A Clique-Based Separator for Intersection Graphs of Geodesic Disks in $\mathbb{R}^2$

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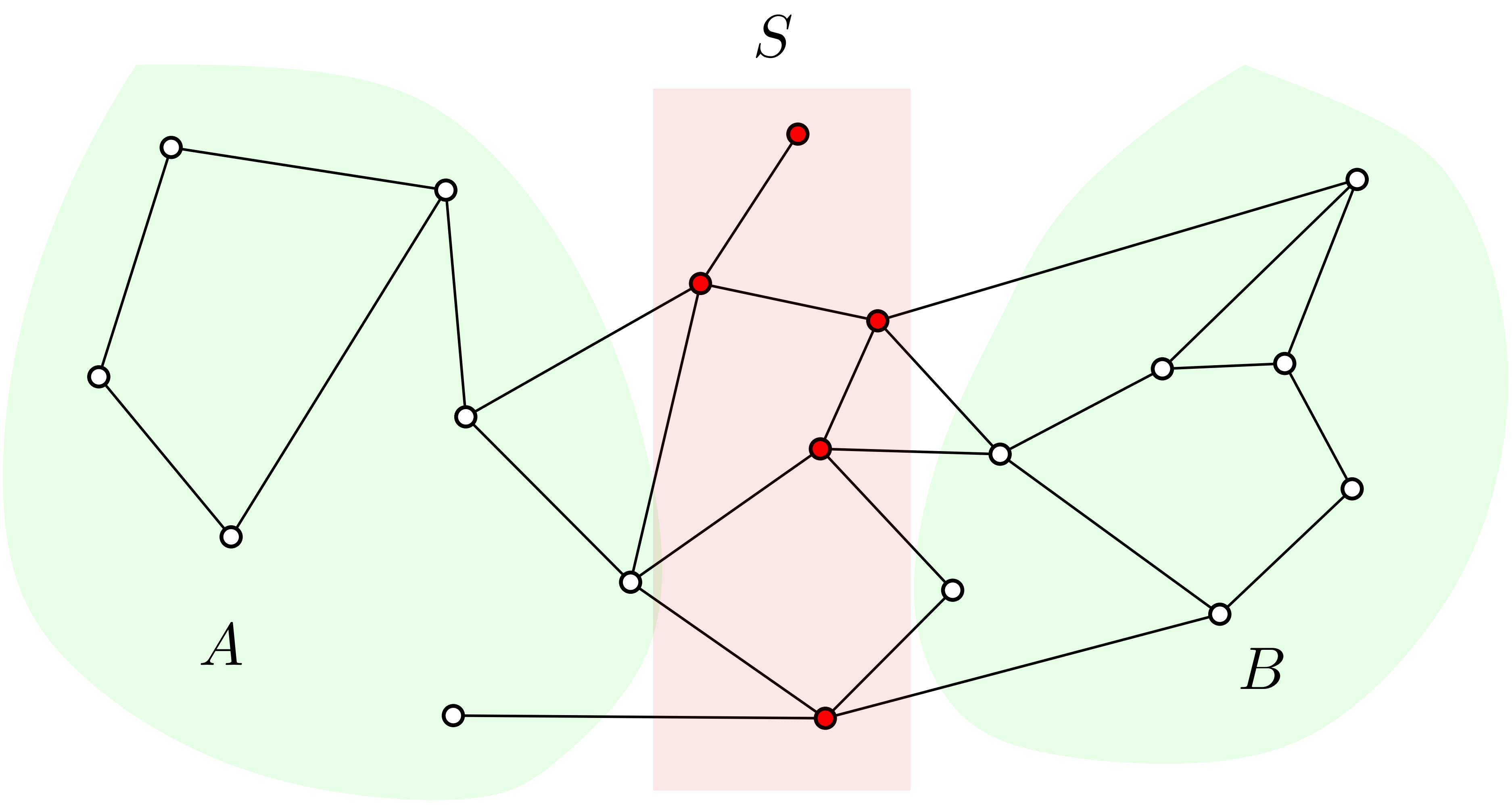
Joint work with:

- Boris Aronov (NYU)
- Mark de Berg (TU/e)

Clique-Based Separators

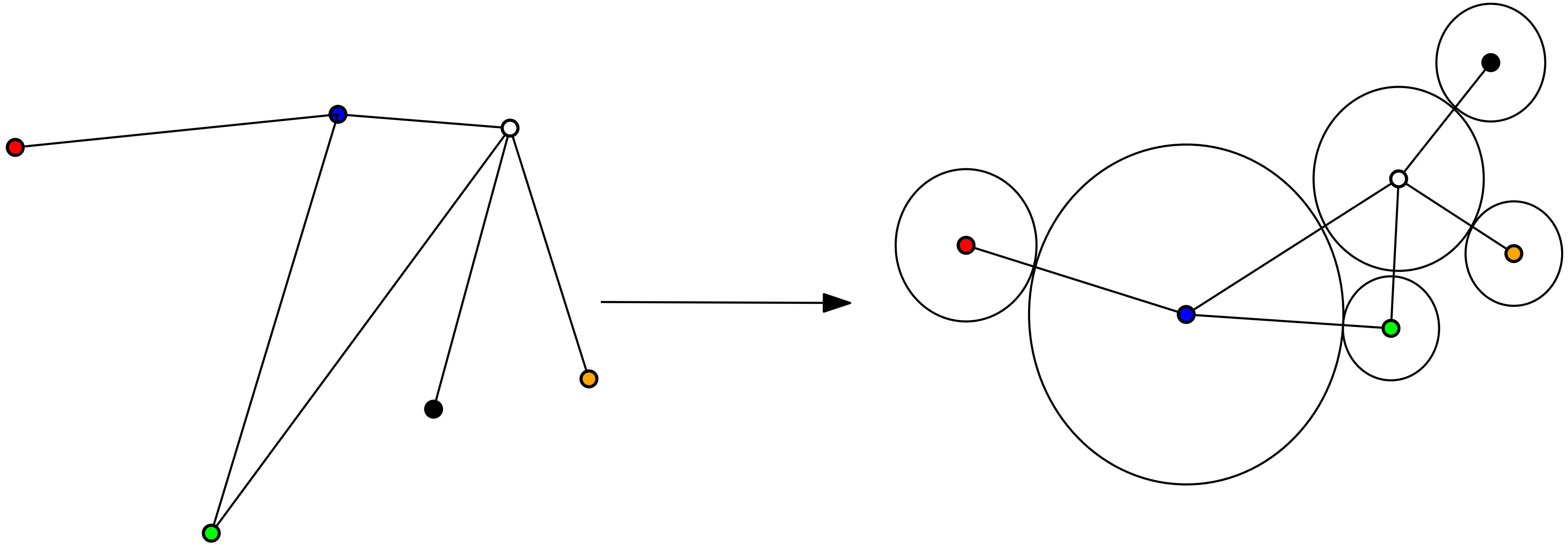
Planar Separator Theorem. [Lipton and Tarjan]
Any planar graph with n vertices has a 2/3-balanced separator of size $O(\sqrt{n})$.



$|A| \leqslant 2n/3$ $\quad |S| = O(\sqrt{n})$ $\quad |B| \leqslant 2n/3$

Clique-Based Separators

Koebe-Andreev-Thurston Theorem.
Any planar graph is the intersection graph of a set of touching disks.



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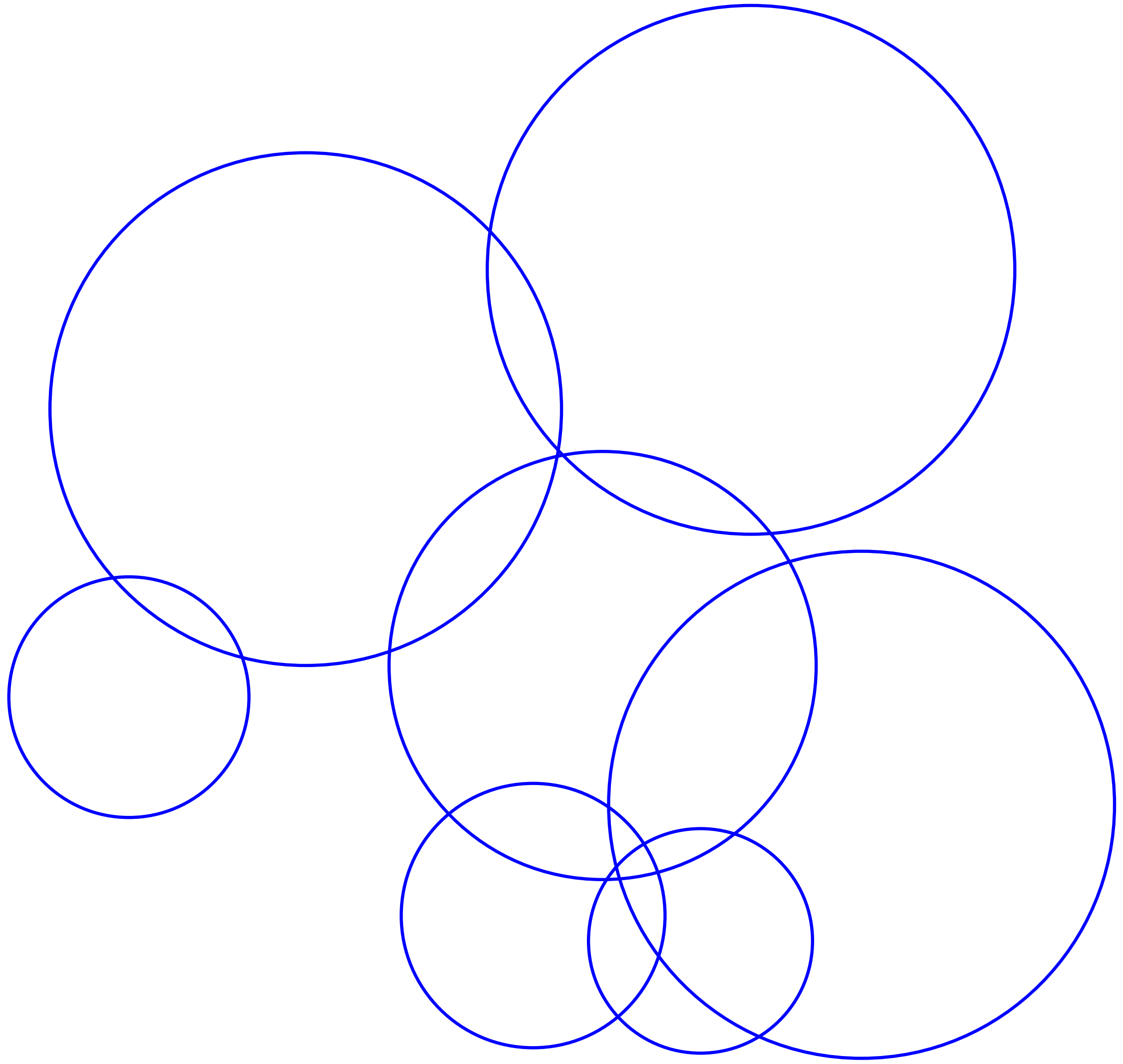
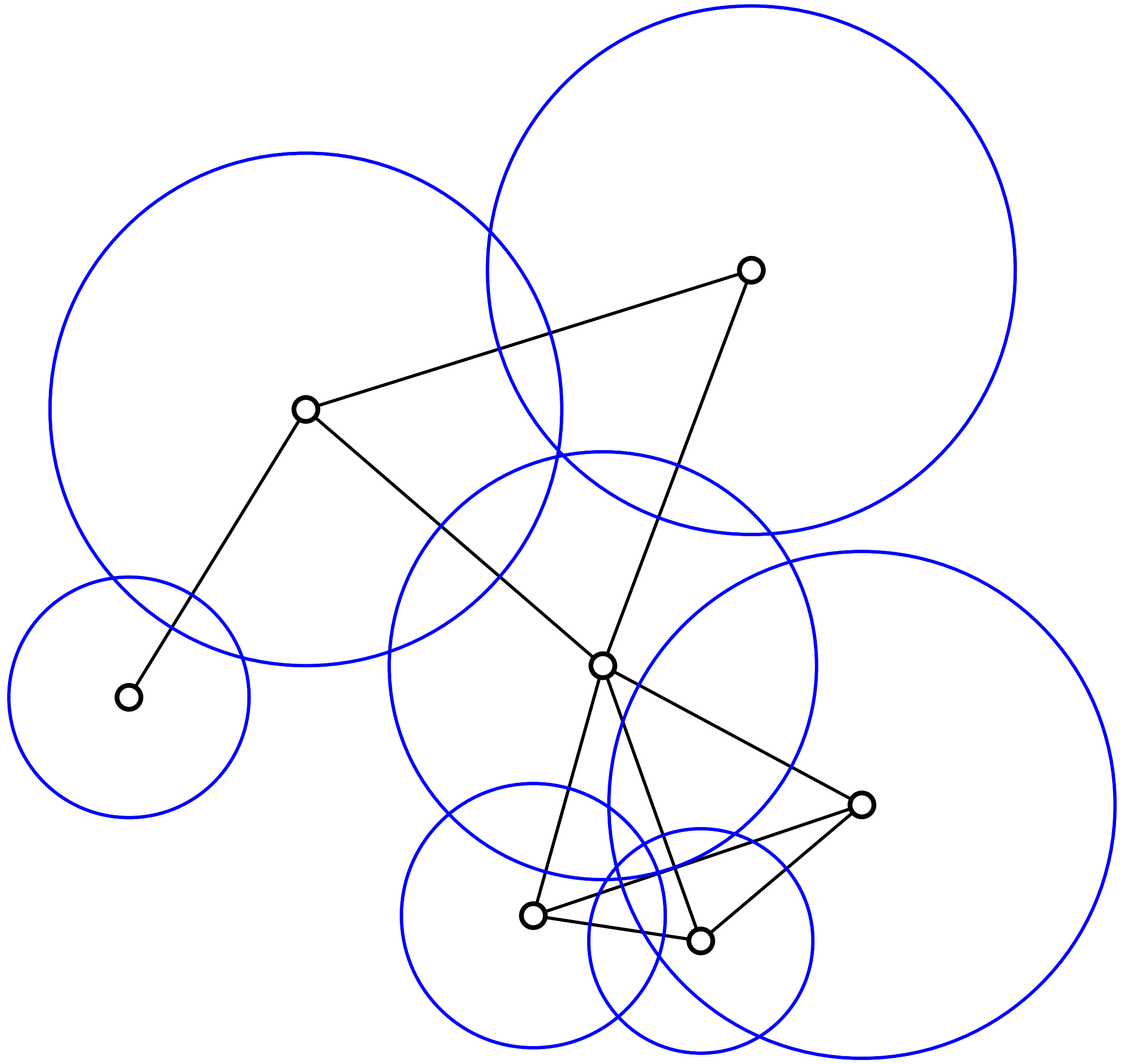


What if we allow disks to intersect?



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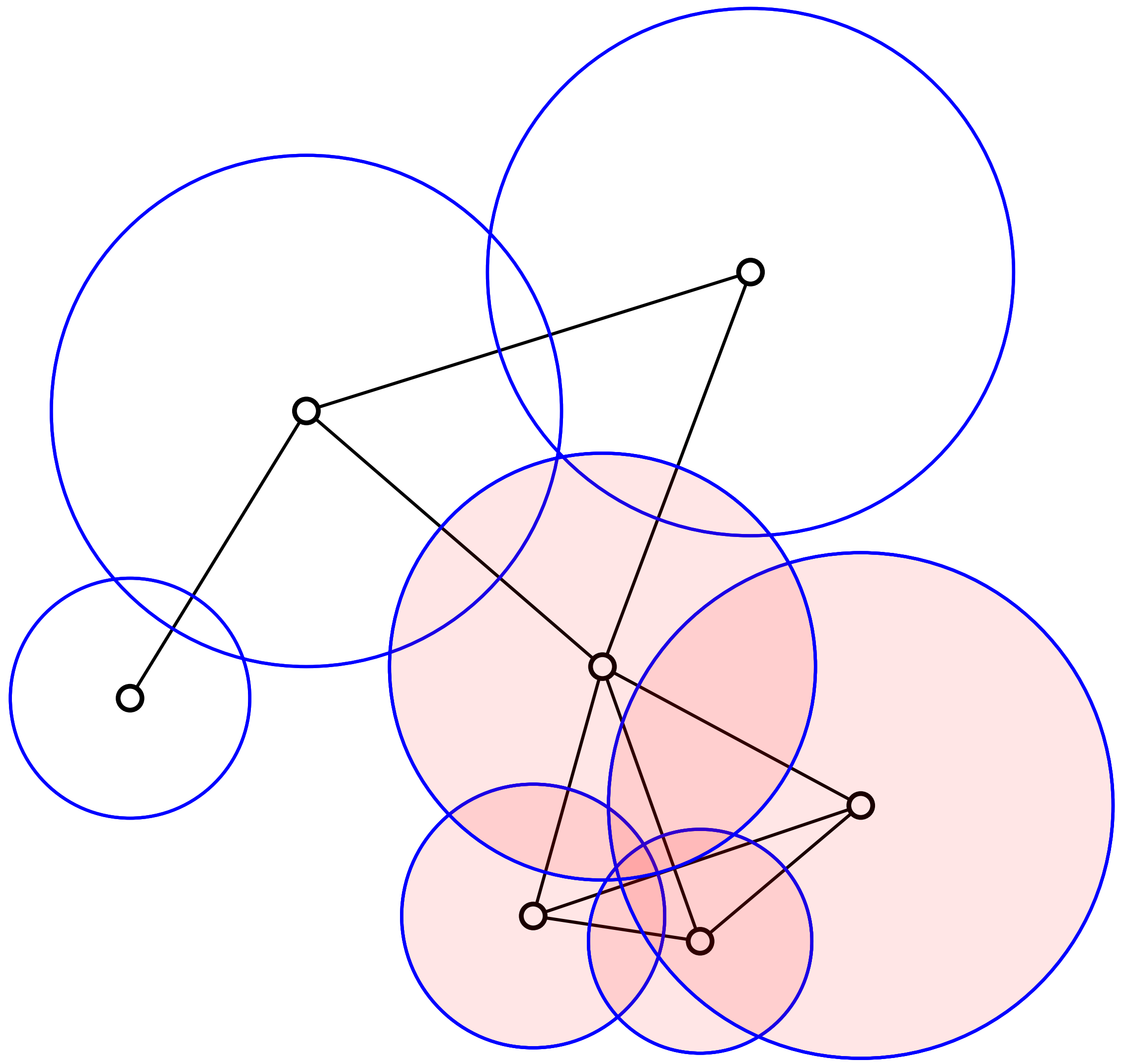

What if we allow disks to intersect?

Arbitrarily large cliques!

Clique-Based Separators

A Separator Theorem for Disk Graphs. [de Berg et al., SICOMP 2020]
For any intersection graph G of n disks, there is a balanced separator S that can be partitioned into $O(\sqrt{n})$ cliques.

size of S: number of cliques it consists of

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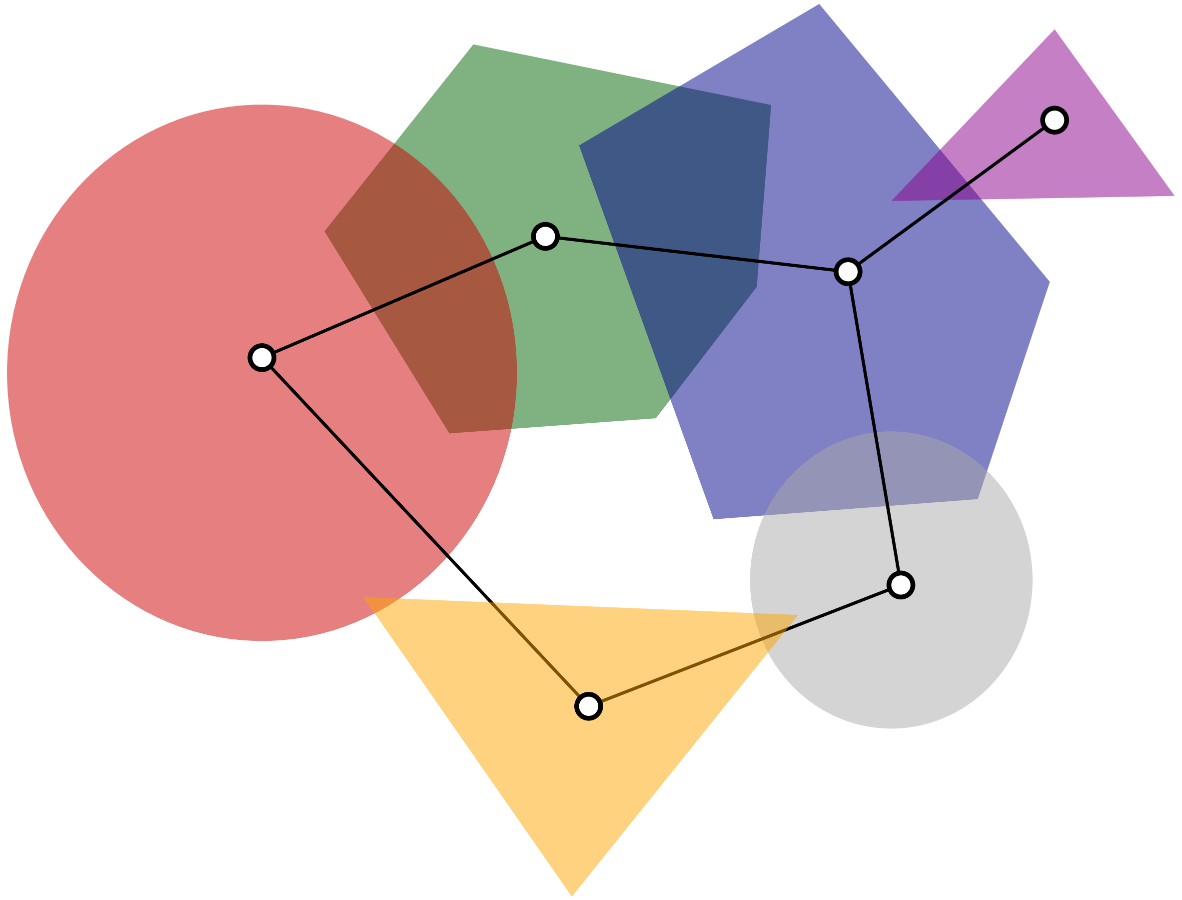
Why are these separators useful?

- Cliques can be handled efficiently for many problems, e.g. Independent Set, q-Coloring.
- Subexponential algorithms for the above (and other) problems. Typically $2^{O(\mathsf{size}(S))}$ running time.

Previous work

Intersection Graphs of	Size of Separator
Convex, Fat Objects	$O(\sqrt{n})$
Pseudodisks	$O(n^{2/3})$
Geodesic Disks in a Simple Polygon	$O(n^{2/3})$



Proof uses a packing argument based on fatness.

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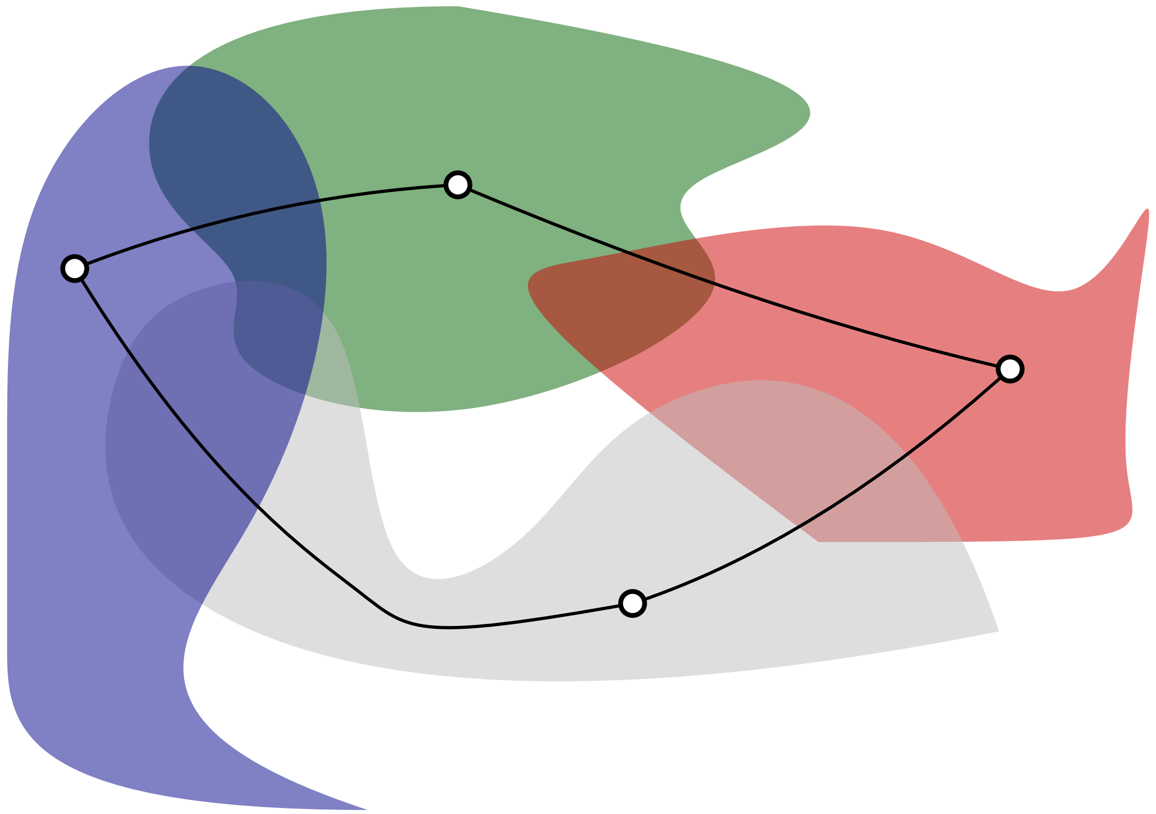

Proof uses linear union complexity of pseudodisks.

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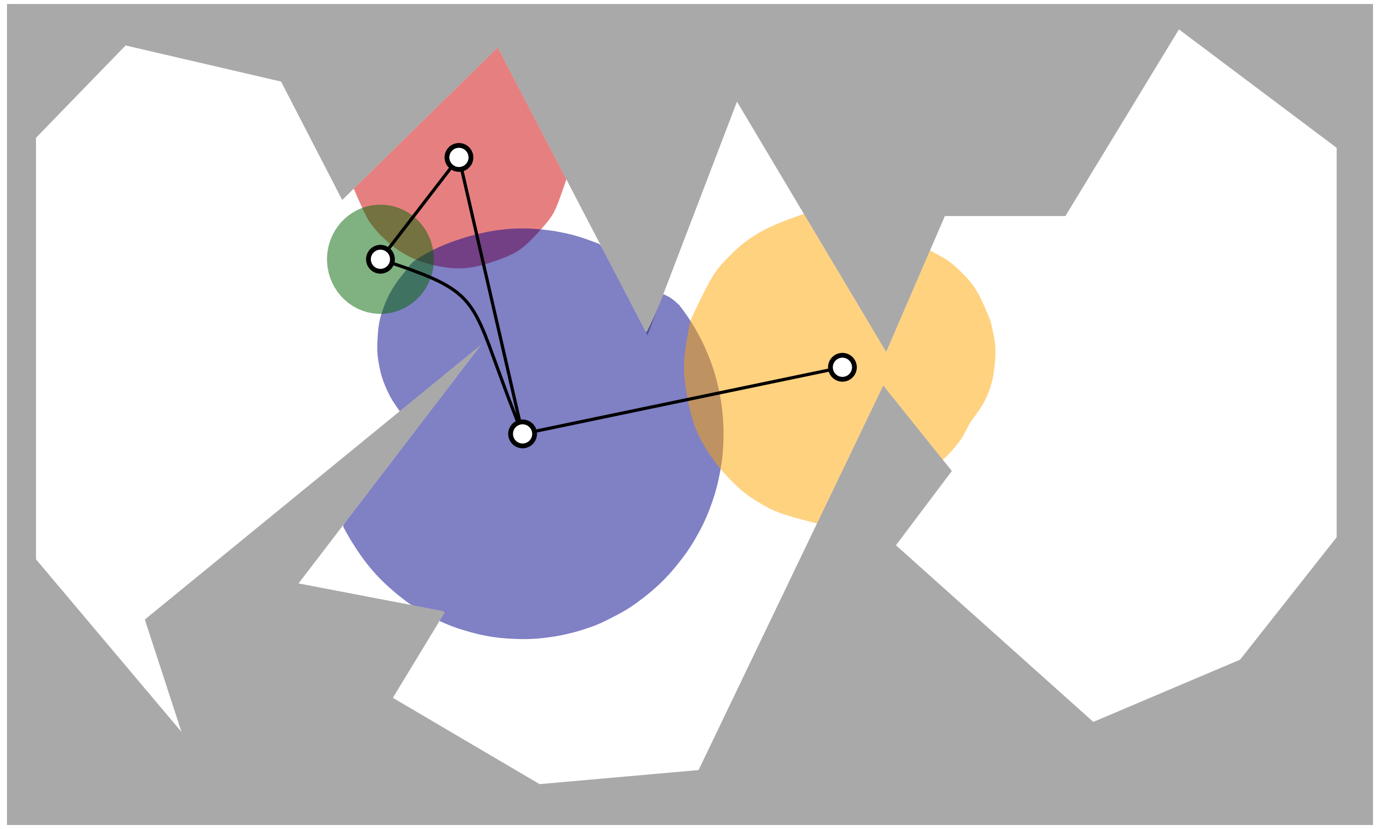

Proof uses that they behave as pseudodisks.

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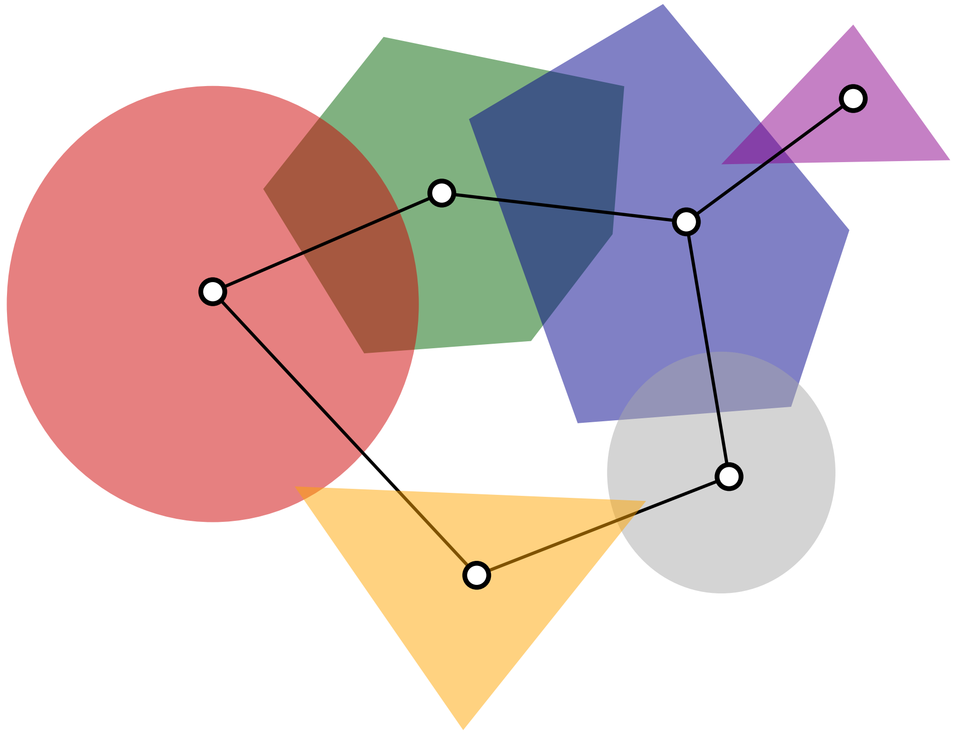
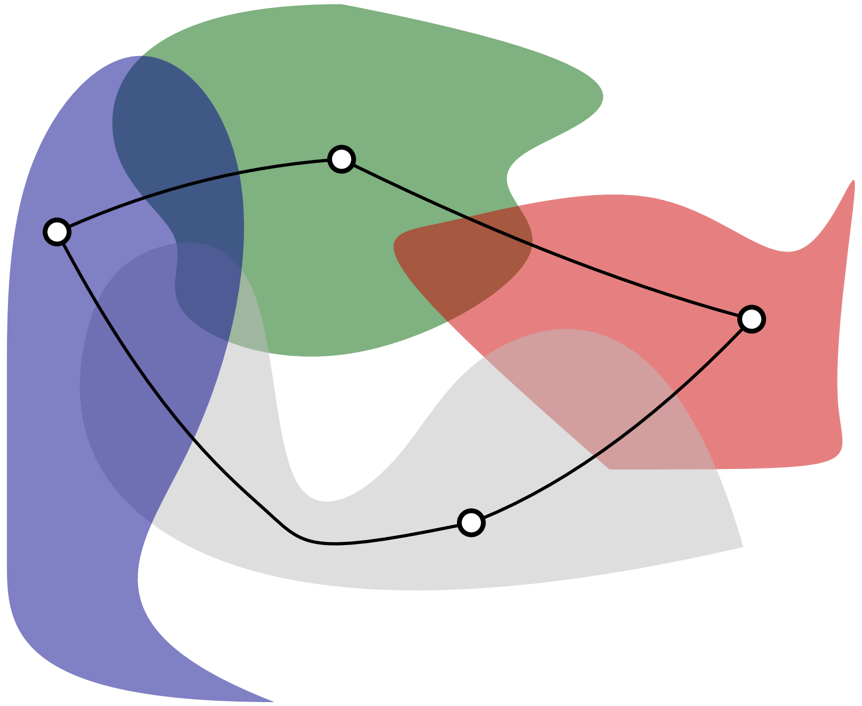
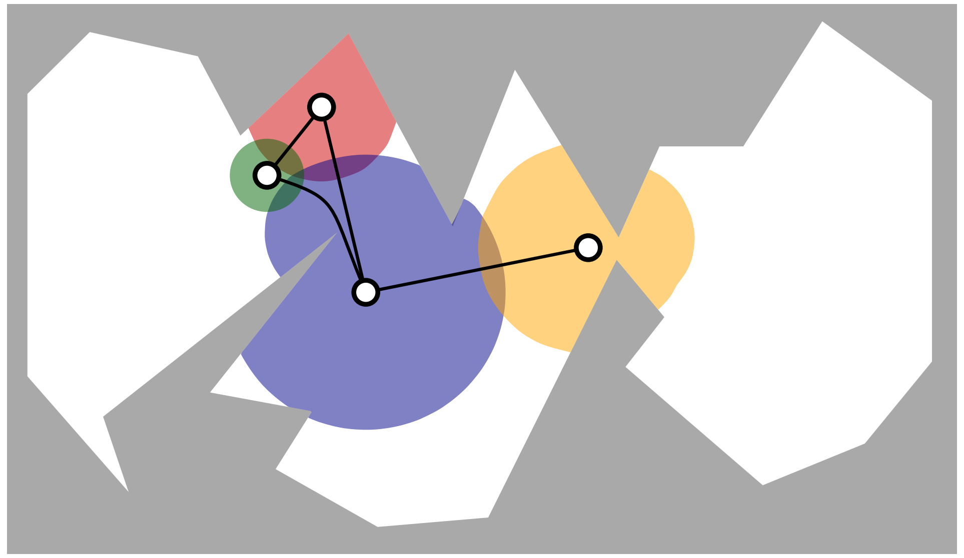
Geodesic Disks in $\mathbb{R}^2$	**?**



packing argument



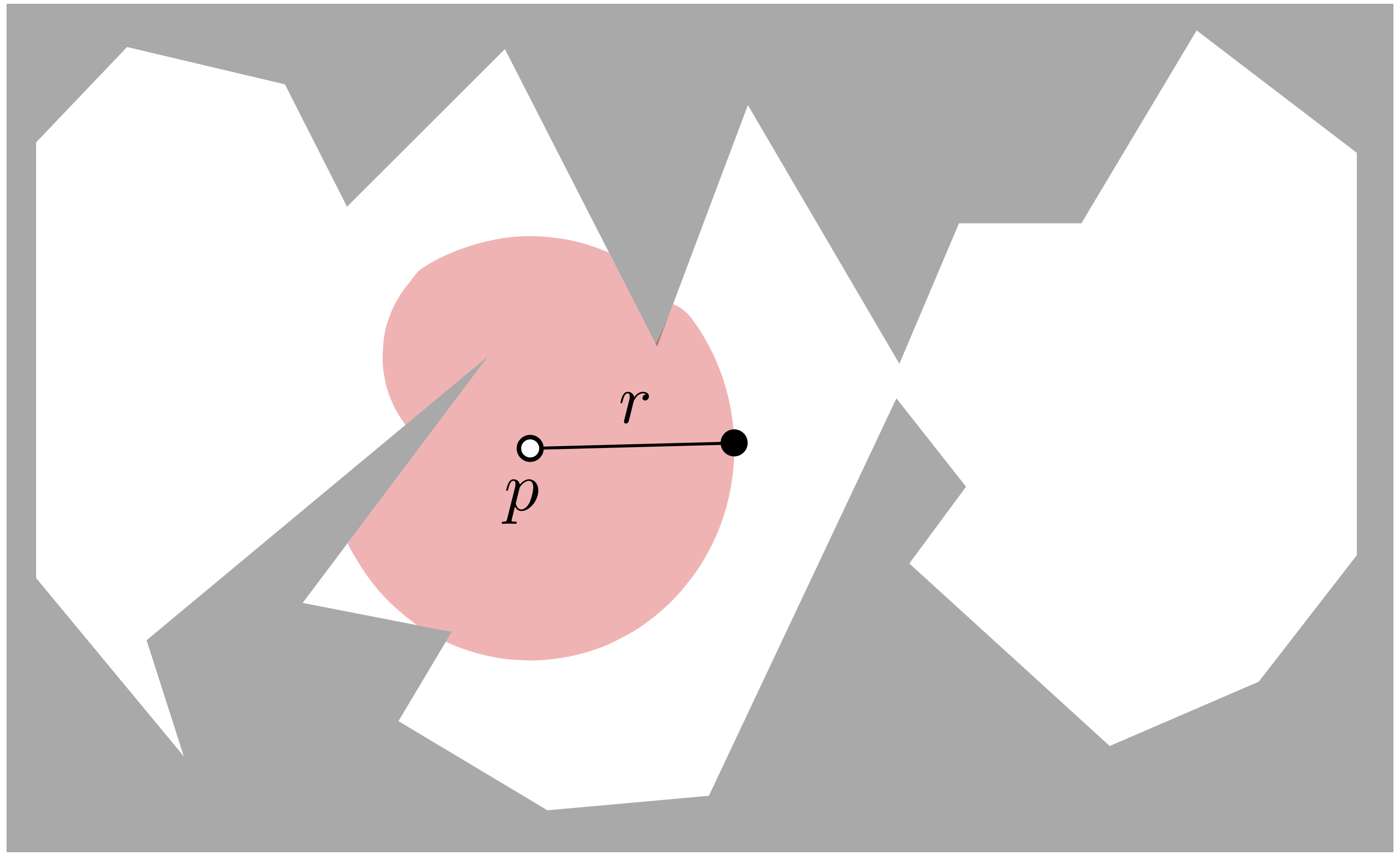
linear union complexity



same as pseudodisks

Our setting

- $F \subset \mathbb{R}^2$: closed and path-connected.
- d: shortest-path metric on F.
- geodesic disk with center $p \in F$ and radius r: all points $q \in F$ such that $d(p,q) \leqslant r$.
- $\mathcal{D}$: set of n geodesic disks in F.



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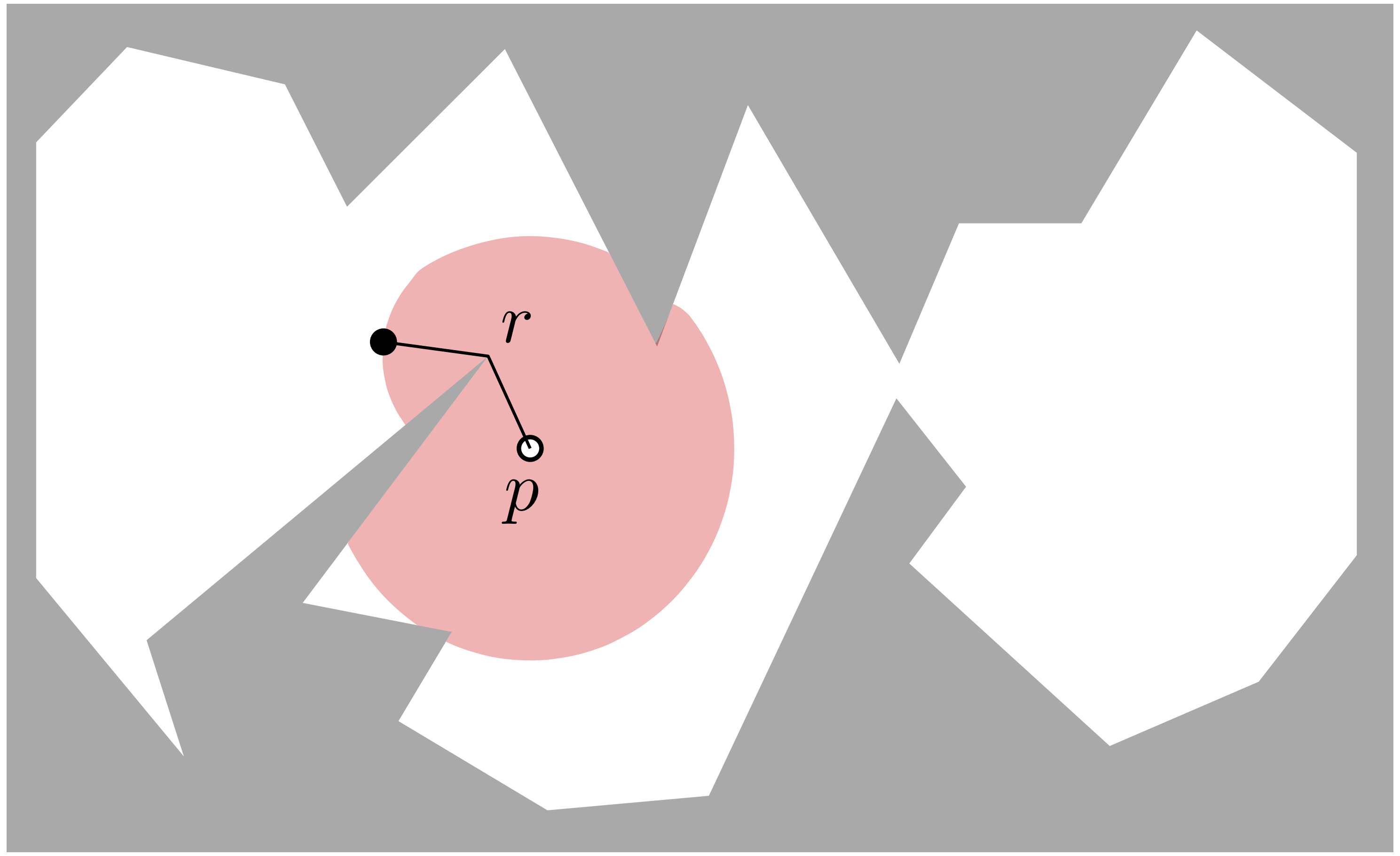

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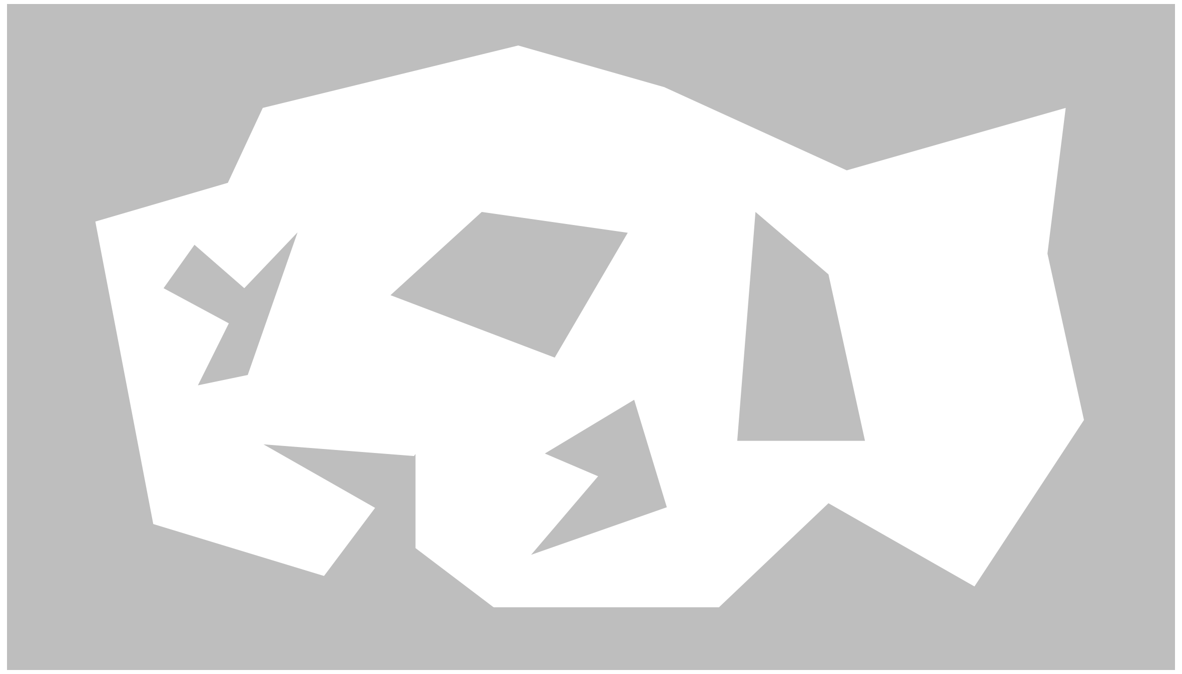
This includes:

- Geodesic disks in a **polygonal domain**
- Geodesic disks on a terrain
- Geodesic disks among weighted regions in the plane



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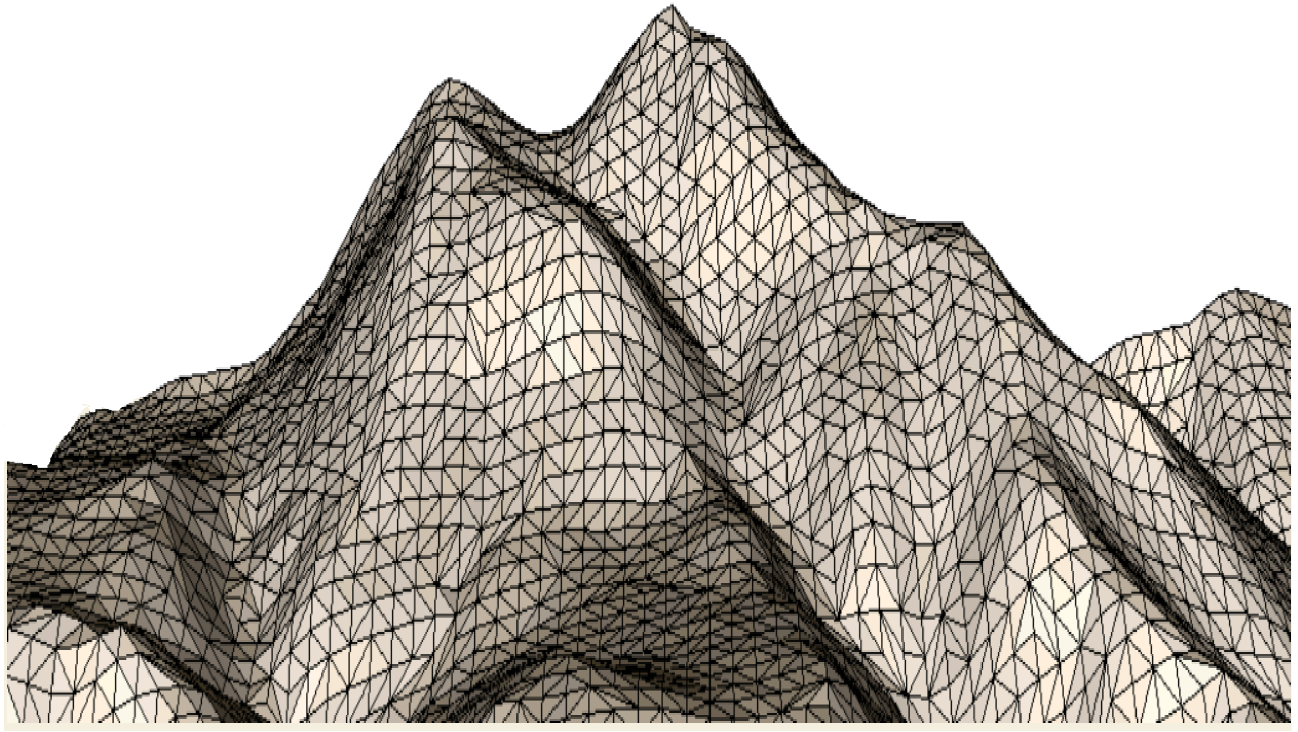

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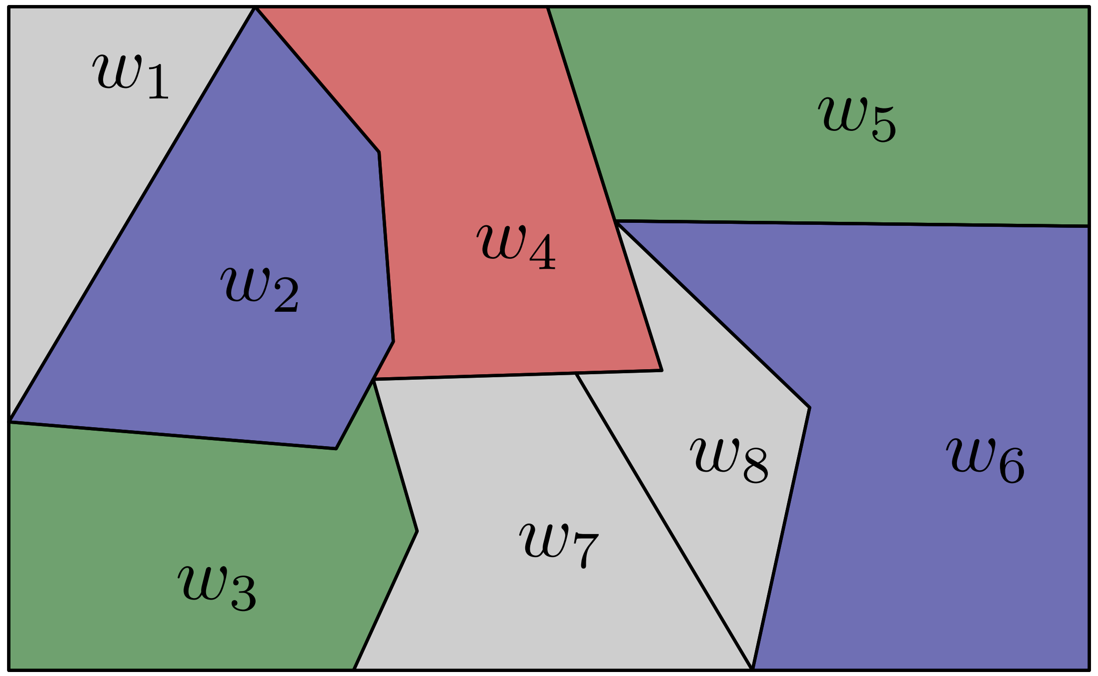
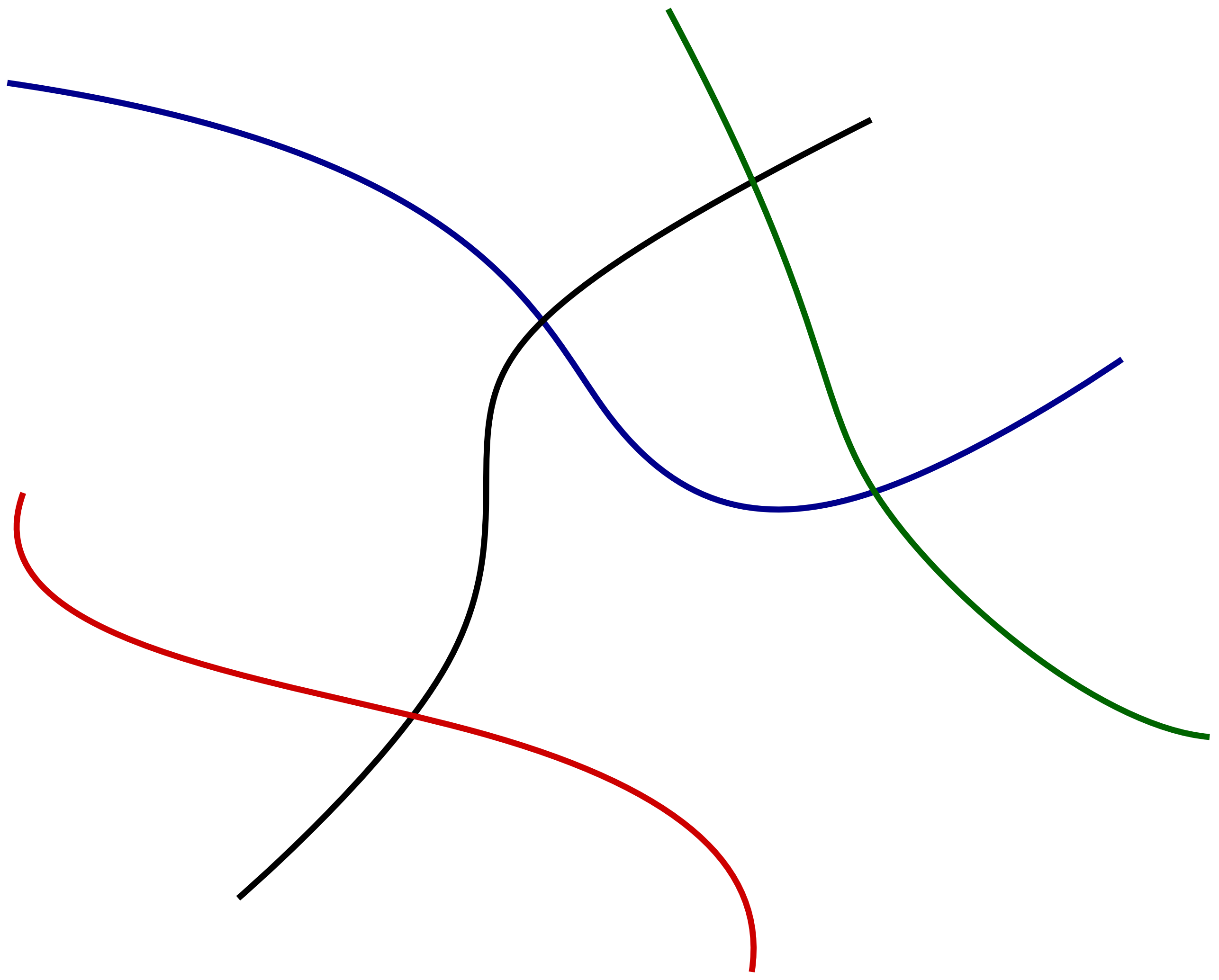


More preliminaries



String graphs:

intersection graphs of curves in the plane

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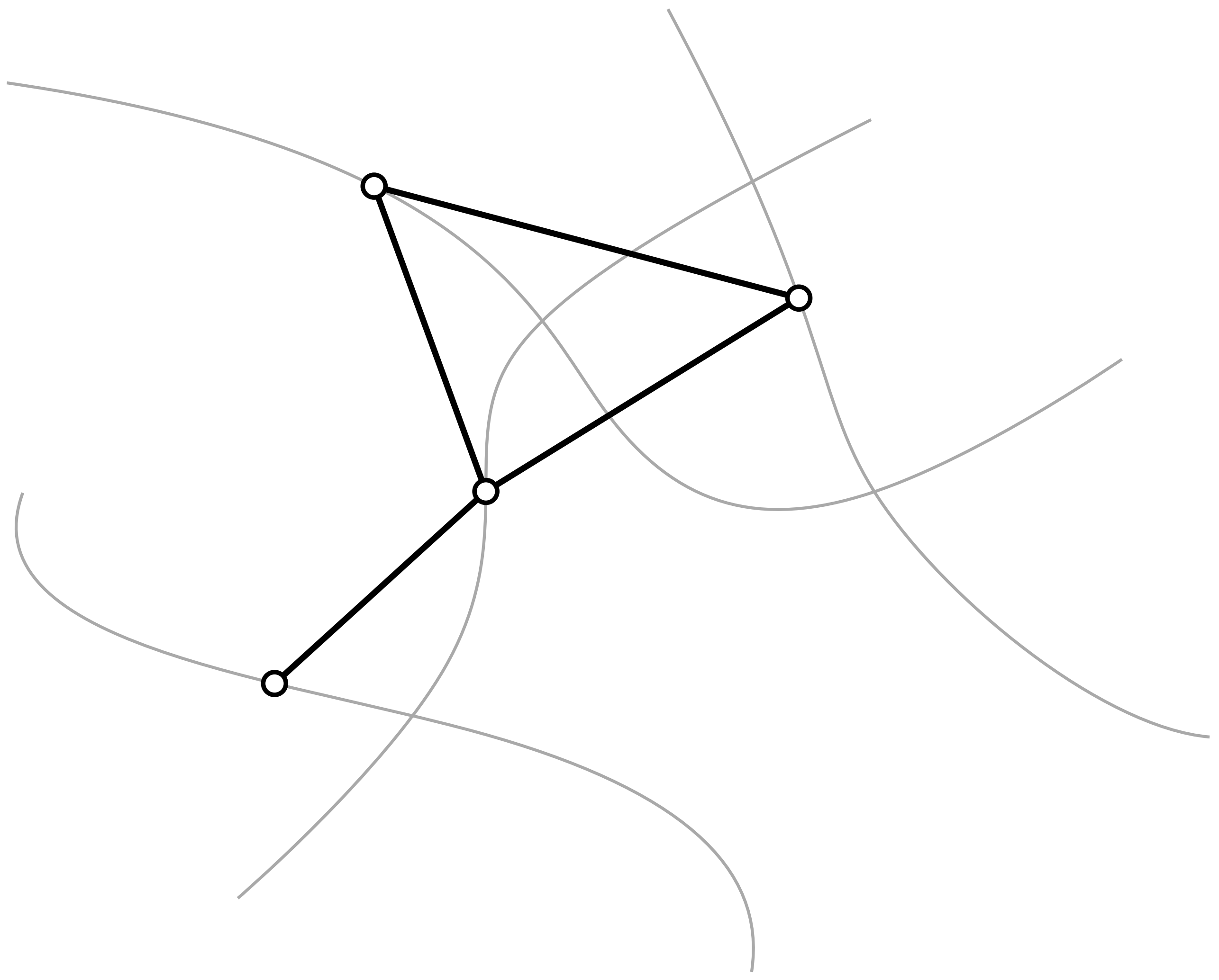

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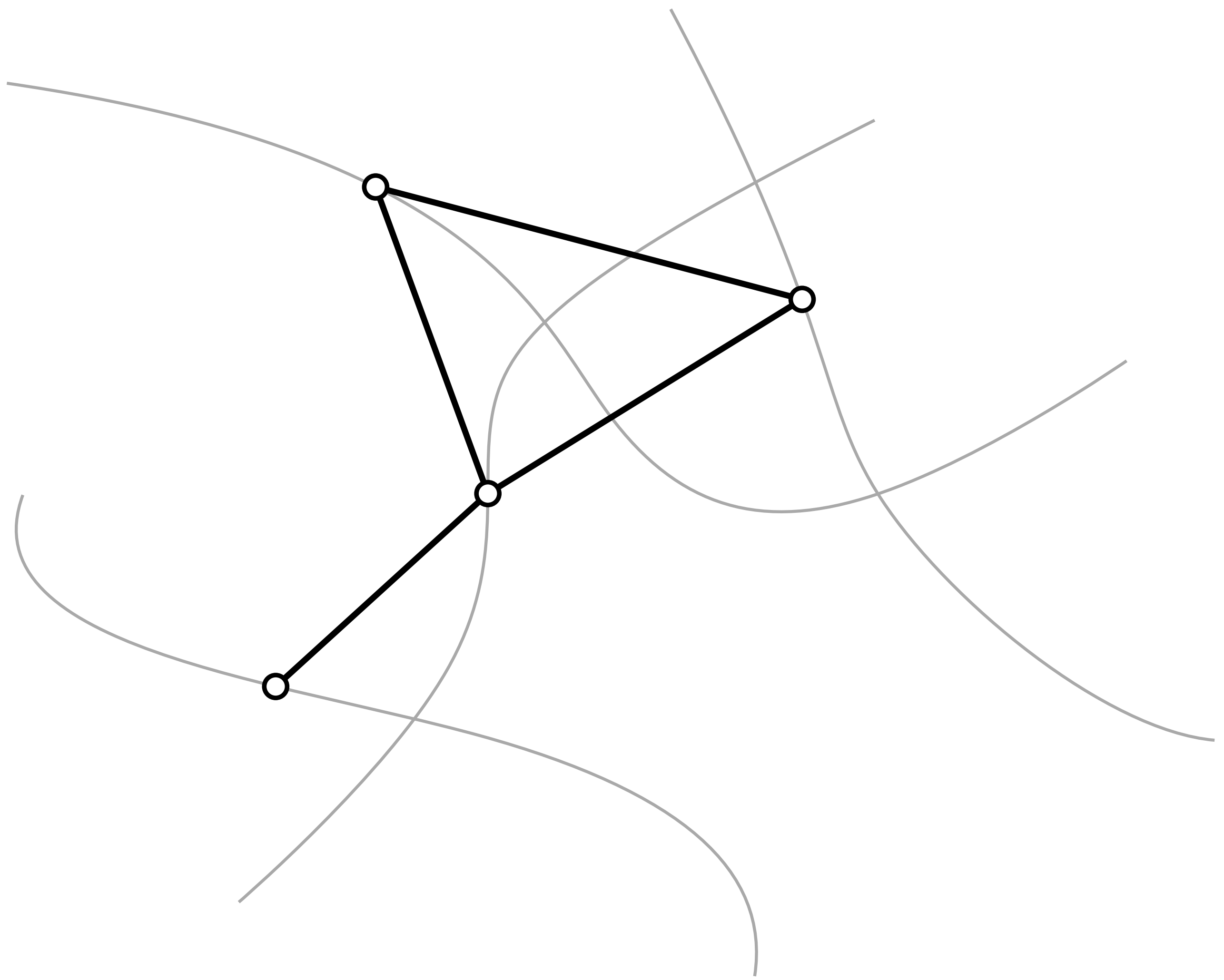

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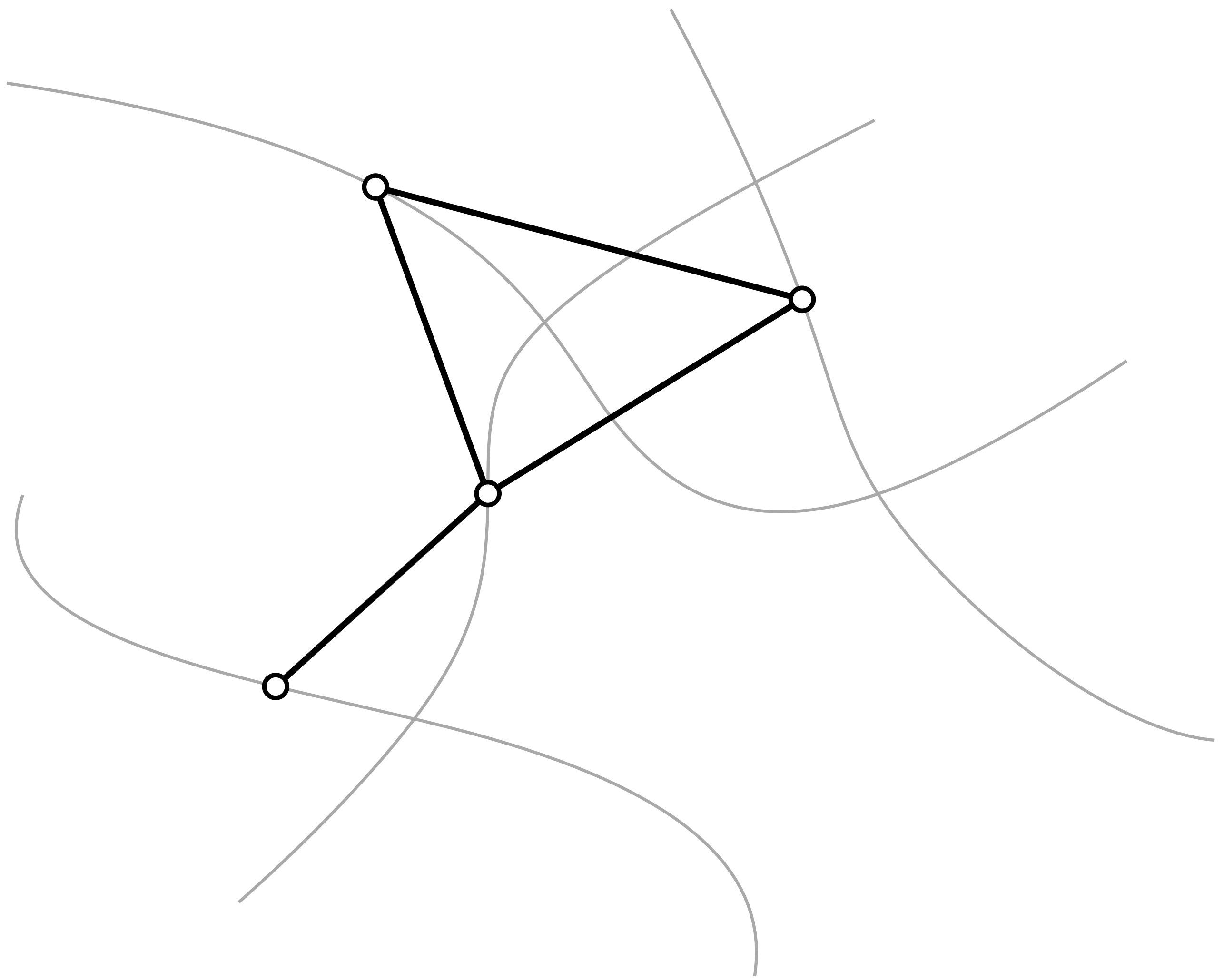


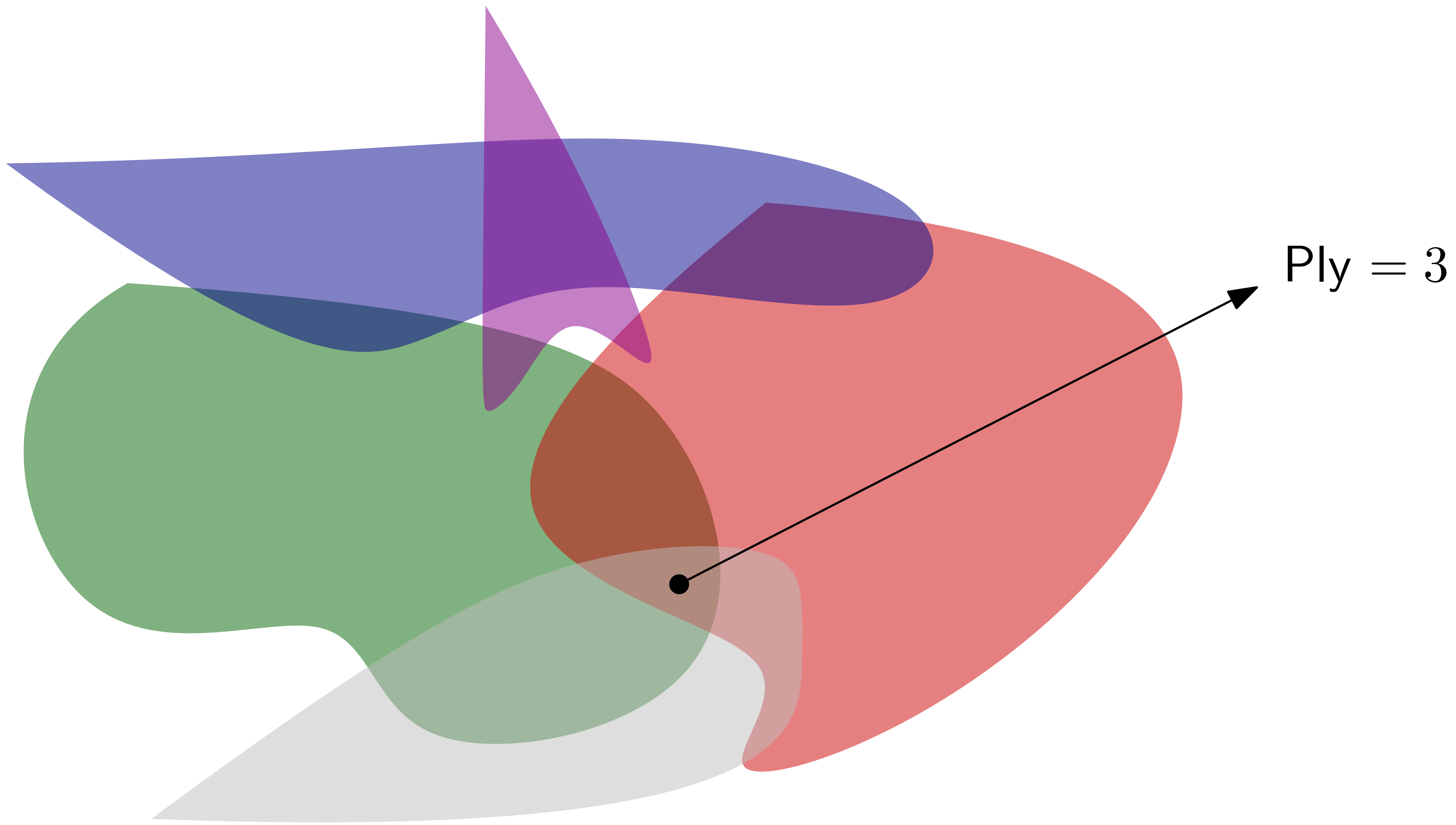
String graphs:
intersection graphs of curves in the plane

Geodesic disk graphs $\subset$ String graphs

Lee's Separator Theorem for String Graphs
Any string graph with m edges has a balanced separator of size $O(\sqrt{m})$.

And a few more

Ply of a set of objects:
maximum number of objects with a common intersection.



Construction

Step 1: Reducing the ply. Repeatedly check whether there exists a $p \in F$ with ply$(p) \geq n^{1/5}$. Remove all such cliques from $\mathcal{D}$ and place them in separator $\mathcal{S}$.

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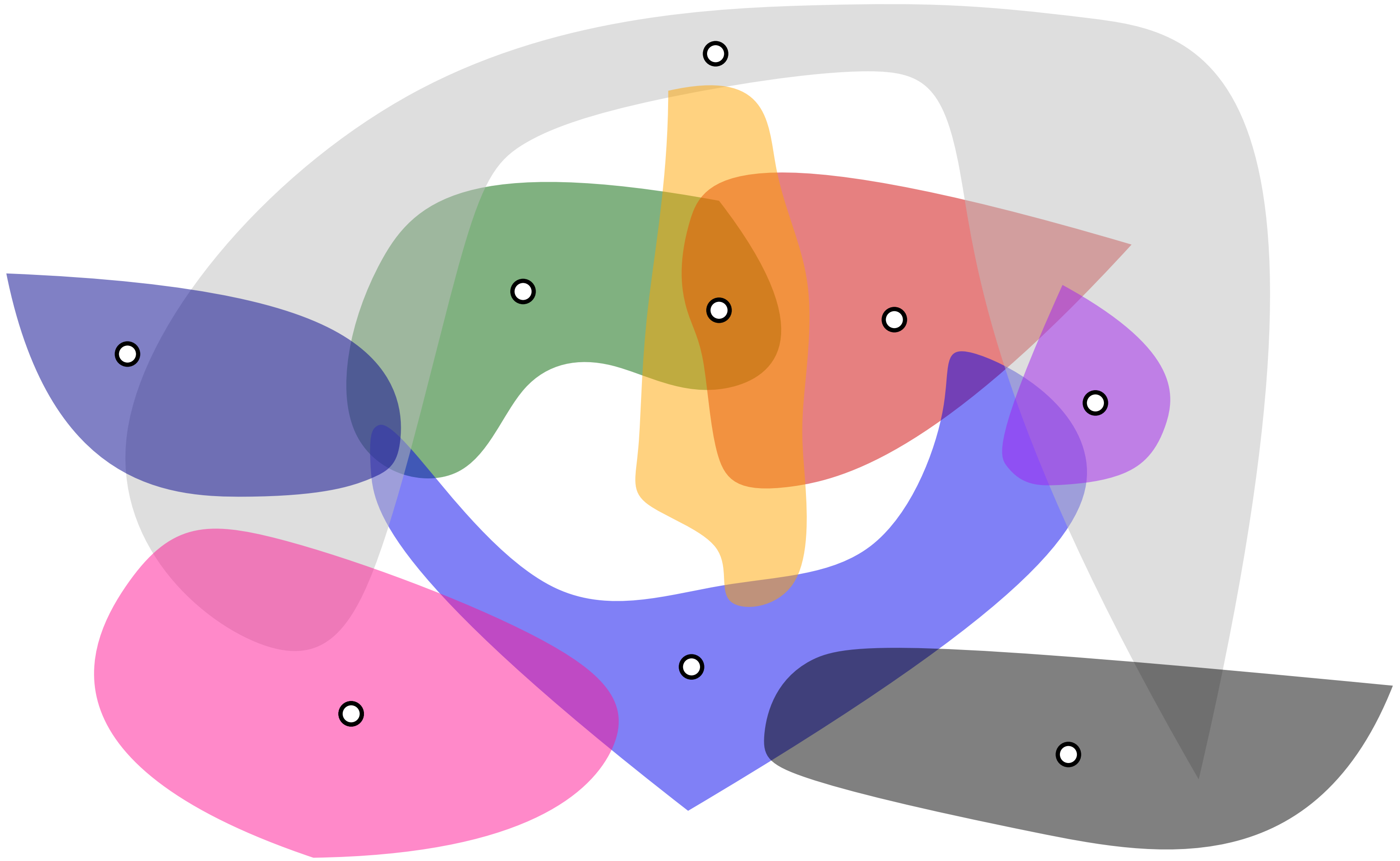

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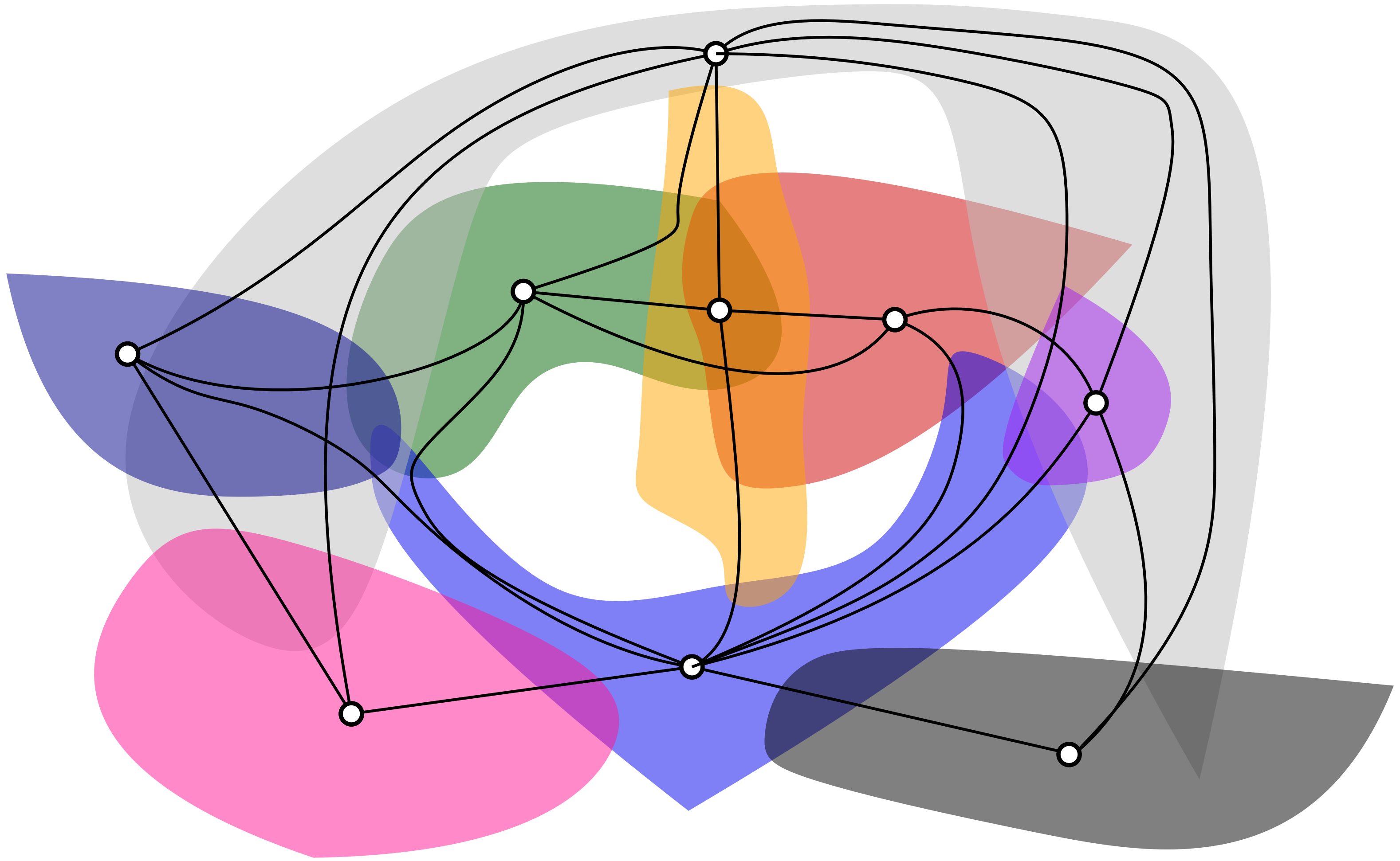

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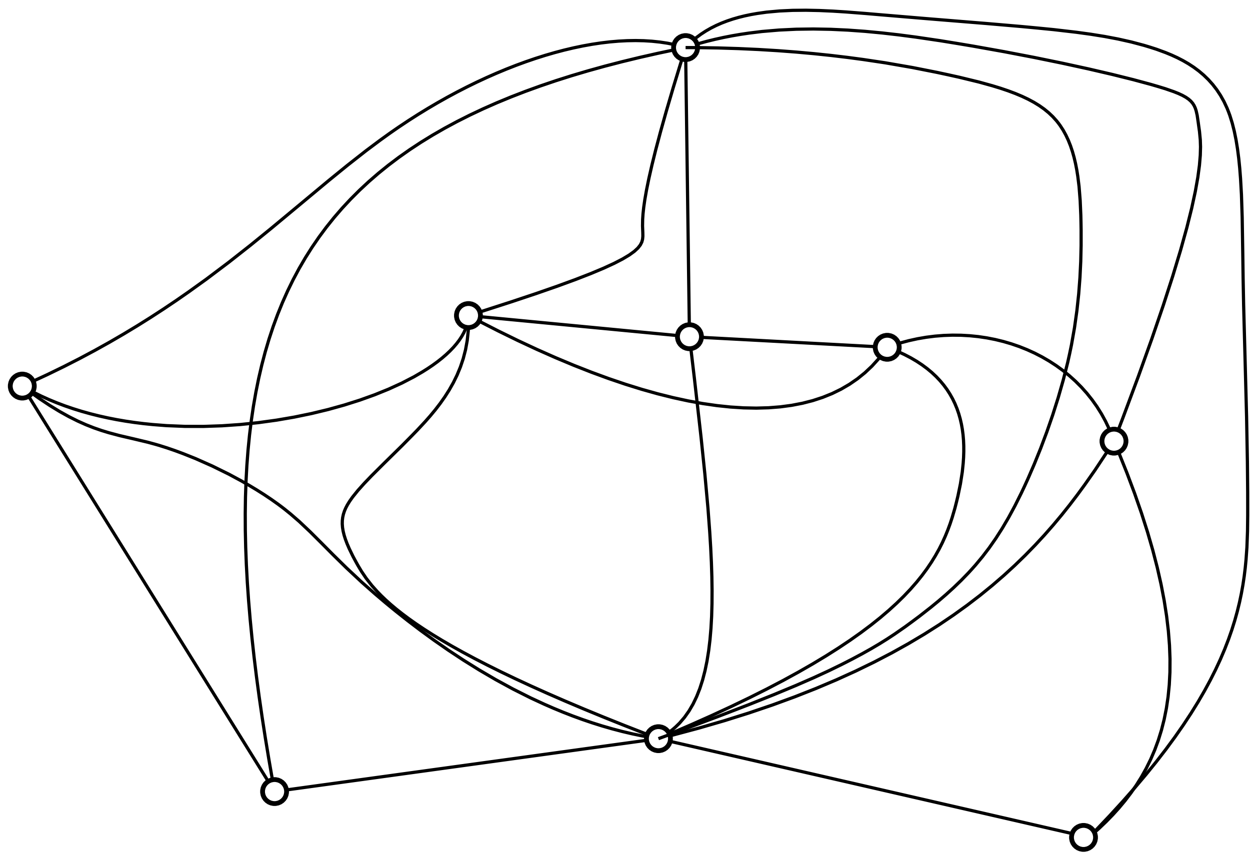

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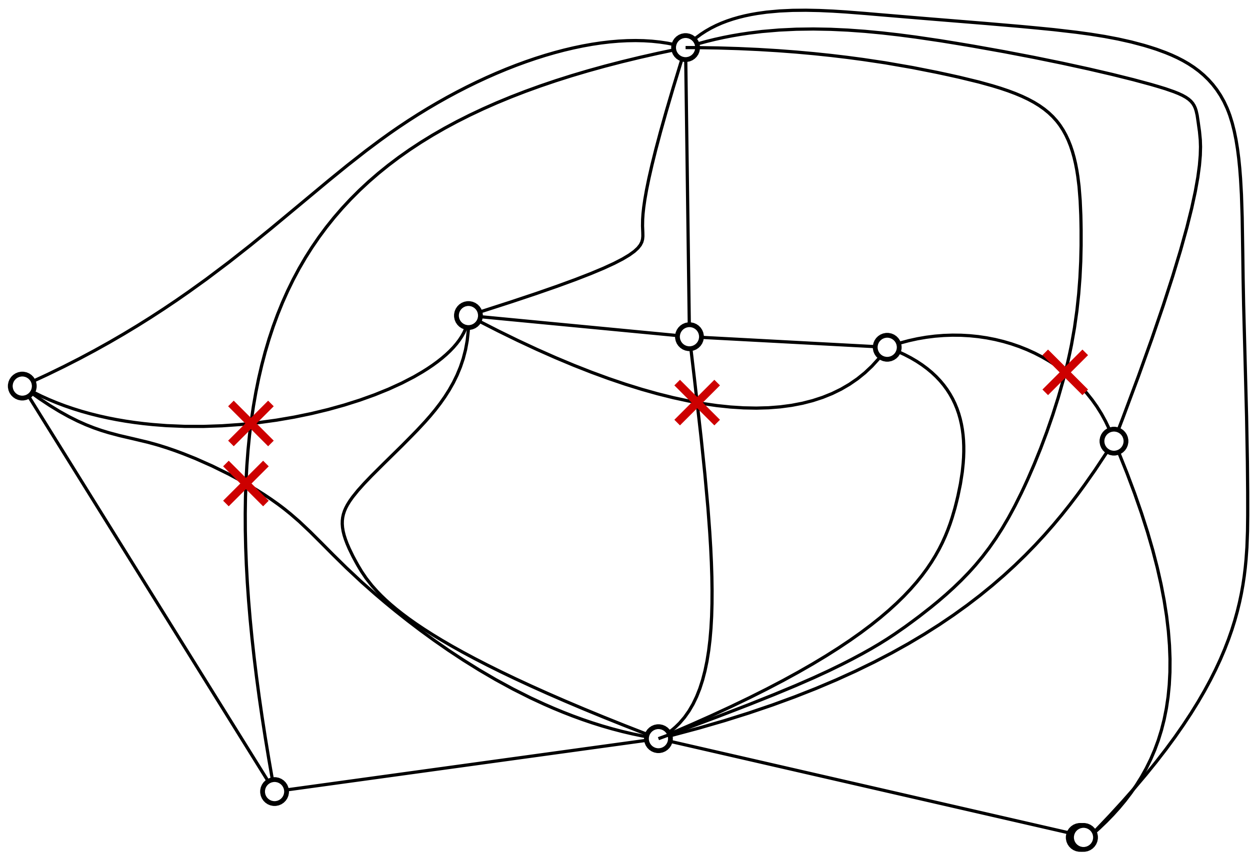

$\mathcal{X}$ = set of crossings

Crossing Lemma.

Any planar drawing of a graph with n vertices and $m \geq n$ edges has $\Omega\left(\frac{m^3}{n^2}\right)$ crossings.

Step 2: Bounding the remaining edges. The number of remaining edges is $O(n^{8/5})$.

Proof by contradiction.

Main idea:

- Assume that $|E| > cn^{8/5}$ edges.
- From Crossing Lemma, $|\mathcal{X}| > c' \frac{|E|^3}{n^2} > \ldots >$ useful bound
- Show that $\sum_{x \in \mathcal{X}} \text{ply}(x) \geq |\mathcal{X}| n^{1/5}$
- Then there exists a crossing $x \in \mathcal{X}$ with $\text{ply}(x) \geq n^{1/5}$, contradiction.

Conclusion

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- Improvement to $O(n^{3/4+\varepsilon})$.
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- Further improving the upper bounds?
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Thank you!

STUDENT PRESENTATION

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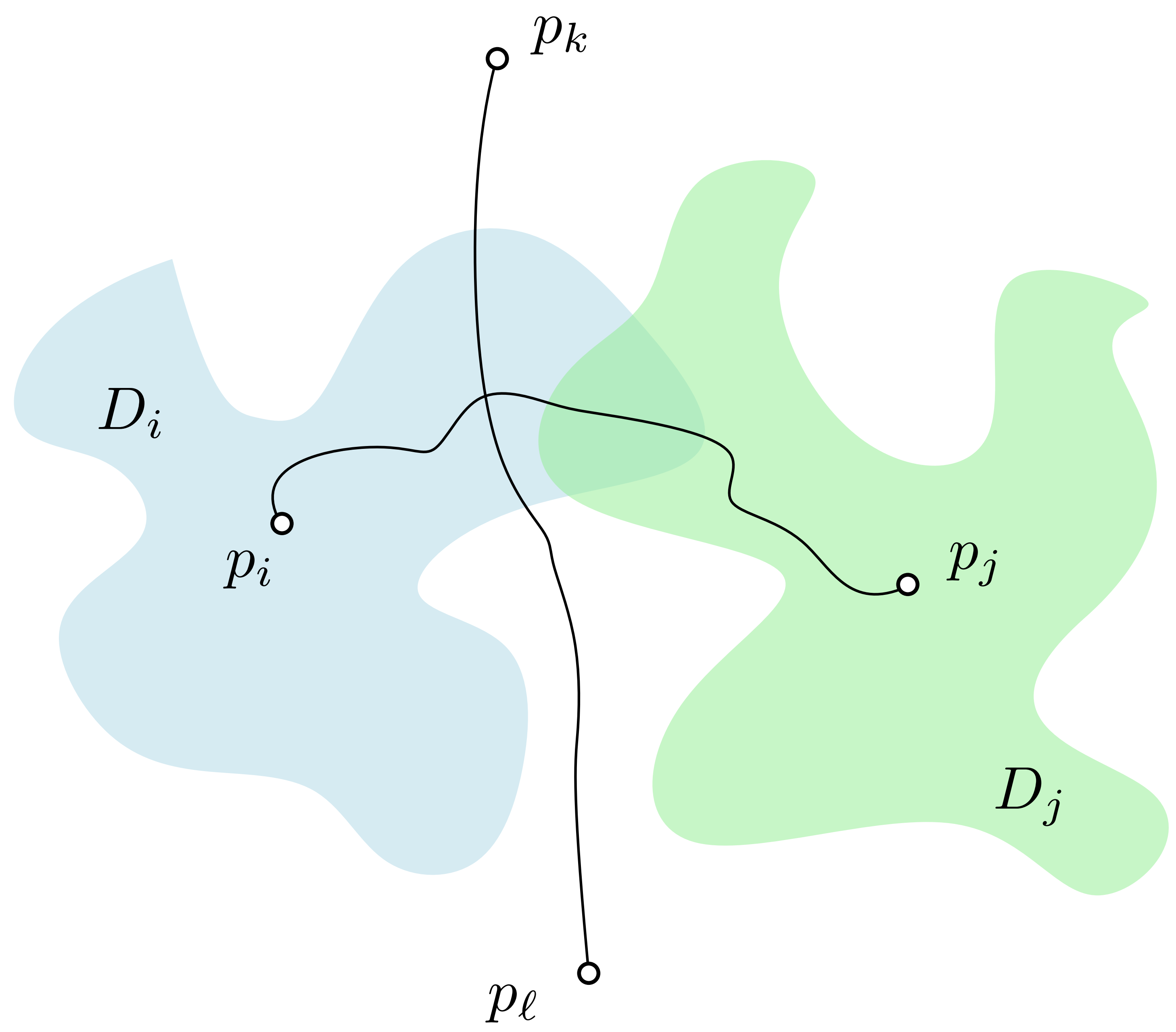

- p_m, r_m: center and radius of D_m respectively

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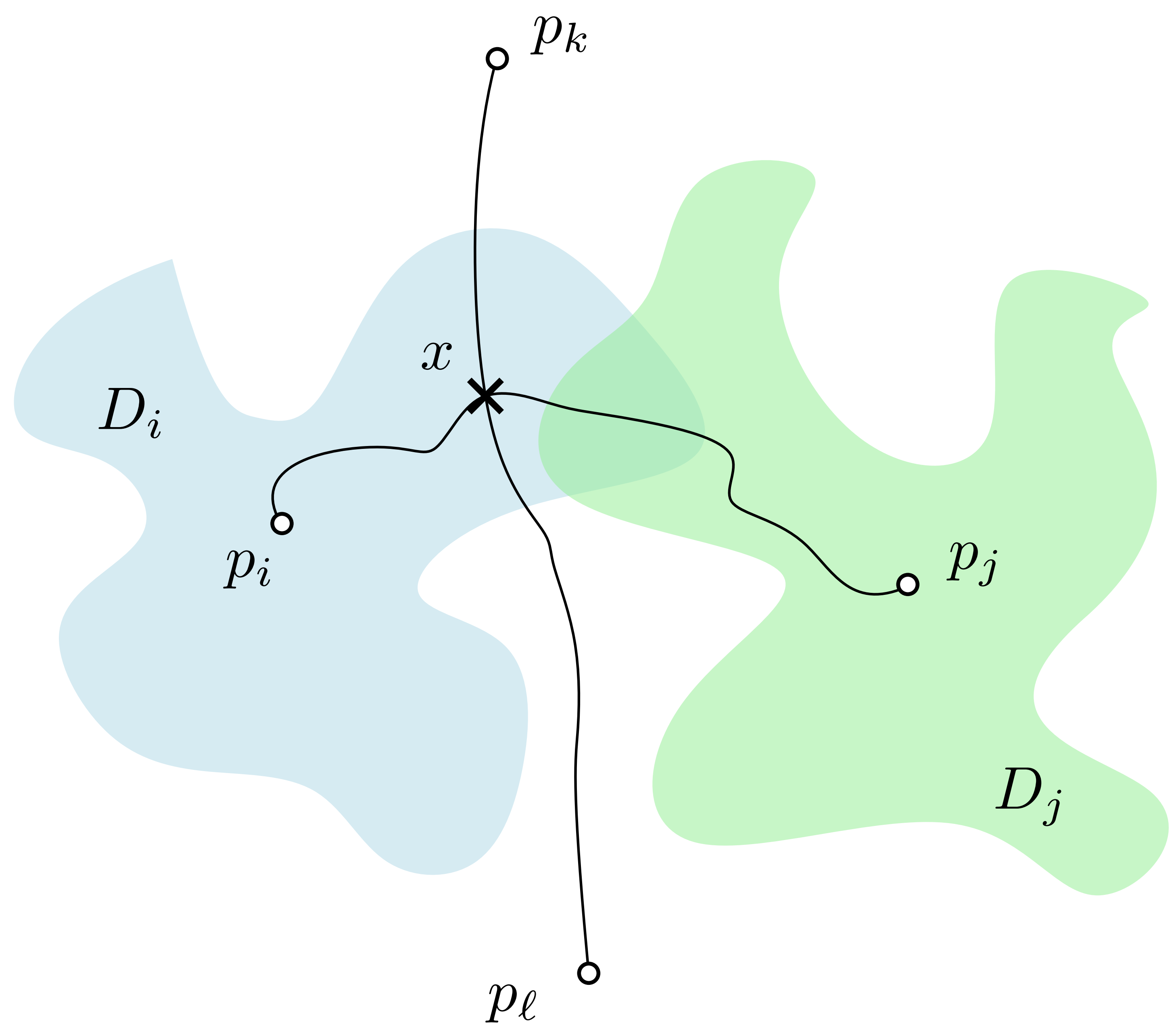

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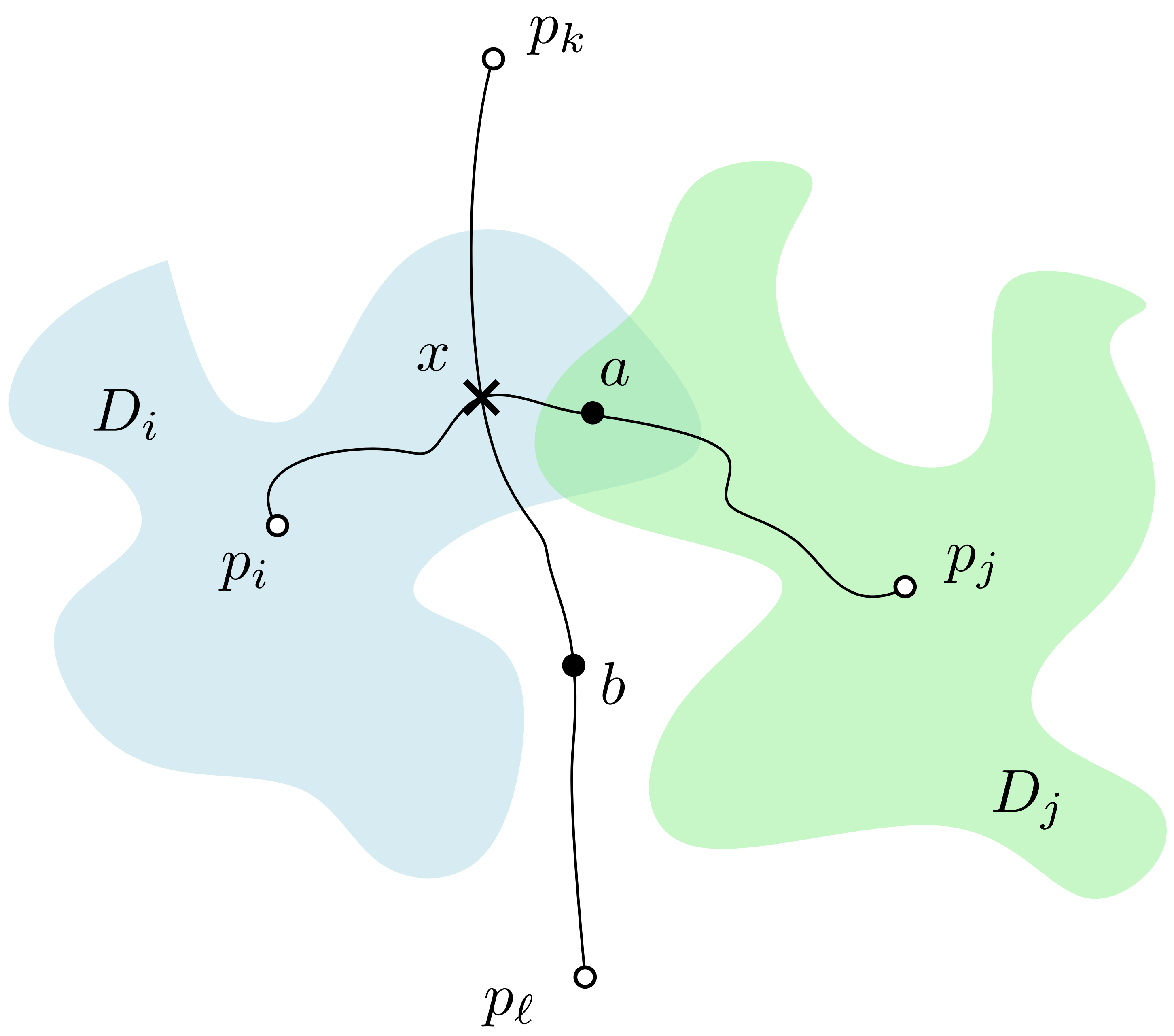

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Split every edge in two **half-edges** by choosing $a \in D_i \cap D_j$, $b \in D_k \cap D_\ell$.

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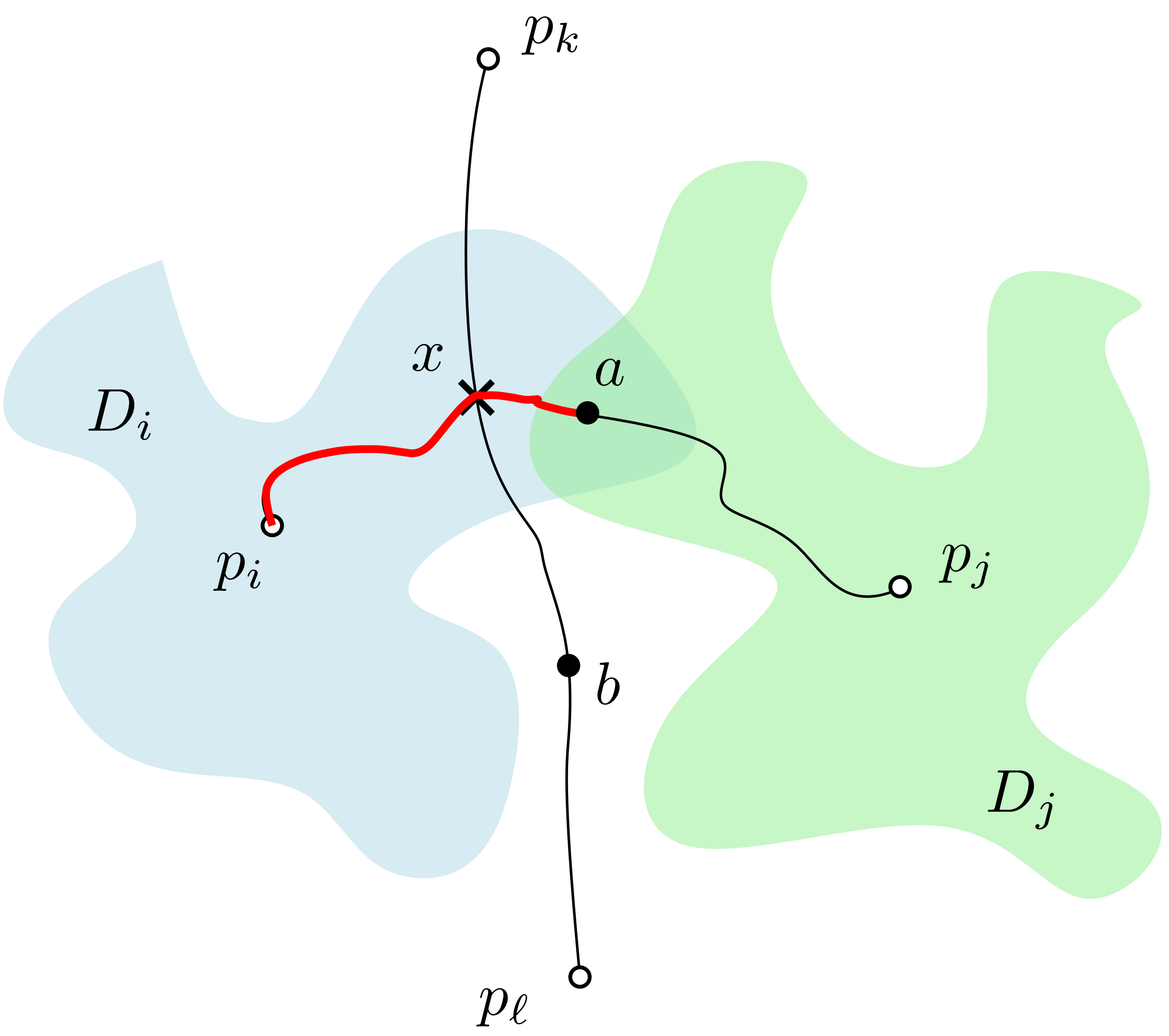

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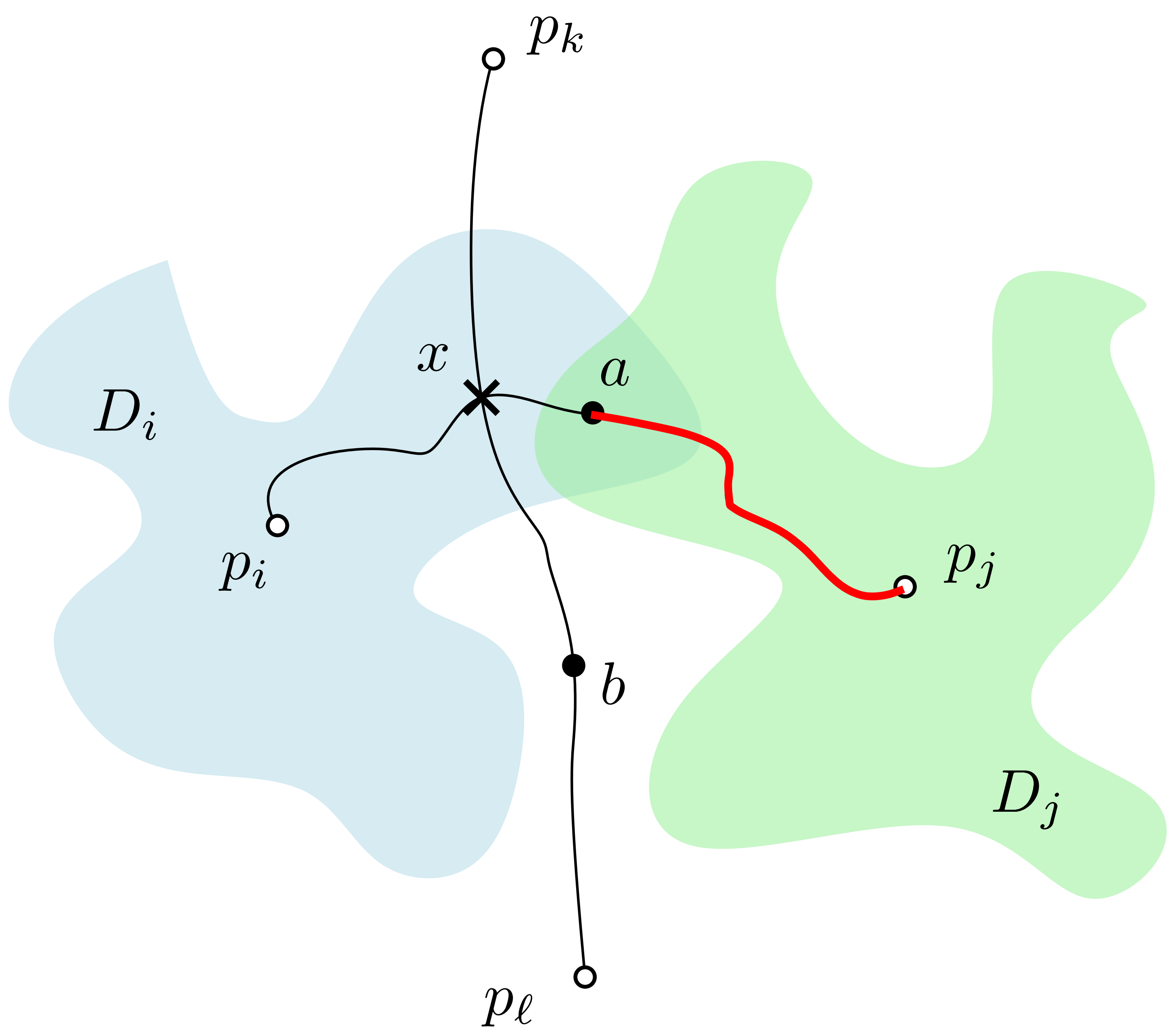

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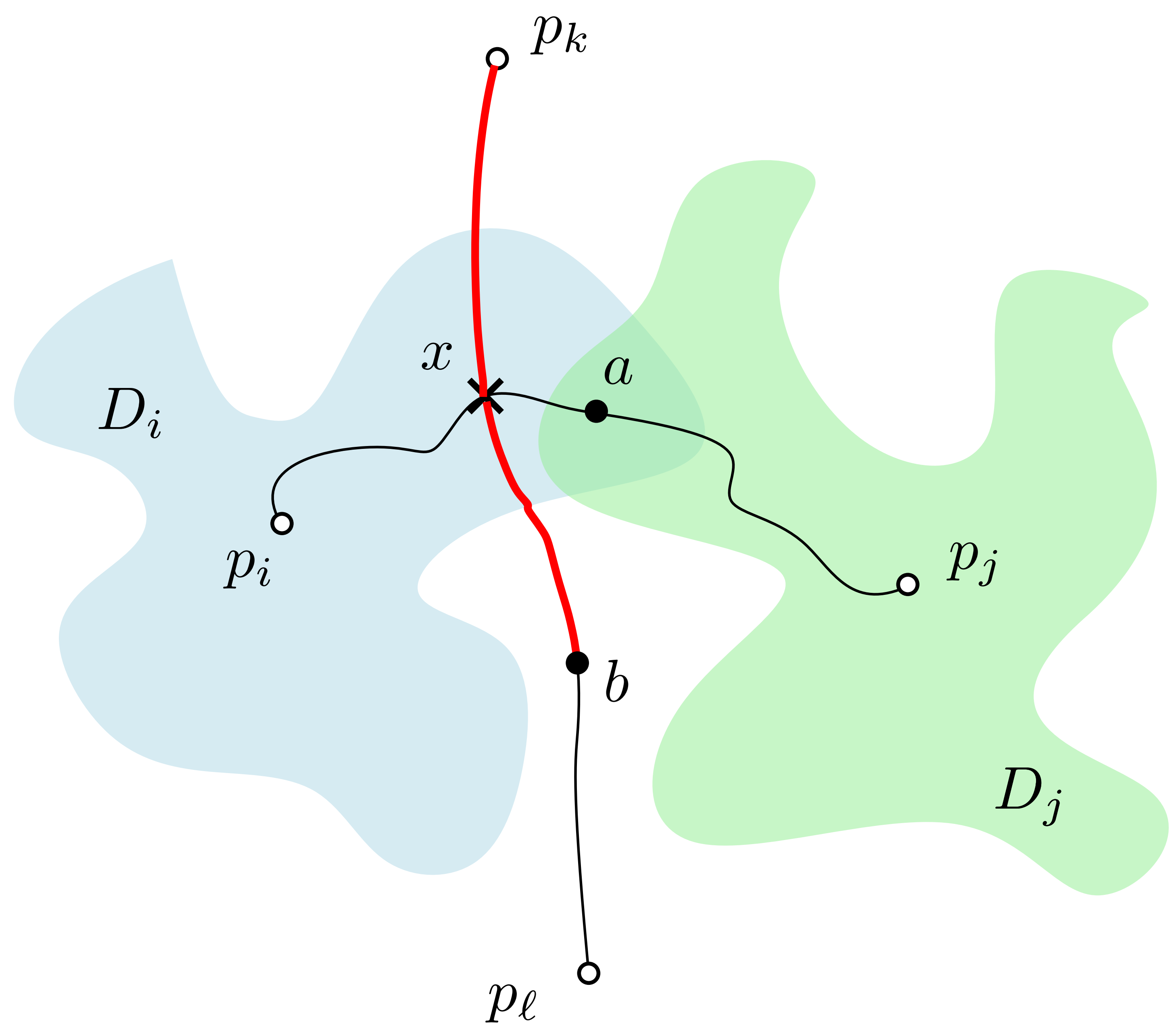

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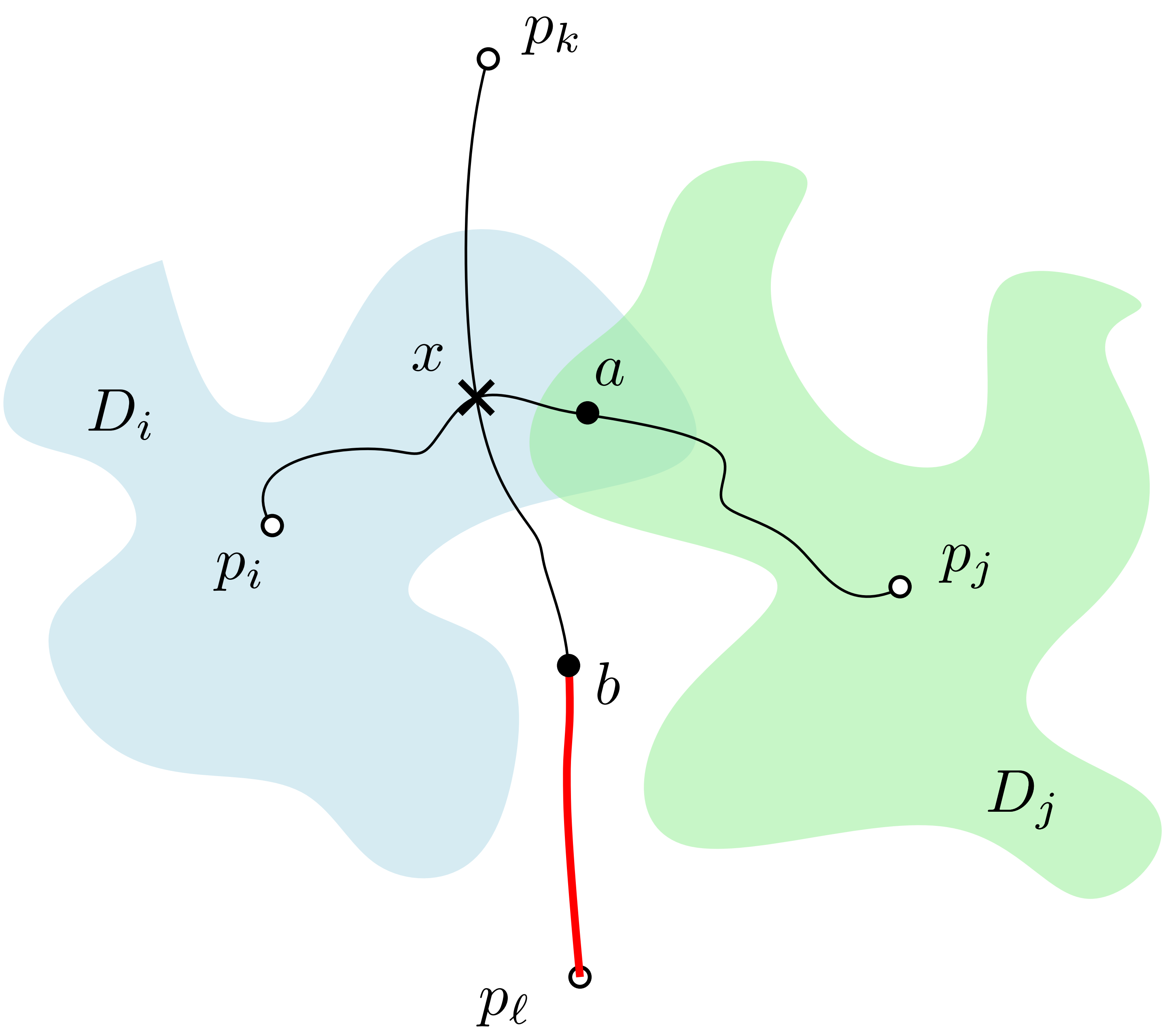

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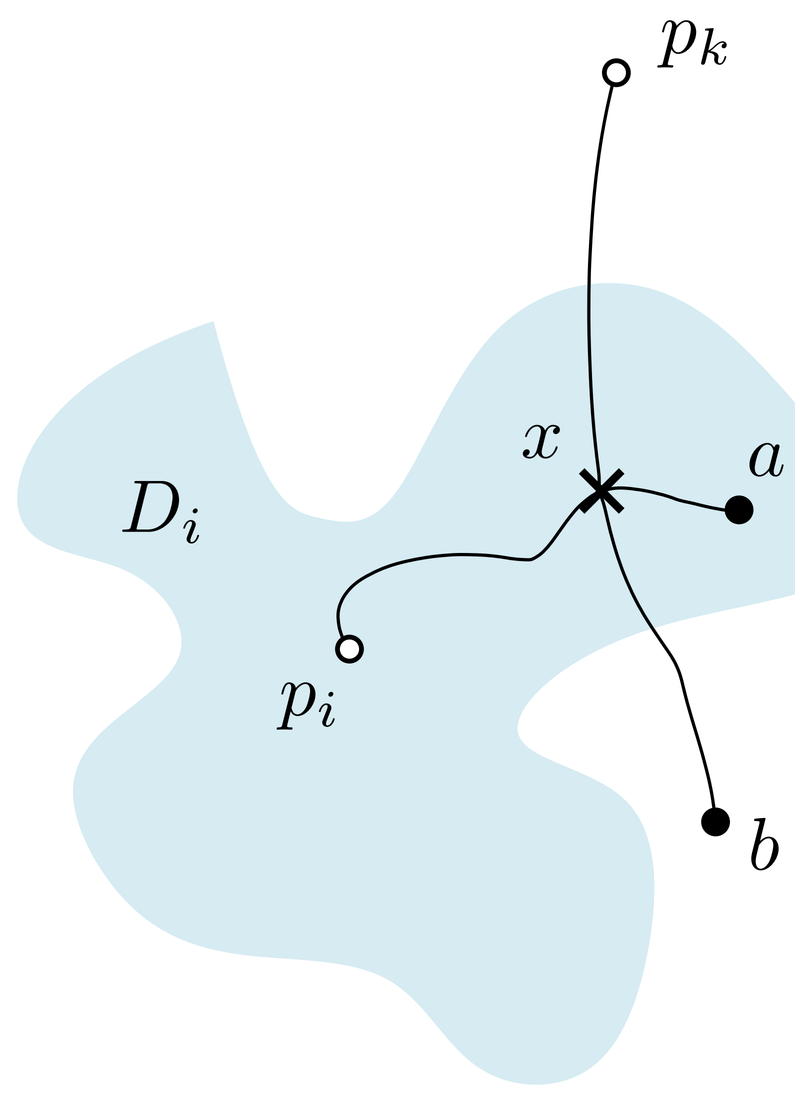

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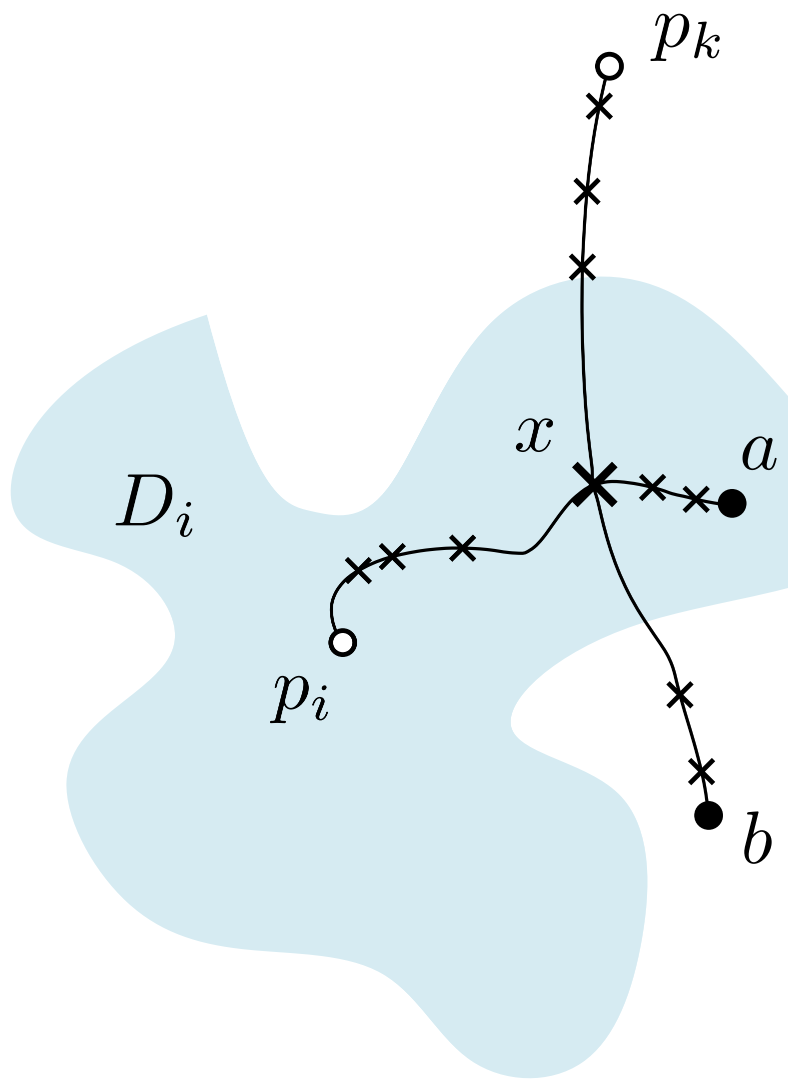

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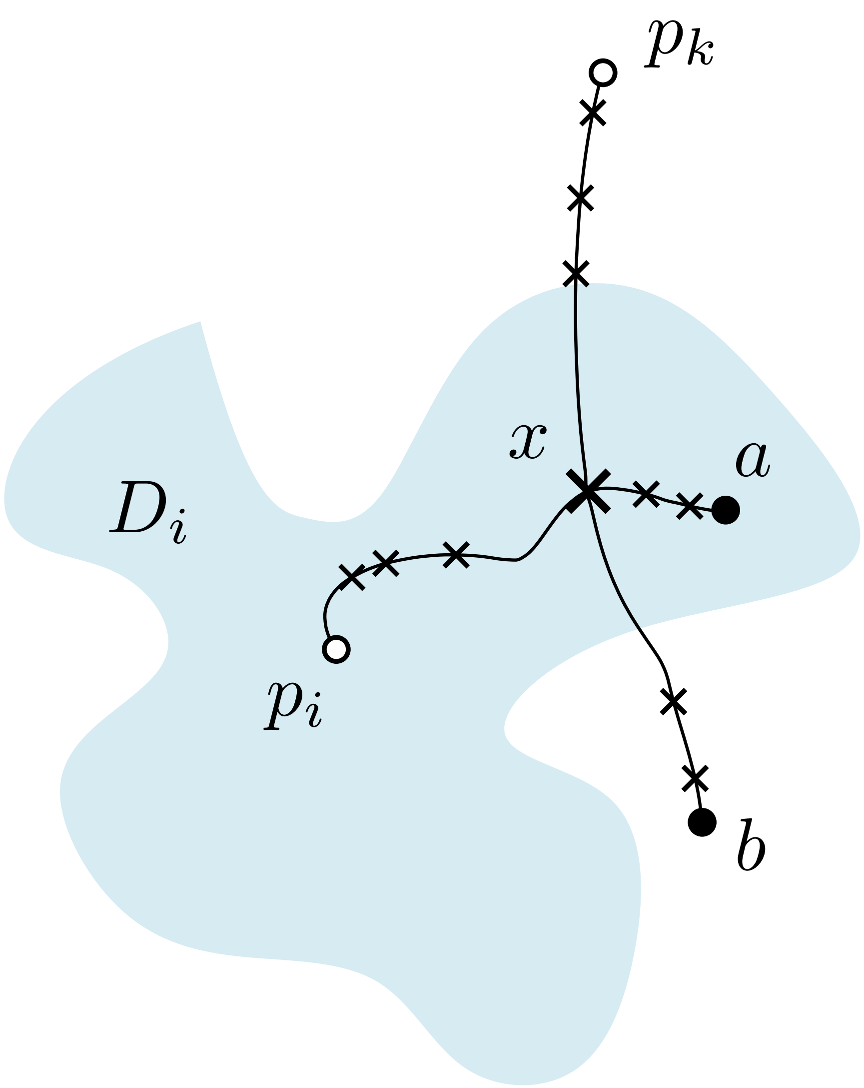

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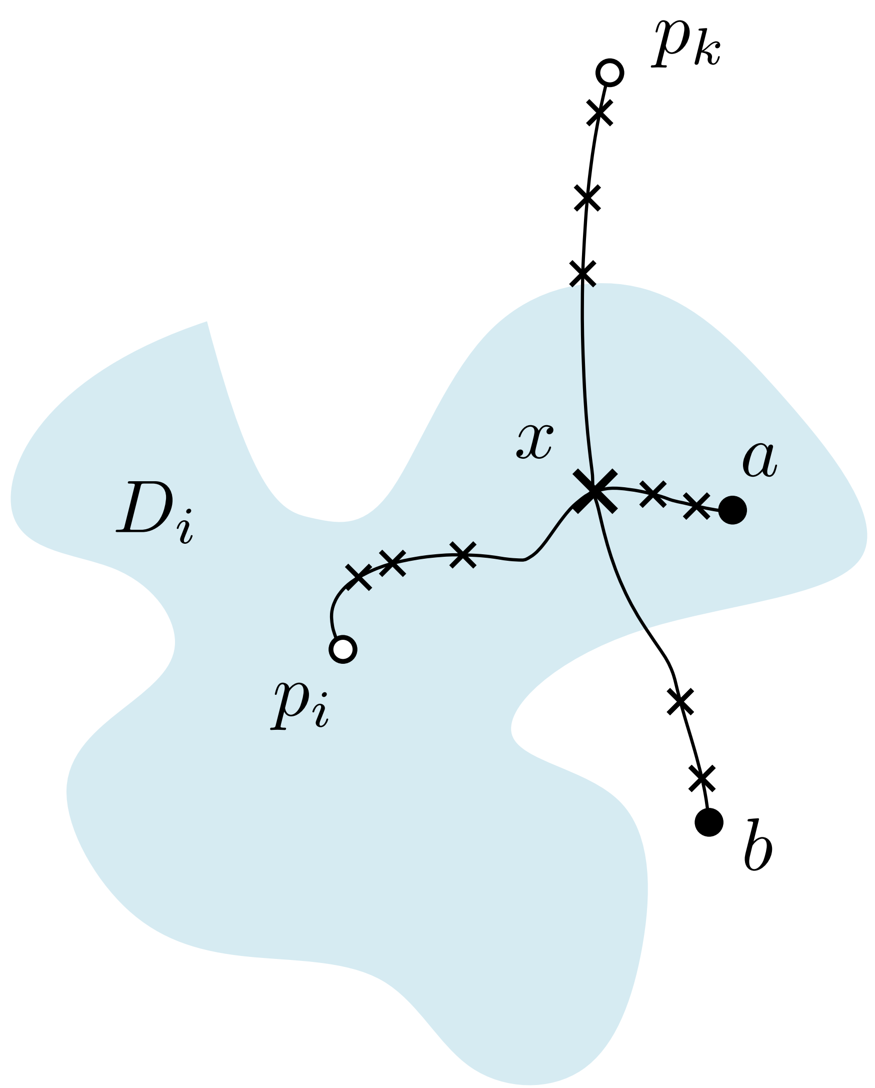
Either:

- $d(x,a) < d(x,b)$ or
- $d(x,b) \leqslant d(x,a)$



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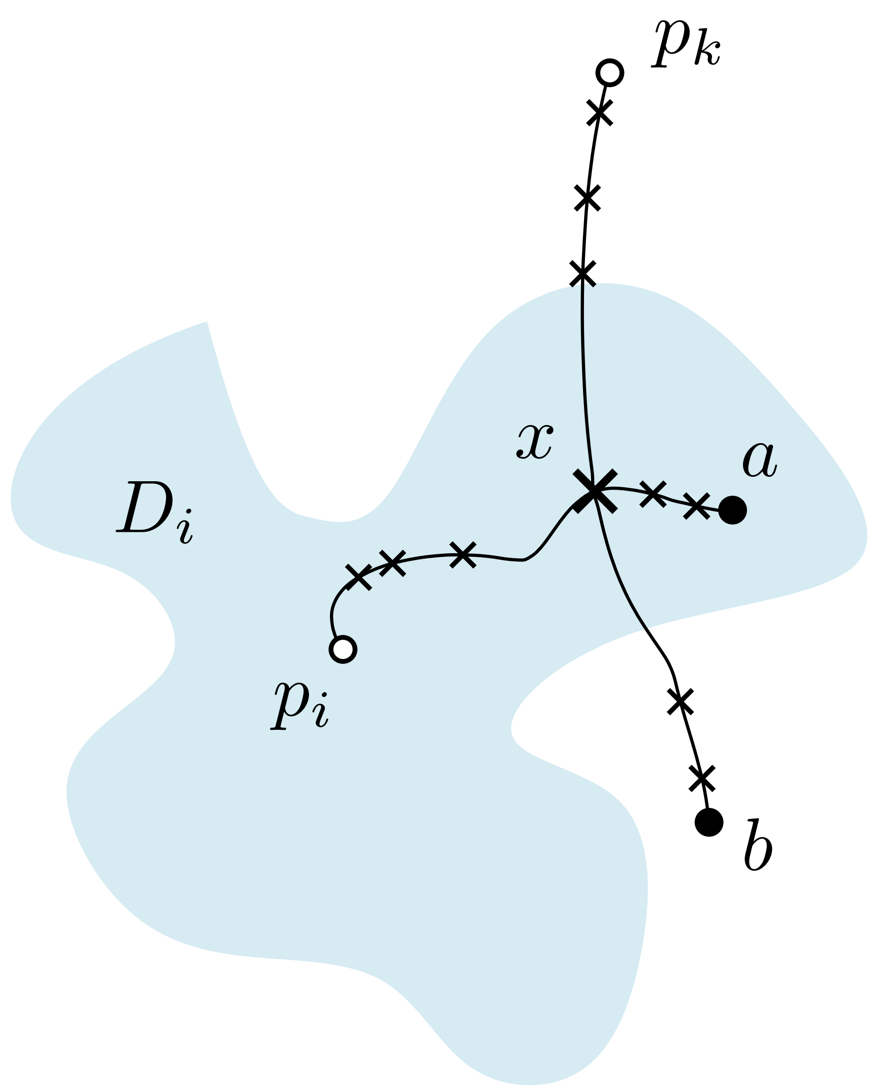

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- $d(x,a) < d(x,b)$

  $\Rightarrow d(p_k,x) + d(x,a) < d(p_k,x) + d(x,b)$

  $\Rightarrow d(p_k,x) + d(x,a) < r_k$

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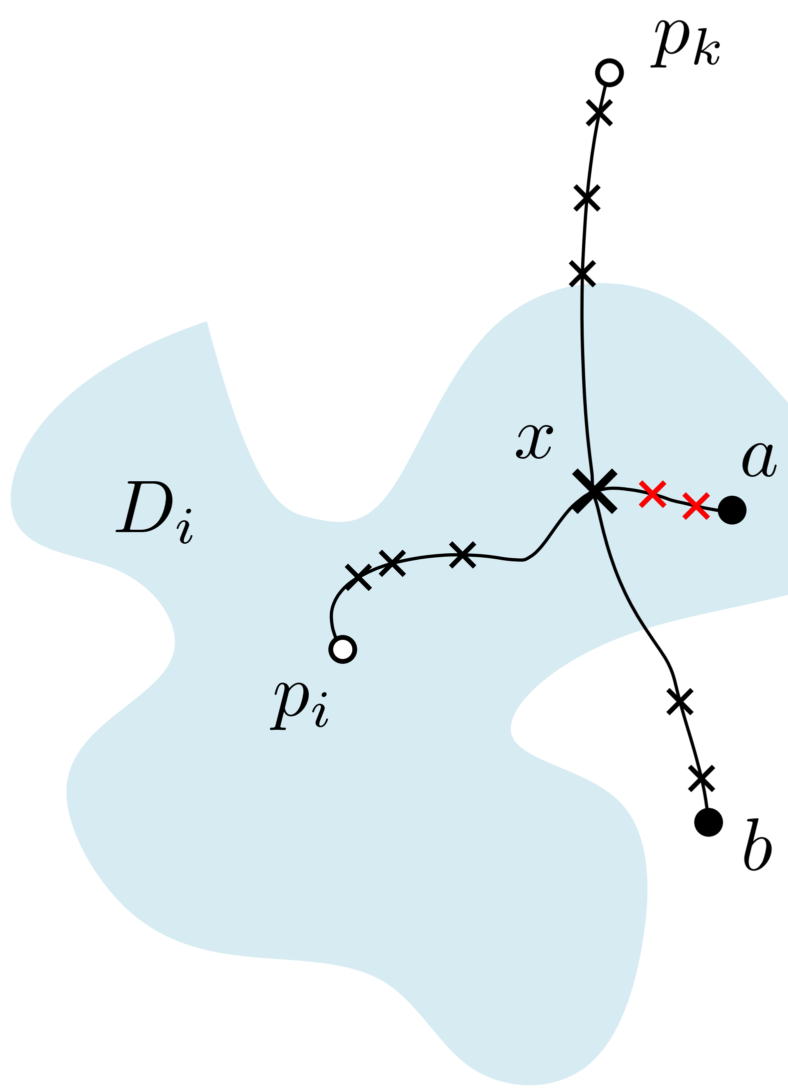

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  $\Rightarrow$ red crossings are in D_k

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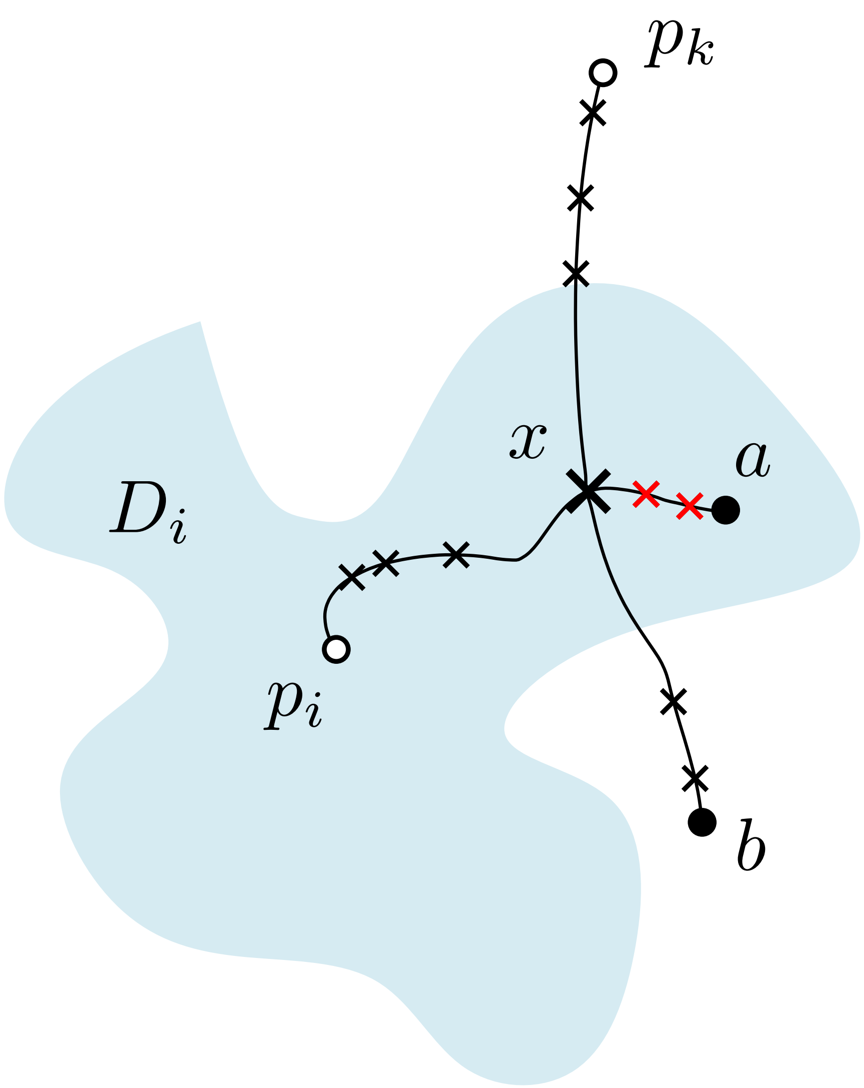

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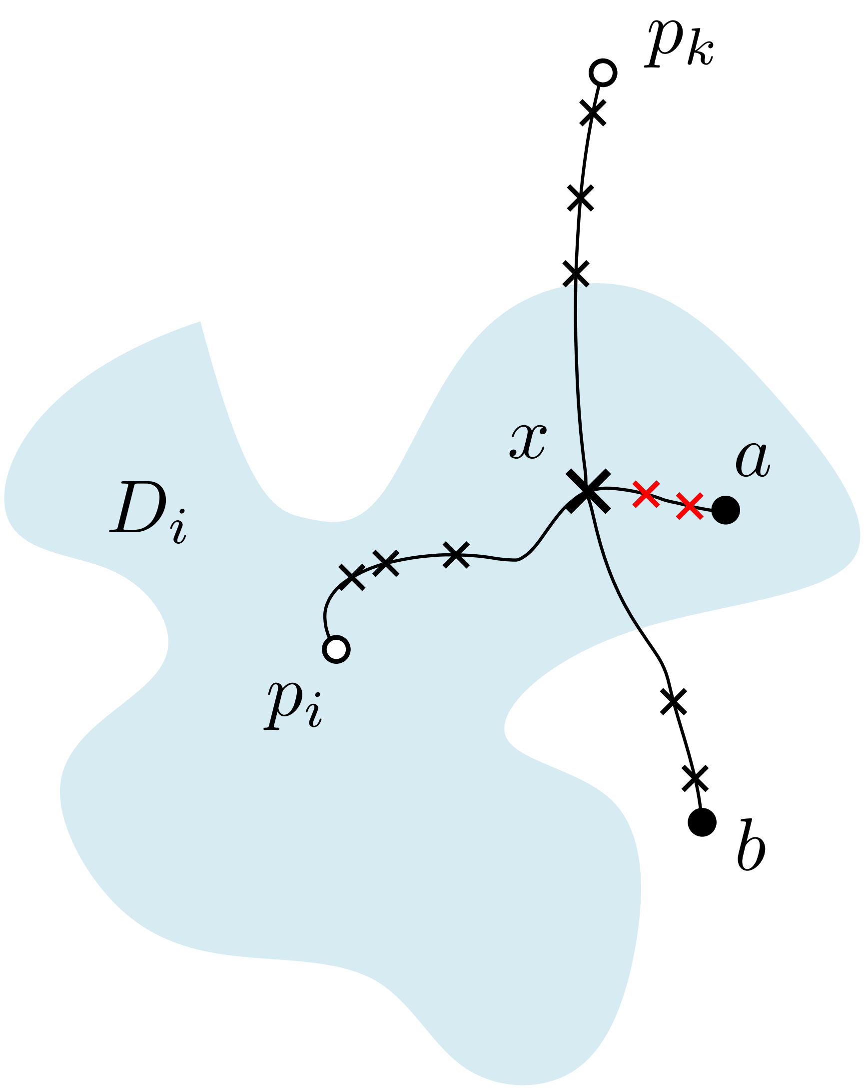
  $\Rightarrow d(p_k,x) + d(x,a) < d(p_k,x) + d(x,b)$

  $\Rightarrow d(p_k,x) + d(x,a) < r_k$

  $\Rightarrow$ red crossings are in D_k

Take-away.

Intuitively, the existence of crossing x is "responsible" for an increase in the ply of these red crossings.



- p_m, r_m: center and radius of D_m respectively

- $d(x,a) < d(x,b)$
  $\Rightarrow d(p_k,x) + d(x,a) < d(p_k,x) + d(x,b)$
  $\Rightarrow d(p_k,x) + d(x,a) < r_k$
  $\Rightarrow$ red crossings are in D_k

Take-away.

Intuitively, the existence of crossing x is "responsible" for an increase in the ply of these red crossings.

Reality is more technical...