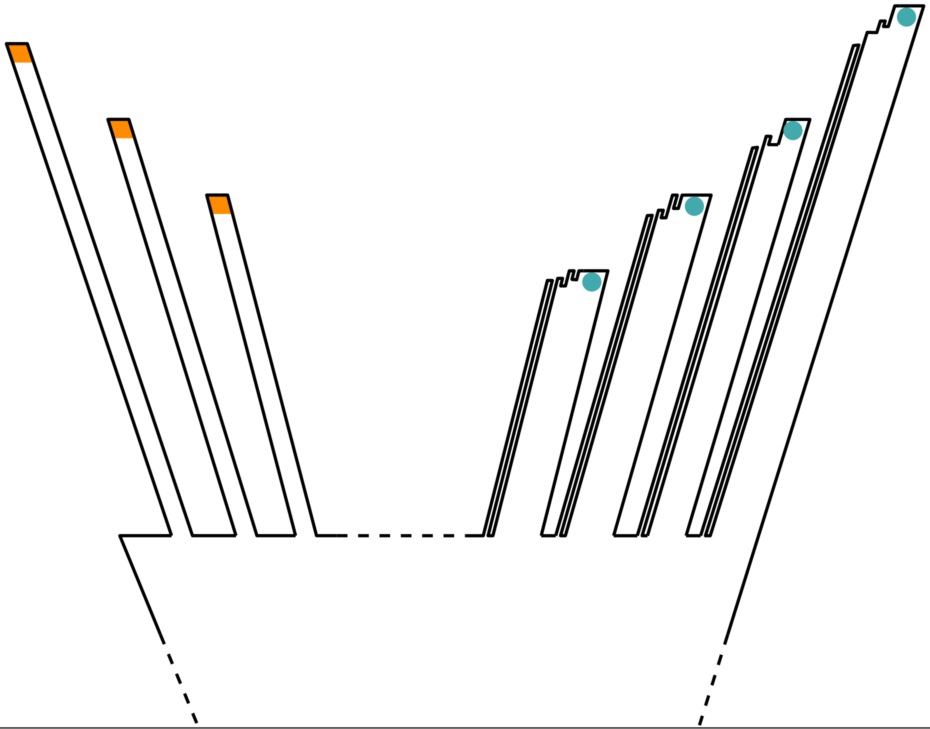


Modification to histogram: slightly stretch it



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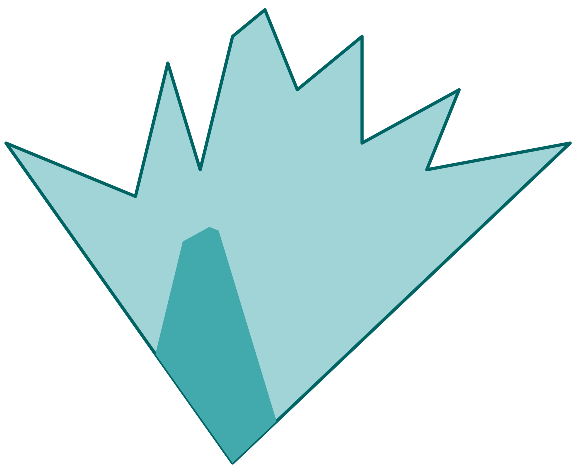

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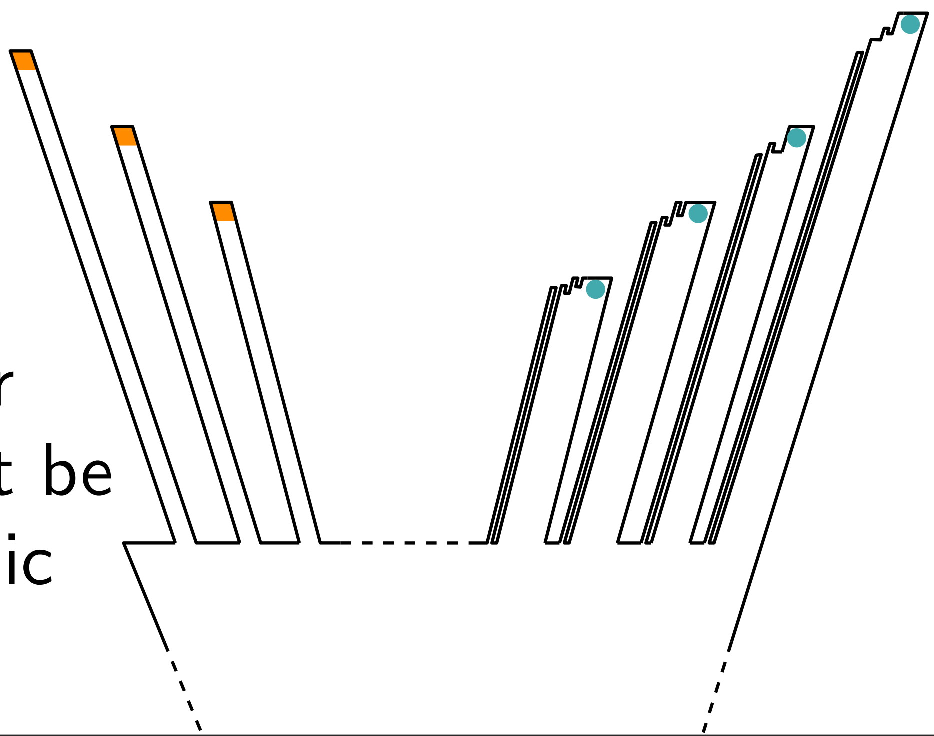

Corollary. For any $k \geq 2$, k-TrWRP(S, P, s) is NP-hard for star-shaped polygons and cannot be approximated within a logarithmic factor $c \log n$, for any $c > 0$.

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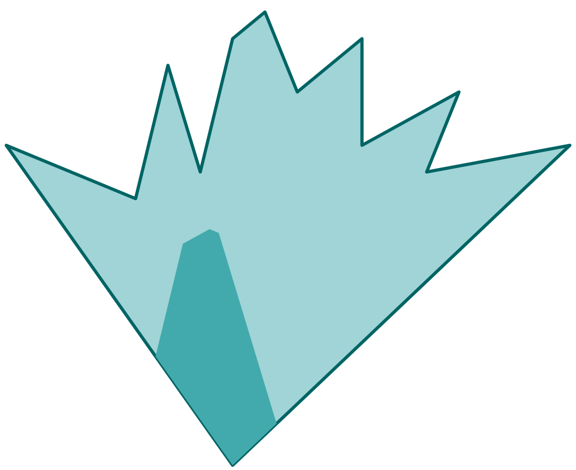

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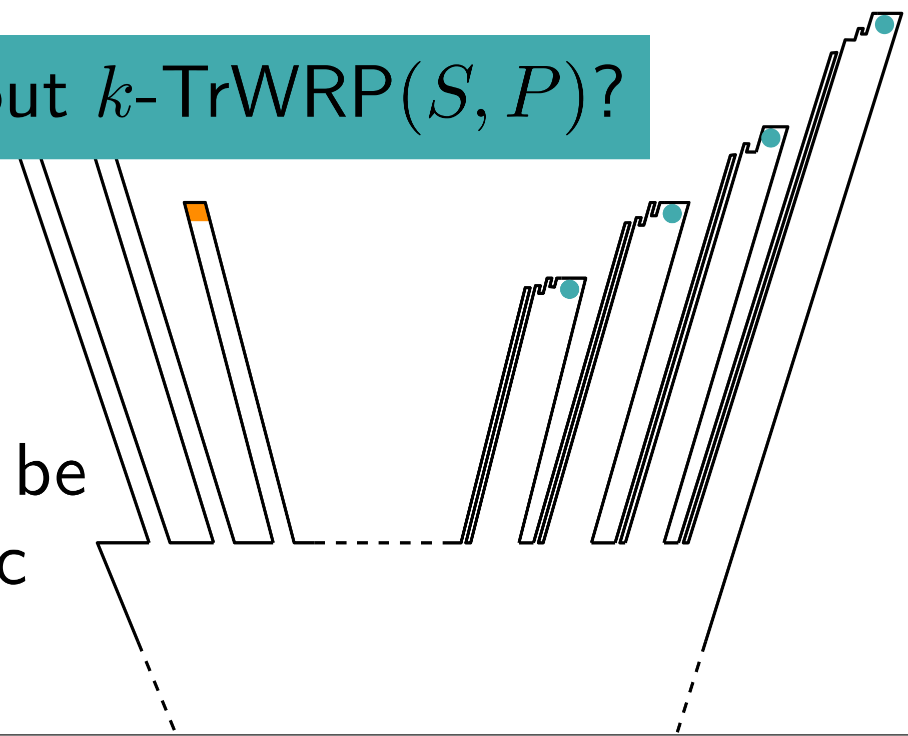
What about k-TrWRP(S, P)?



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Summary & Open Problems

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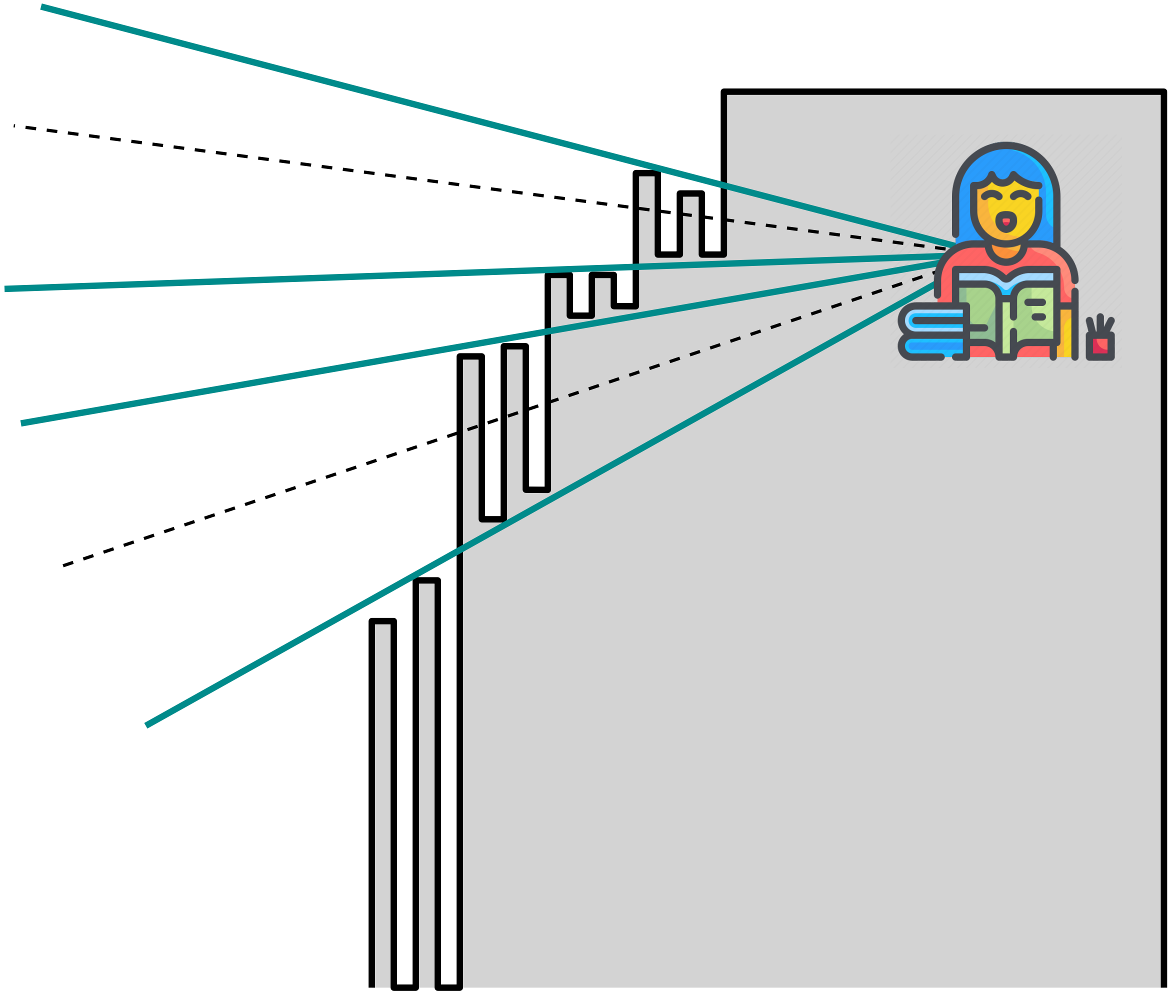
NP-Hardness for Histograms

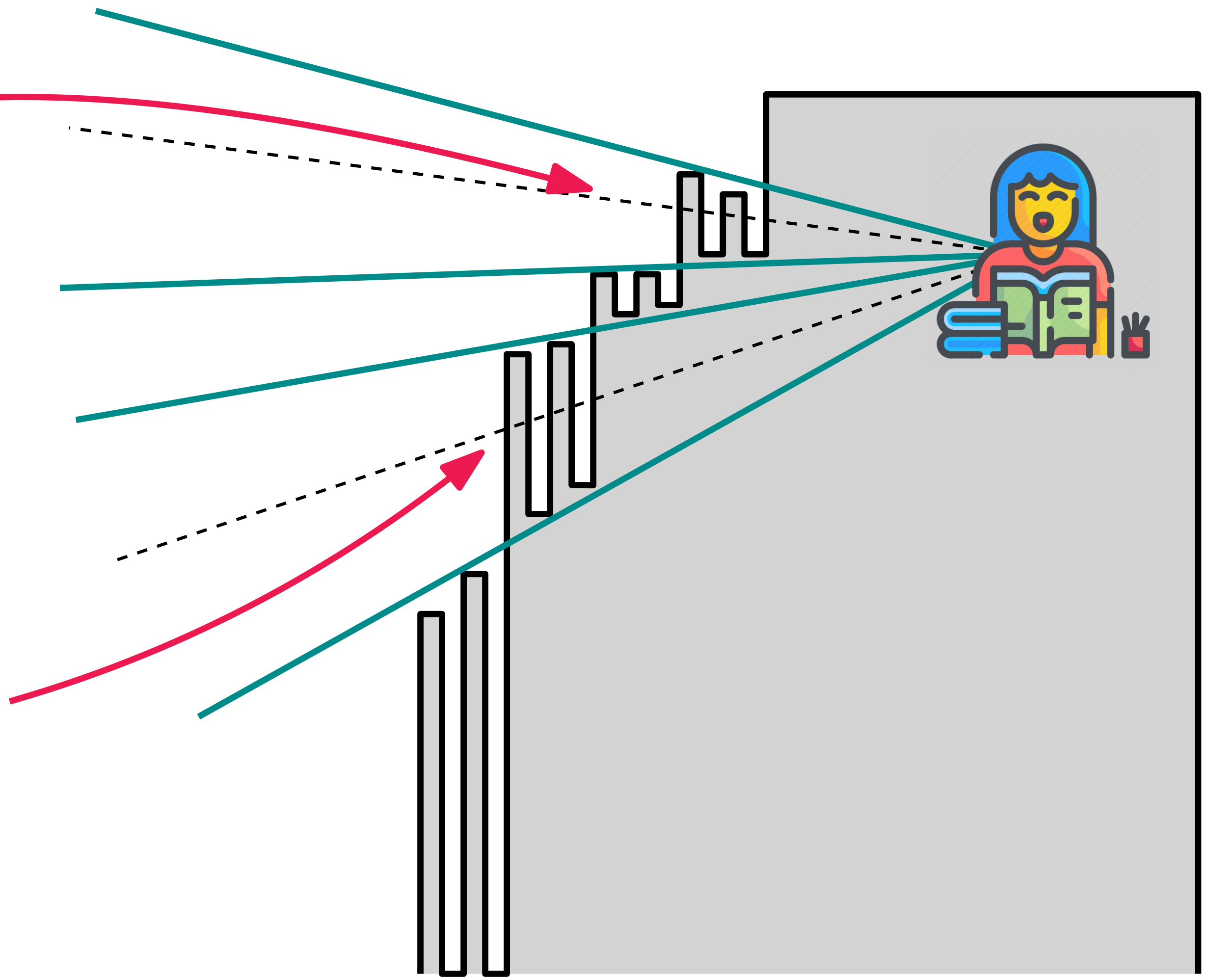
block visibility



NP-Hardness for Histograms

block visibility

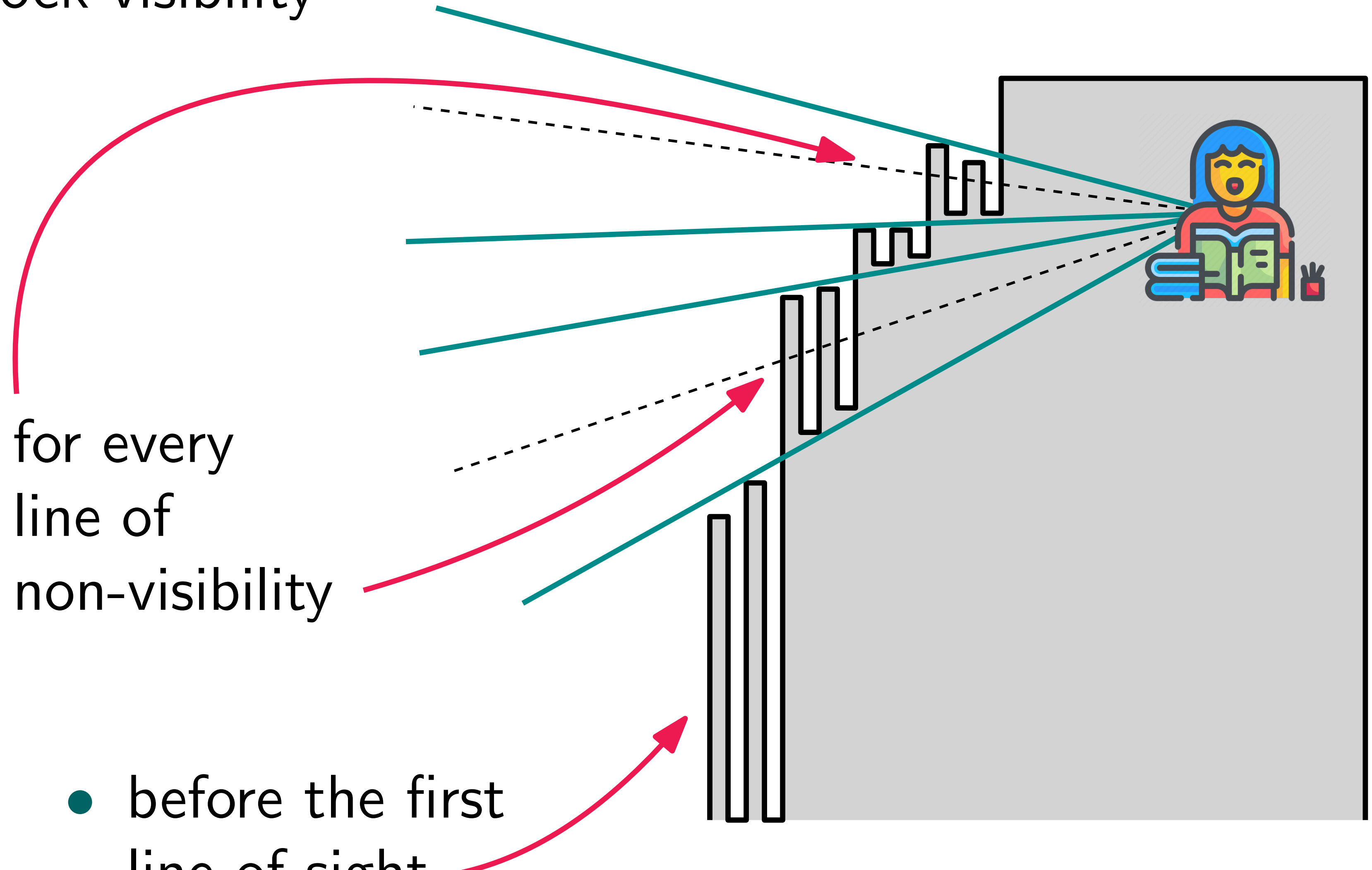
- for every line of non-visibility



NP-Hardness for Histograms

block visibility

- for every line of non-visibility
  - before the first line of sight



NP-Hardness for Histograms

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- between lines of sight

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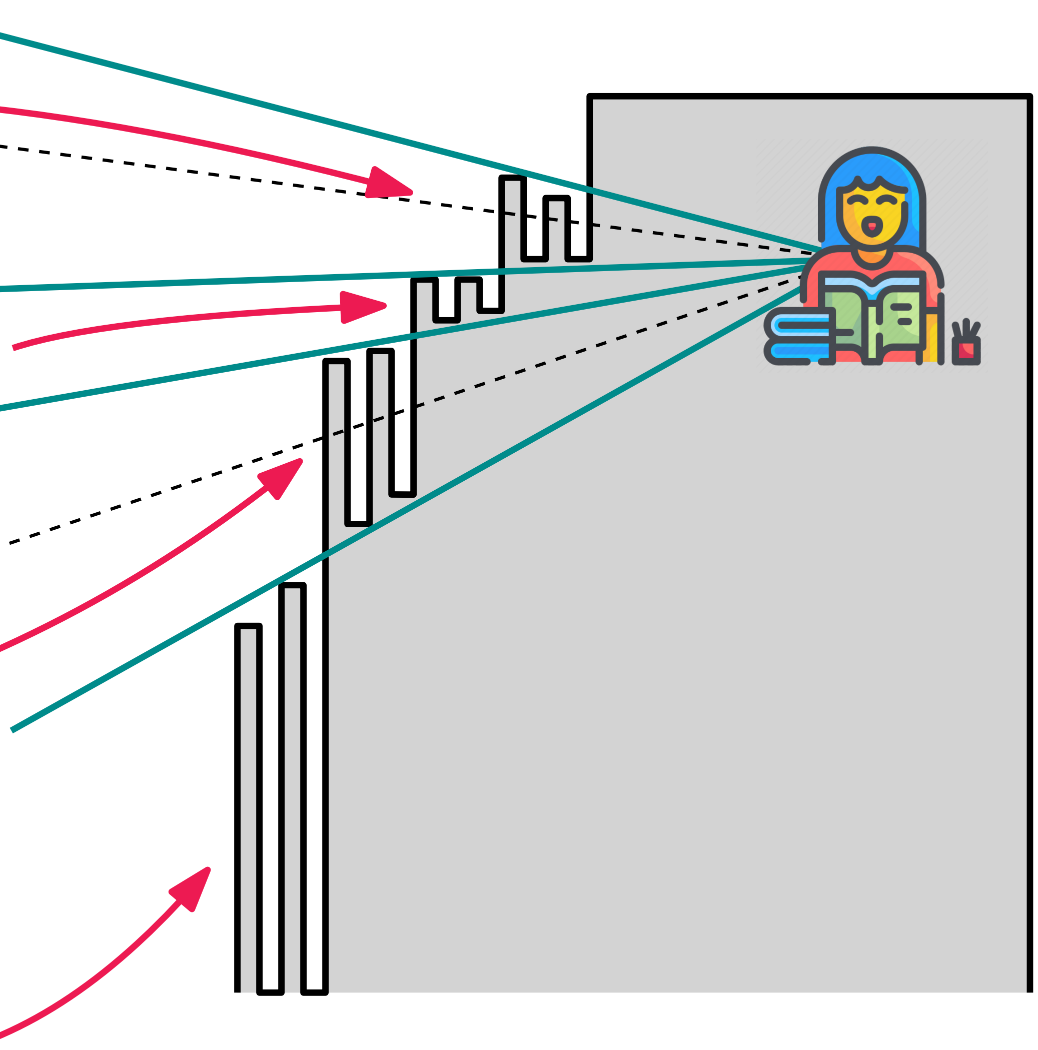
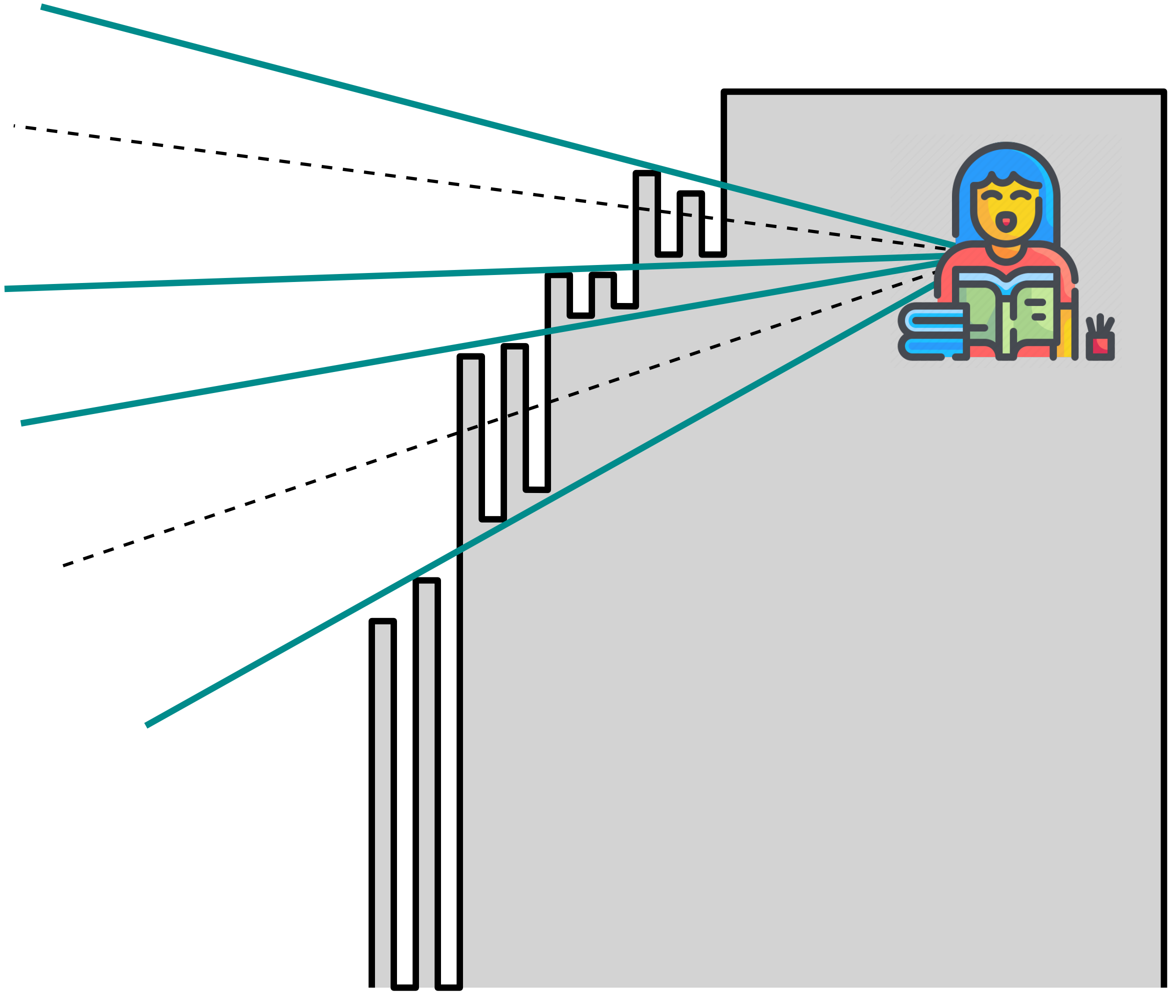


NP-Hardness for Histograms

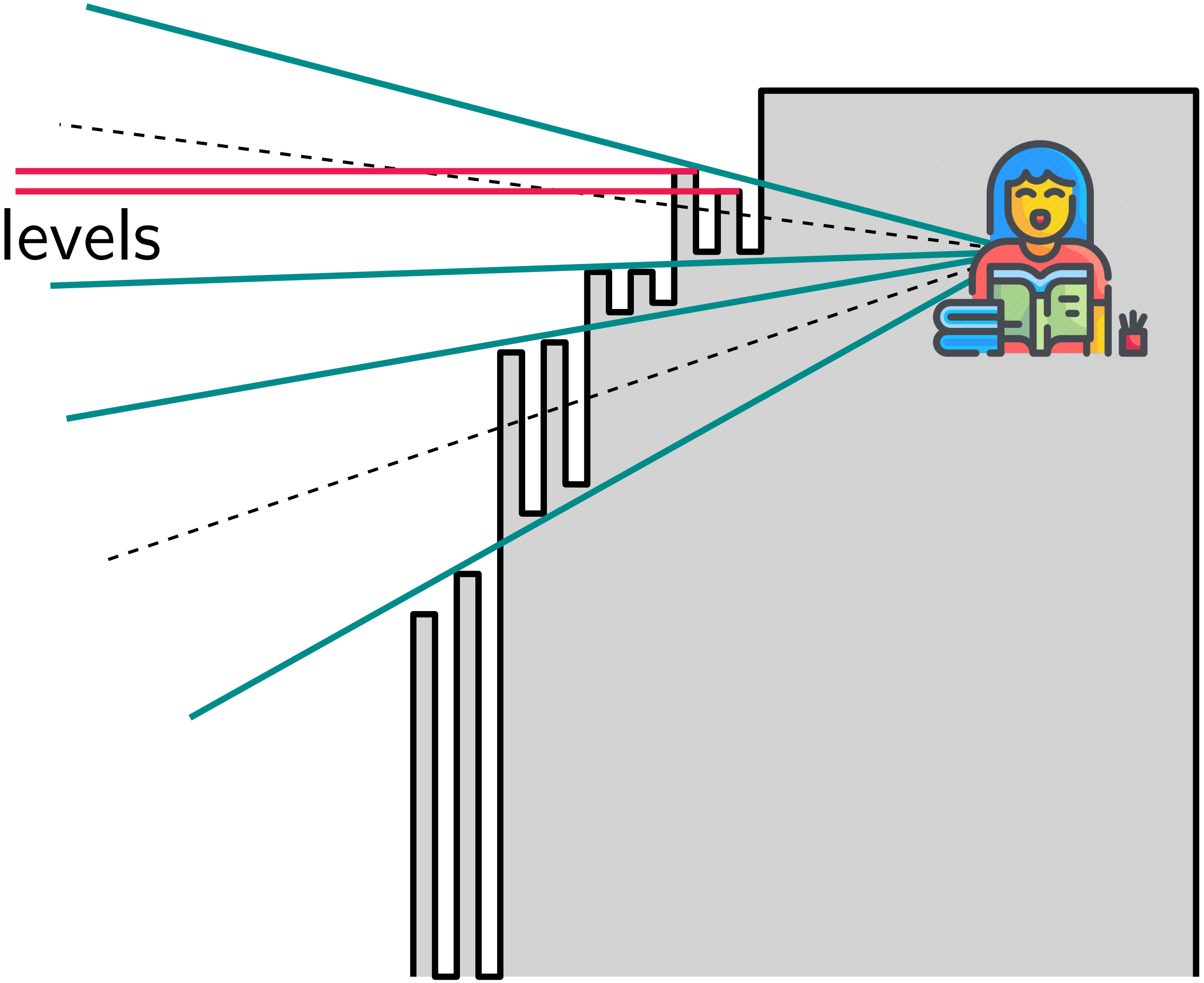
Integer Coordinates



NP-Hardness for Histograms

Integer Coordinates

$O(k \cdot |\mathcal{R}|)$
integral y-levels



NP-Hardness for Histograms

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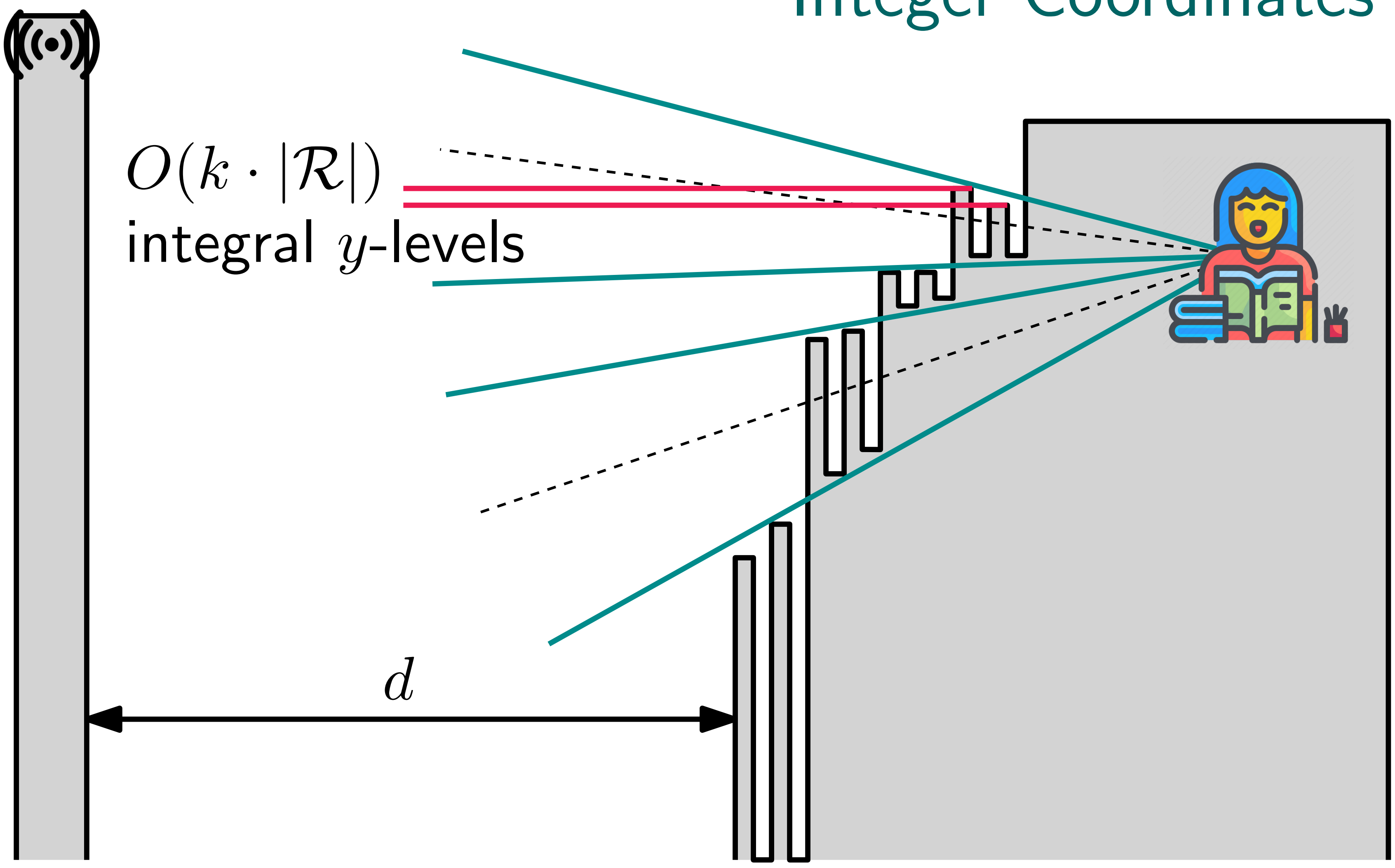

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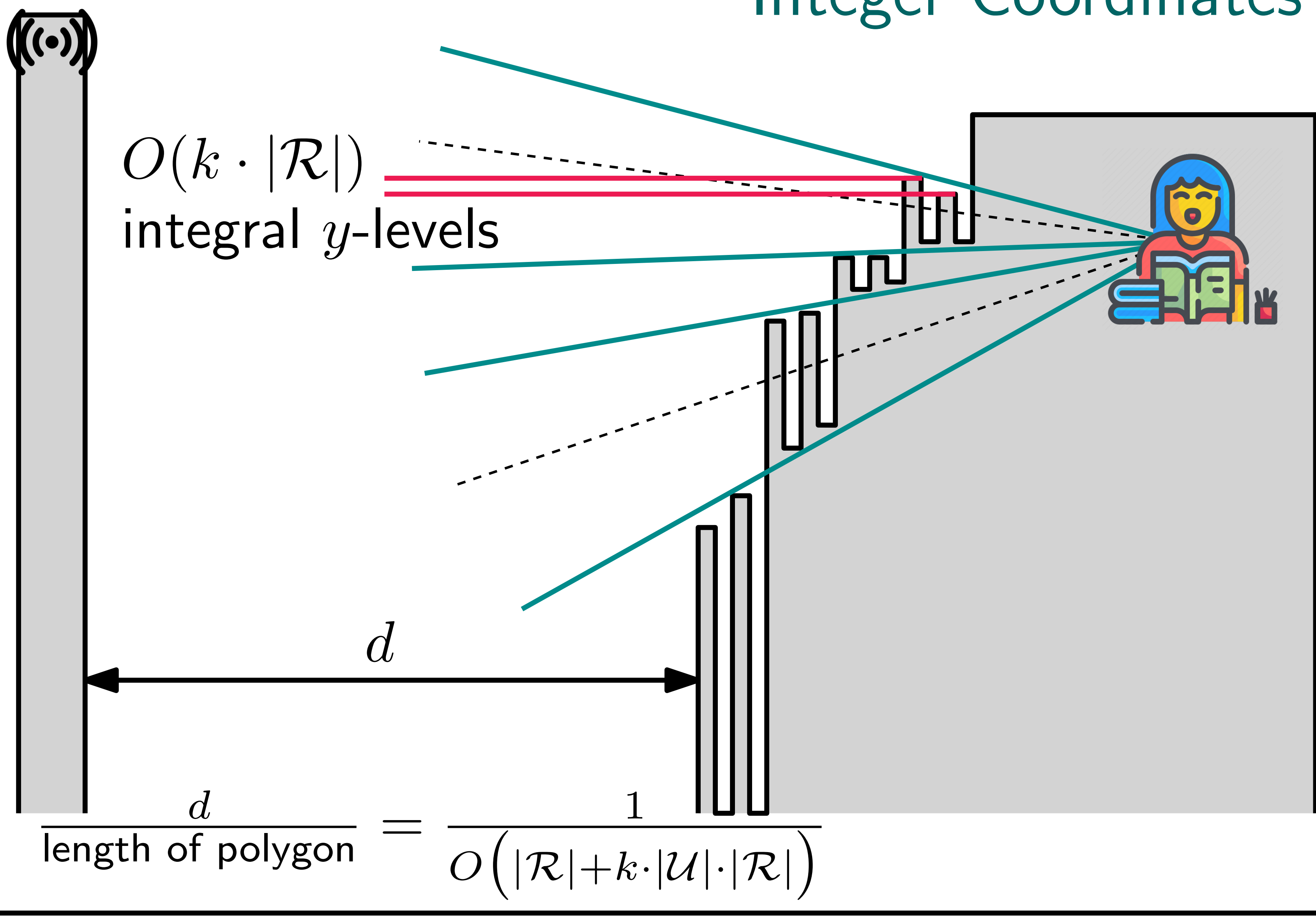

$$\frac{d}{\text{length of polygon}} = \frac{1}{O\left(|\mathcal{R}|+k\cdot|\mathcal{U}|\cdot|\mathcal{R}|\right)}$$

NP-Hardness for Histograms

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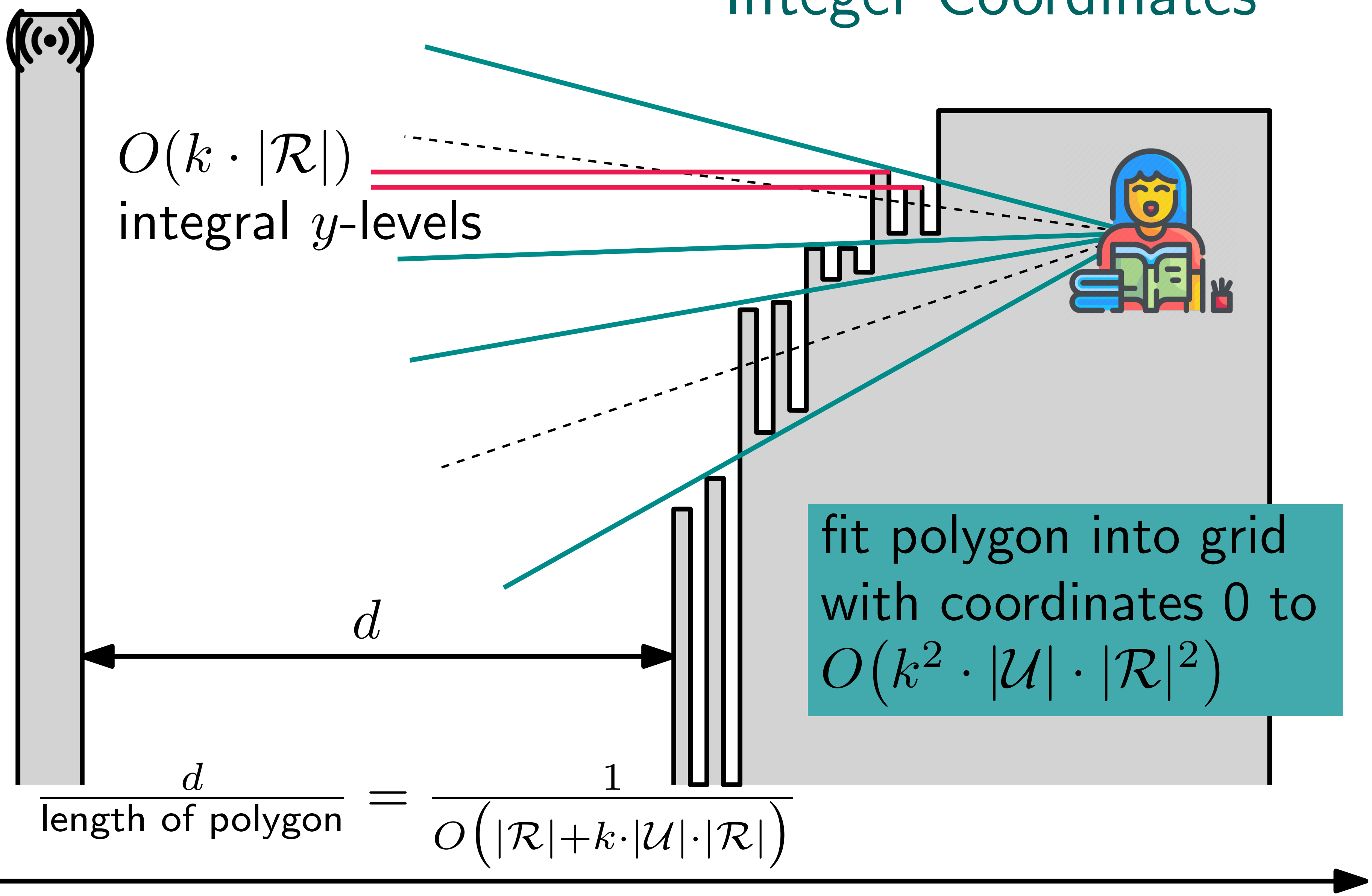

fit polygon into grid with coordinates 0 to $O\left(k^2 \cdot |\mathcal{U}| \cdot |\mathcal{R}|^2\right)$

$$\frac{d}{\text{length of polygon}} = \frac{1}{O\left(|\mathcal{R}|+k\cdot|\mathcal{U}|\cdot|\mathcal{R}|\right)}$$