Deltahedral Domes over Equiangular Polygons

MIT CompGeom Research Group

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Coauthors

MIT CompGeom Group[1], Hugo A. Akitaya[2], Erik D. Demaine[3], Adam Hesterberg[4], Anna Lubiw[5], Jayson Lynch[6], Joseph O'Rourke[7], Frederick Stock[8], and Josef Tkadlec[9]

1 Artificial 1st author to highlight that coauthors worked as an equal group.
2 U. Mass. Lowell, hugo_akitaya@uml.edu
3 MIT, edemaine@mit.edu
4 Harvard U., achesterberg@gmail.com
5 U. Waterloo, alubiw@uwaterloo.ca
6 MIT, jaysonl@mit.edu
7 Smith College, jorourke@smith.edu. *Corresponding author.*
8 U. Mass. Lowell, fbs9594@rit.edu
9 Charles U., jtkadlec@ist.ac.at

Deltahedral Domes

Definition. *Delta dome* $\mathcal{D}$ over P:

(1) P: convex polygon in xy-plane
(2) $\mathcal{D}$: convex surface of unit equilateral triangles
(3) Triangles can be coplanar
(4) $P \cup \mathcal{D}$ is a convex polyhedron
(5) $P \cap \mathcal{D} = \partial P$
(6) No triangle of $\mathcal{D}$ in xy-plane; $\mathcal{D}$ above P

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(3) $\Rightarrow$ faces convex *polyiamonds*
(6) $\Rightarrow$ polyhedron positive volume

Rectangle domes

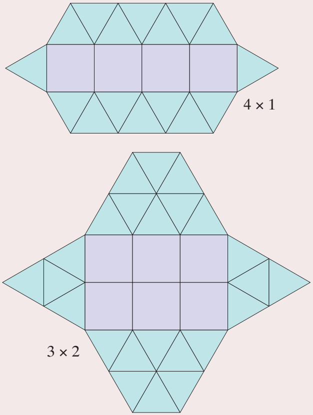


Figure: Integral rectangle $a \times b$: roof faces: trapezoids and triangles.

Main Theorem

Theorem

(a) The only equiangular convex polygons with integral edge lengths that can be domed have n vertices, where $n \in \{3, 4, 5, 6, 8, 10, 12\}$.

(b) Moreover, for each of these n, we completely characterize which integral edge-length patterns can be domed.

Glazyrin & Pak

Question: Richard Kenyon, 2005

Answered negatively: 2022

DOMES OVER CURVES

ALEXEY GLAZYRIN* AND IGOR PAK◇

DOMES OVER CURVES

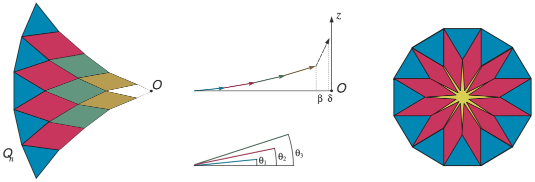


FIGURE 7. Left: Nearly planar tiling of a portion of Q_n with rhombi. Middle: Vertical slice. Right: Example of Q_{12} with triangles and nearly-flat rhombi R_1.

Can every "curve" be "spanned"? NO.

Differences between Doming & Spanning

(a) Our P is a planar convex polygon. Their P is a 3D possibly self-intersecting polygonal chain.

(b) Our dome $\mathcal{D}$ is embedded (non-self-intersecting) and convex. Their PL-surface is (in general) nonconvex, immersed, and self-intersecting.

Glazyrin & Pak Results (2022)

Thm. 1.2 : There is a nonplanar unit rhombus that cannot be “spanned.”

Thm. 1.4 : Every planar regular n-gon can be “spanned.”

Regular Polygons $\bar{P}_n$

$$n \in \{3, 4, 5, 6, 8, 10, 12\}$$

$$(\text{Not: } n = 7, 9, 11, \geq 13.)$$

$n = 3, 4, 5$

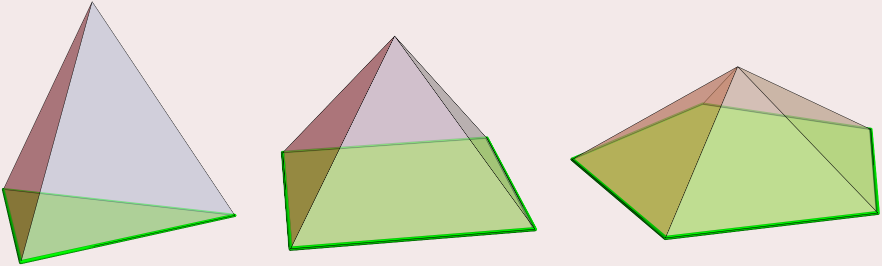


Figure: Pyramids over regular $\bar{P}_n$, $n = 3, 4, 5$.

Hexagon

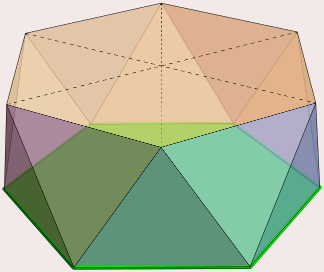


Figure: Hexagonal Antiprism: $\bar{P}_6$.

(Hexagonal pyramid: not a dome $\mathcal{D}$.)

Octagon

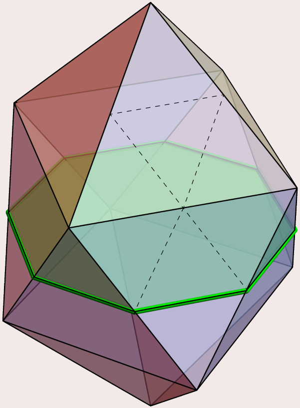


Figure: Gyro Elongated Square Diprism → Octagon $\bar{P}_8$.

Decagon

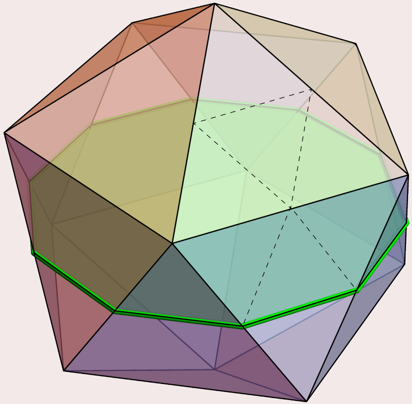


Figure: Icosahedron → Decagon $\bar{P}_{10}$.

Dodecagon

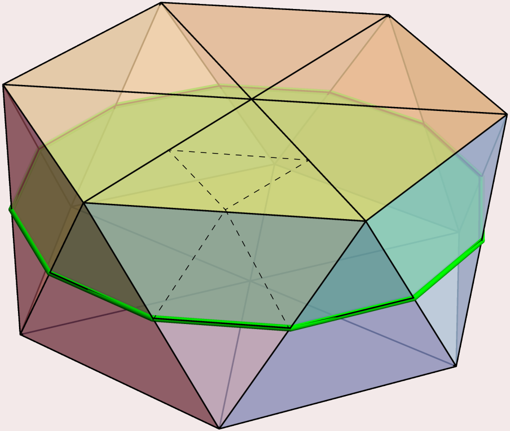


Figure: Hexagonal Antiprism → Dodecagon $\bar{P}_{12}$.

Pentagon: Dome not Unique

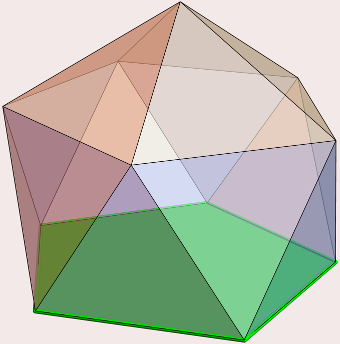


Figure: A different dome over $\bar{P}_5$.

Equiangular decagon

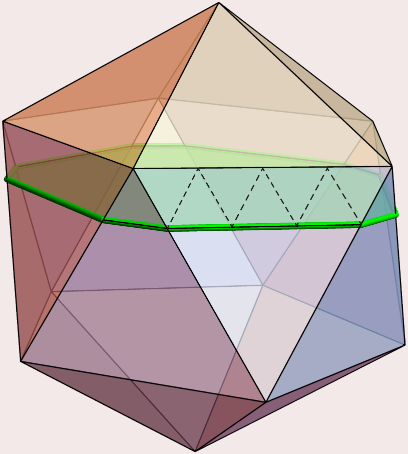


Figure: Equiangular decagon with edge lengths alternating $1, 3$.

Main Theorem (a)

Theorem

(a) The only equiangular convex polygons with integral edge lengths that can be domed have n vertices, where $n \in \{3, 4, 5, 6, 8, 10, 12\}$.

(b) Moreover, for each of these n, we completely characterize which integral edge-length patterns can be domed.

Impossible: $n = 7, 9, 11, \geq 13$.

Proof Steps for Theorem (a)

(1) Each base vertex has three incident dome triangles.

(2) Curvature constraints imply that there is a dome with at most 6 (non-base) dome vertices.

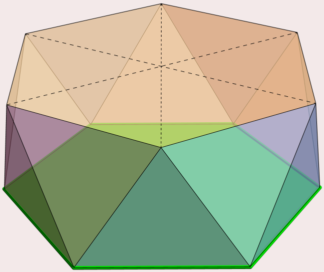
(3) Of the n dome faces incident to base edges, at least half tilt toward the outside of the base and have a “private” dome vertex. Furthermore, for n odd we strengthen this to *all* dome faces incident to base edges.

(4) Thus, since there are at most 6 dome vertices, $n \leq 12$, and for n odd, there are no solutions for $n \geq 6$.

Step (1): Three triangles per base vertex

Lemma

In a dome over an equiangular n-gon P_n, $n \geq 7$, each base vertex b_i is incident to three dome triangles.

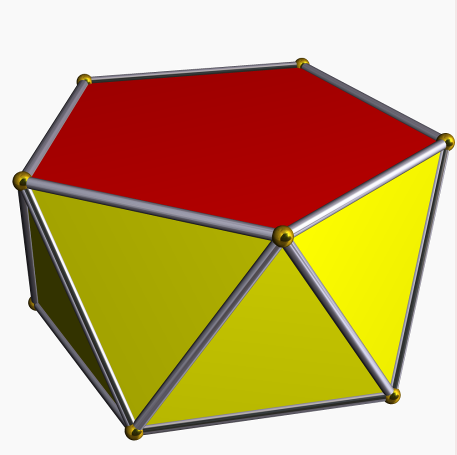


Step (2): Curvature $2\pi \Rightarrow \leq 6$ dome vertices

Lemma

For an equiangular base P_n, $n \geq 7$, there can be at most 6 dome vertices.

Pentagonal Antiprism: ±Normals



Step (3): ±Normals

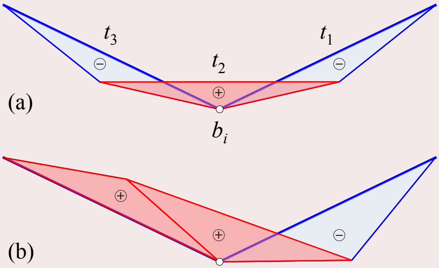


Figure: (a) Both t_1 and t_3 downward. (b) Only t_1 downward.

±Normals

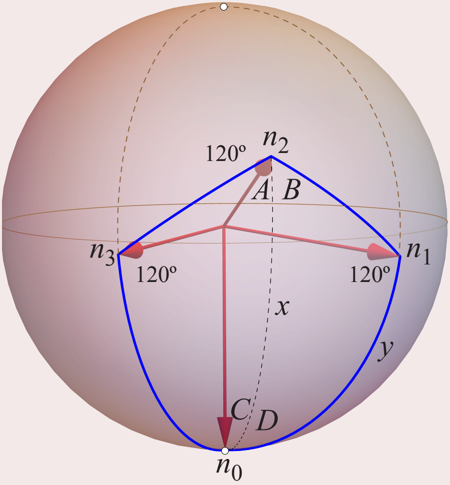


Figure: Gauss map for base vertex b_2. n_2: upward normal. Both n_1, n_3: downward.

Step (4): $n \leq 12$; None odd $n \geq 7$

Lemma

(1) If P is a domeable convex n-gon with all angles $\geq 120^\circ$, then $n \leq 12$.

(2) For odd $n \geq 7$, there is no domeable equiangular n-gon.

Main Theorem

Theorem

(a) The only equiangular convex polygons with integral edge lengths that can be domed have n vertices, where $n \in \{3, 4, 5, 6, 8, 10, 12\}$.

(b) Moreover, for each of these n, we completely characterize which integral edge-length patterns can be domed.

Open Problems

(1) Is there any convex 7-gon that can be domed?

- We have constructed 9- and 11-gons (non-equilateral) that can be domed.

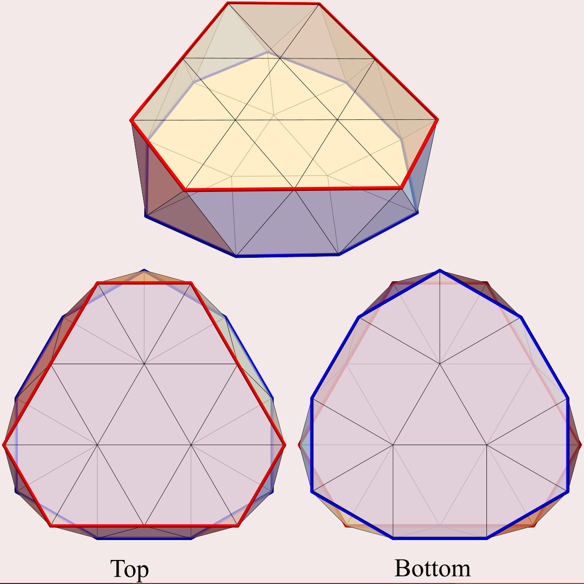
(2) Is there any convex n-gon with $n > 12$ that can be domed?

- Our strongest proved upper bound is $n = 24$.

(3) Can any non-equilateral triangle be domed?

- Glazyrin & Pak conjectured that a $2 \times 2 \times 1$ isosceles triangle cannot be spanned (and so cannot be domed).

Irregular 9-gon Domed



The End

Thanks!