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Ioannina, Greece

University of Manitoba

2-Coloring Point Guards in a *k*-Guarded Polygon

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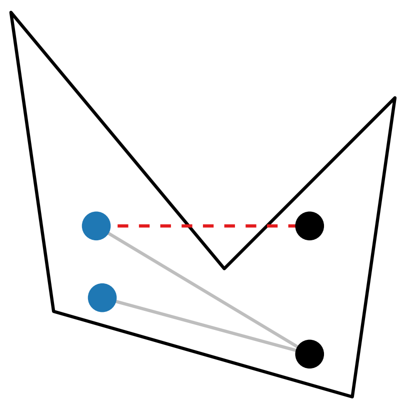
March, 2024

Preliminaries

The ***k*-Guarding Art Gallery** problem:

- given a simple polygon P in the plane,
- find a set G of points in P that *guard* P, so that each point in P is visible to at least k guards.



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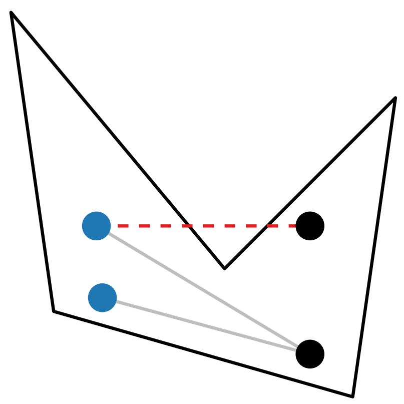

Some results for a variant where guards are on the vertices:

- $\lfloor n/3 \rfloor$ guards are enough and sometimes necessary for $k = 1$, [Chvátal, '73] [Fisk, '78]
  $\lfloor 2n/3 \rfloor$ guards are sometimes necessary for $k = 2$,
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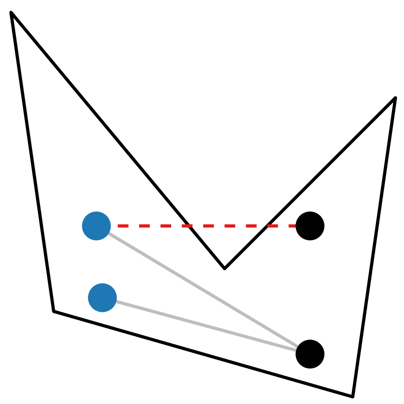

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- $O(k \log \log OPT_k(P))$-approximation algorithm. [Busto, 2013]

Preliminaries

Example:

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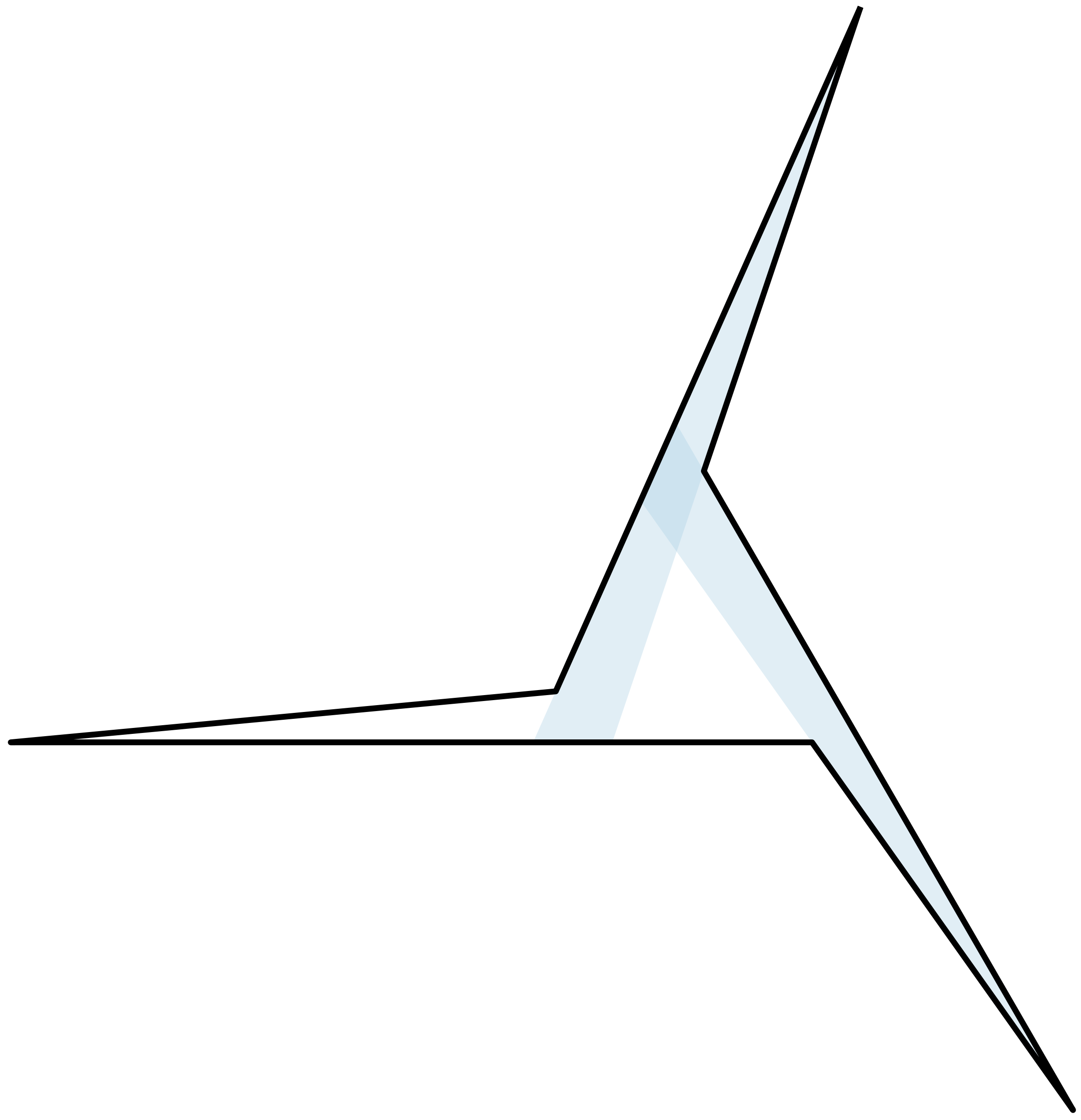

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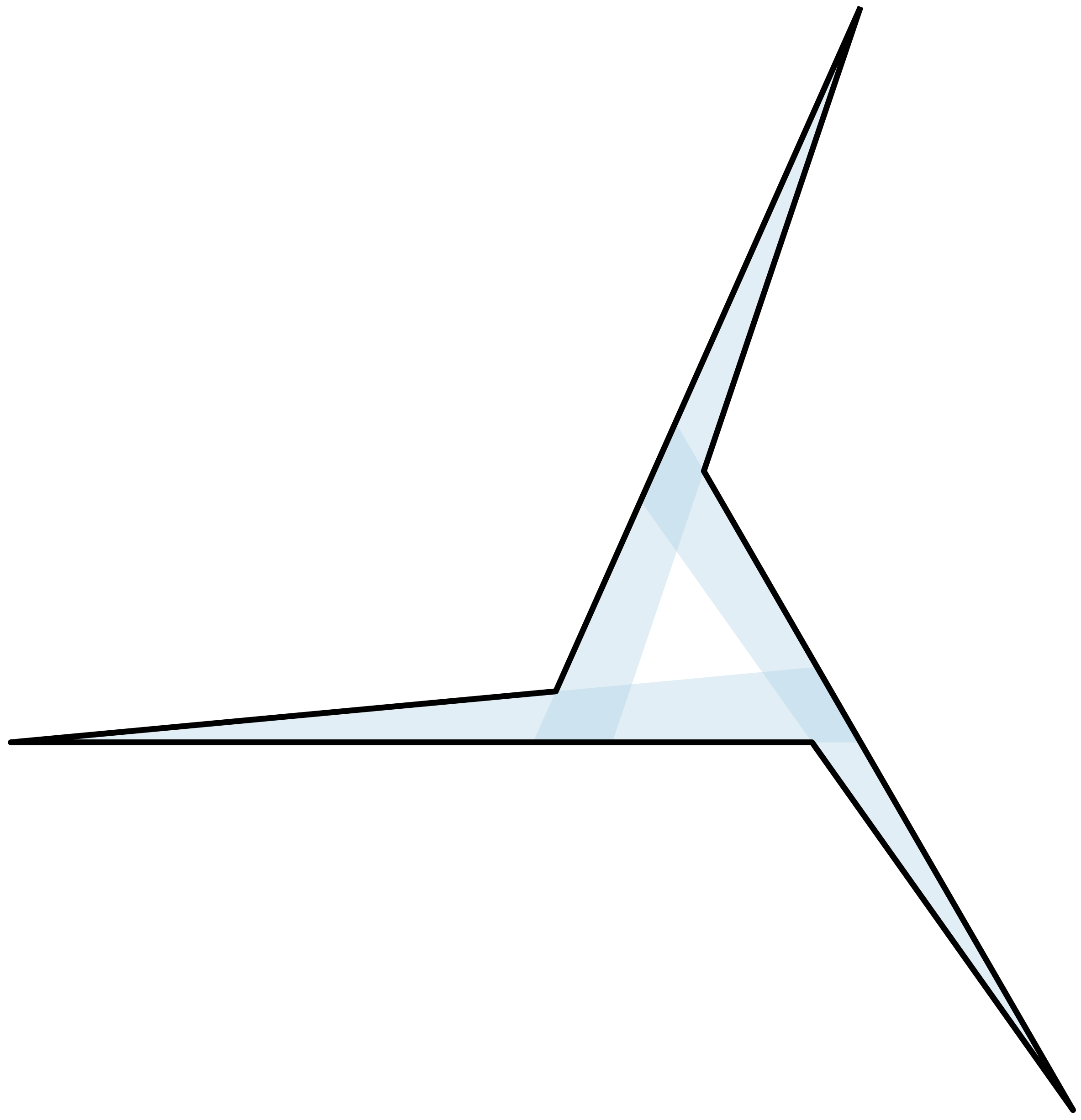

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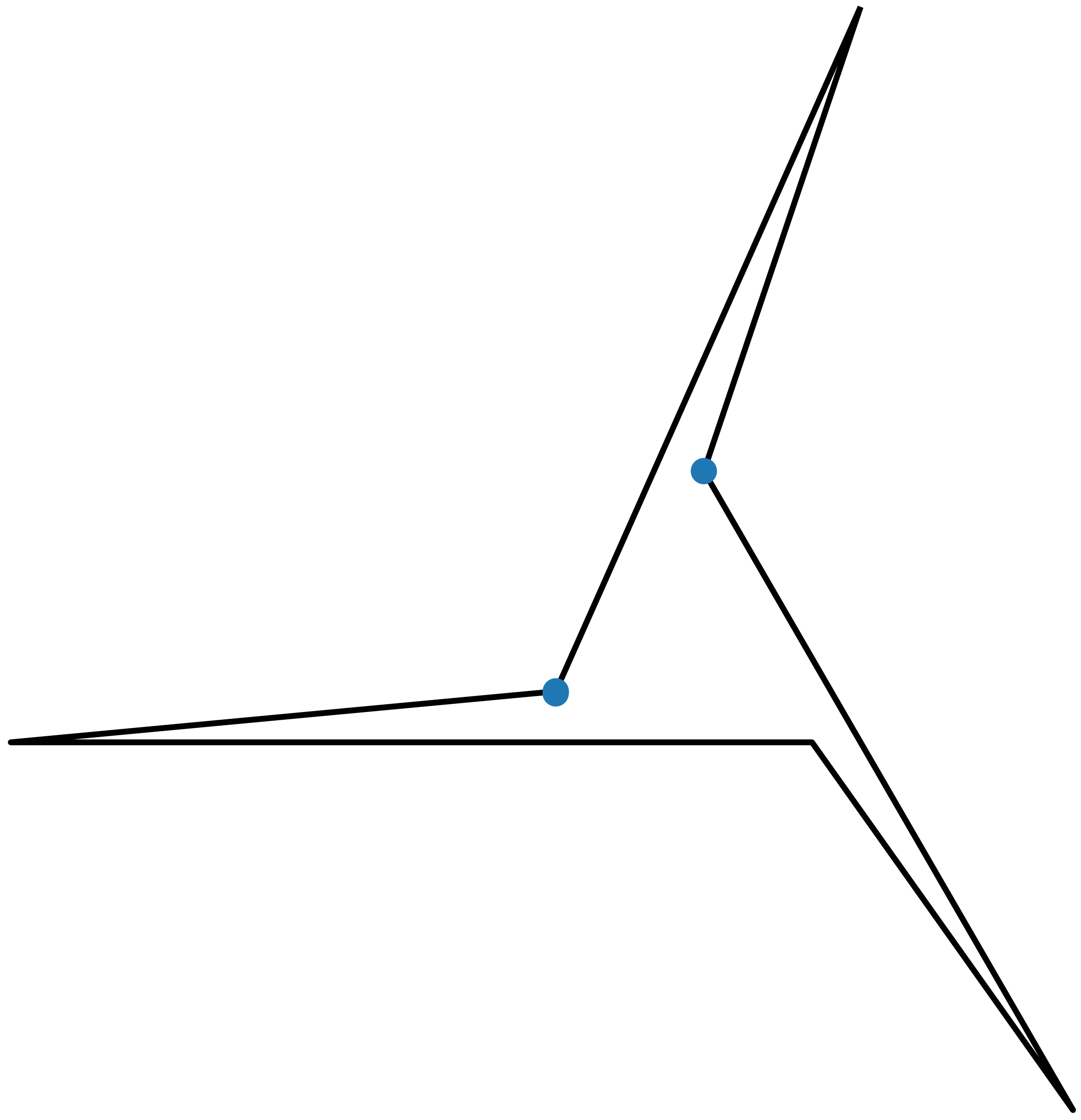

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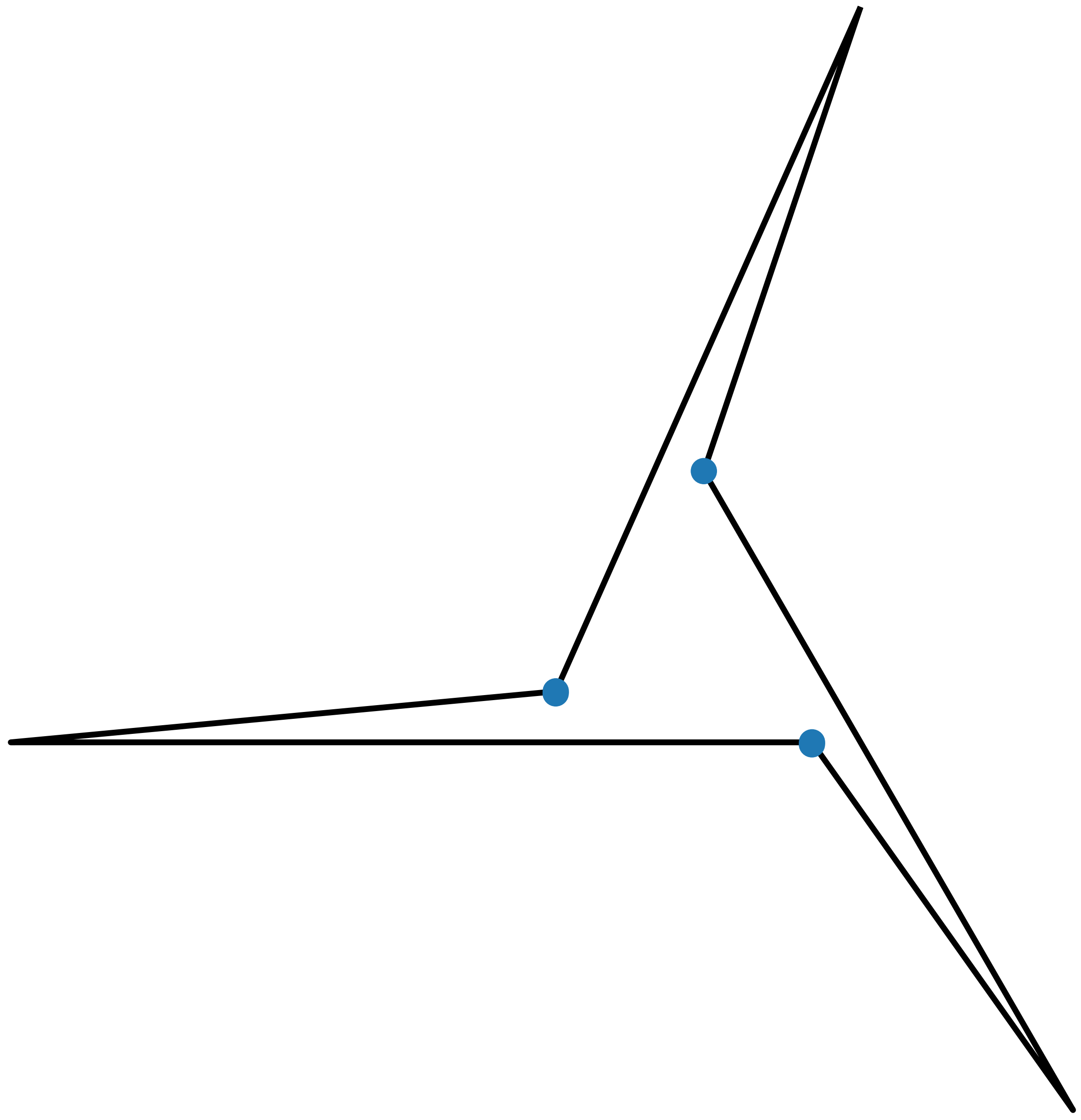

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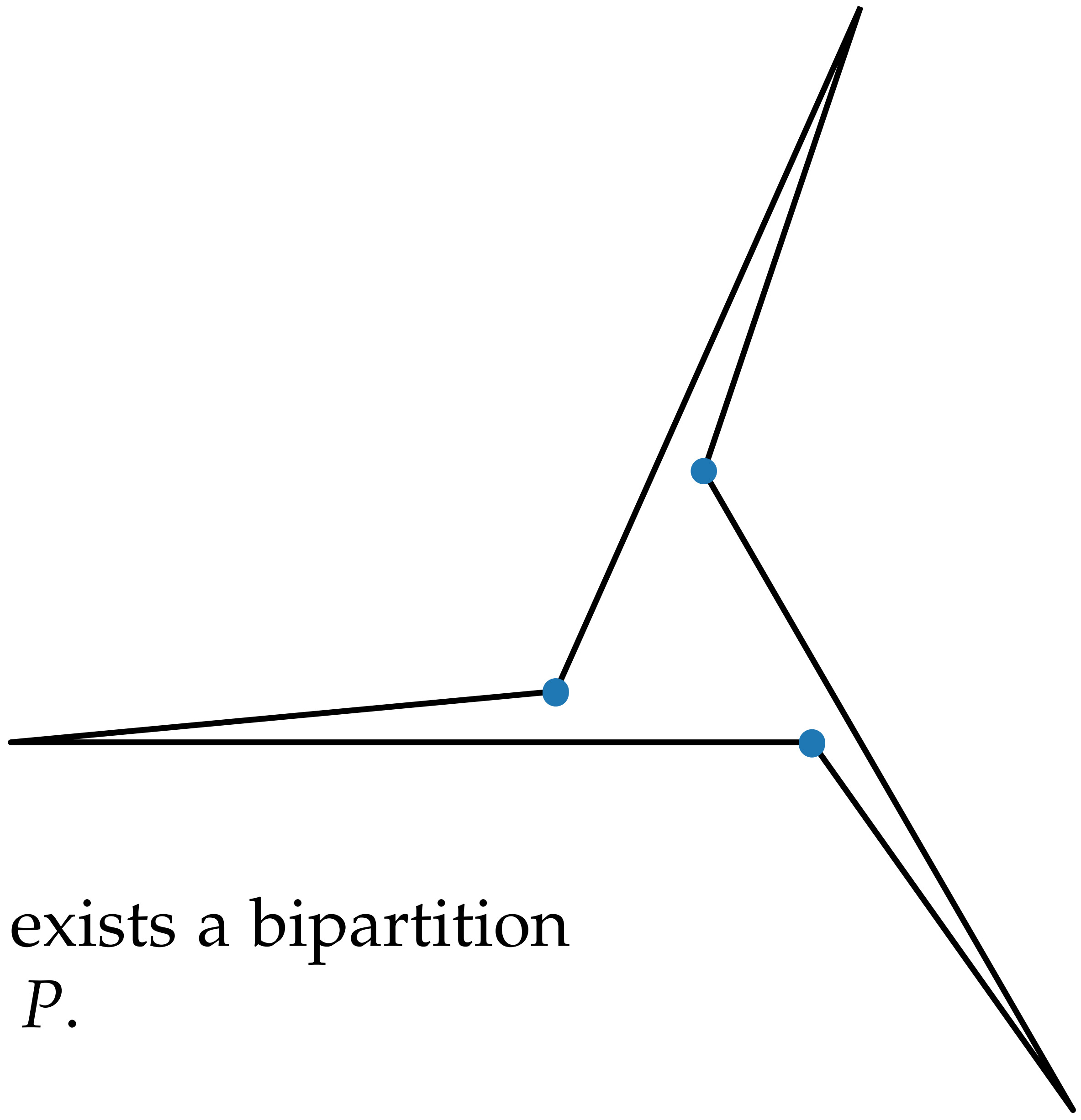

The set of guards G is **2-colorable** if there exists a bipartition of G into two sets such that each 1-guards P.

k-guardability vs. 2-colorability of a guard set G

Question: Does there exist k such that for each polygon P and each guard set G that k-guards P, there exists a 2-coloring of G?

[Morin, Bose, Carmi, WADS 2023]

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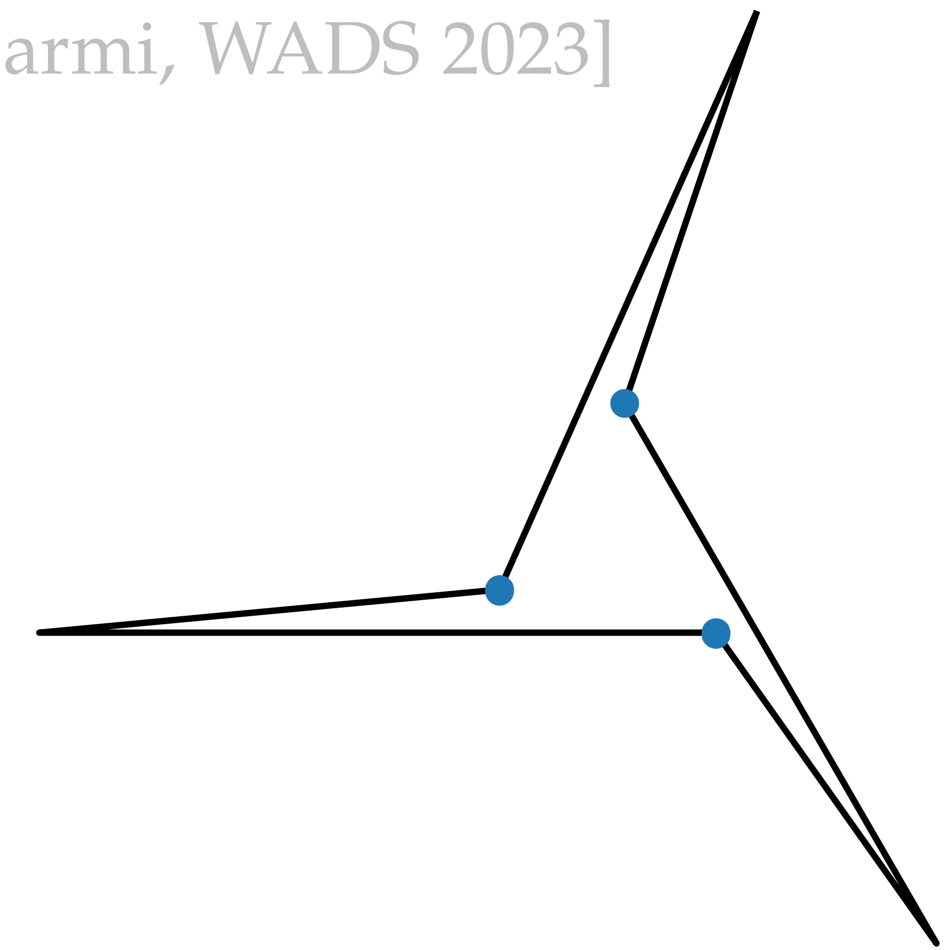

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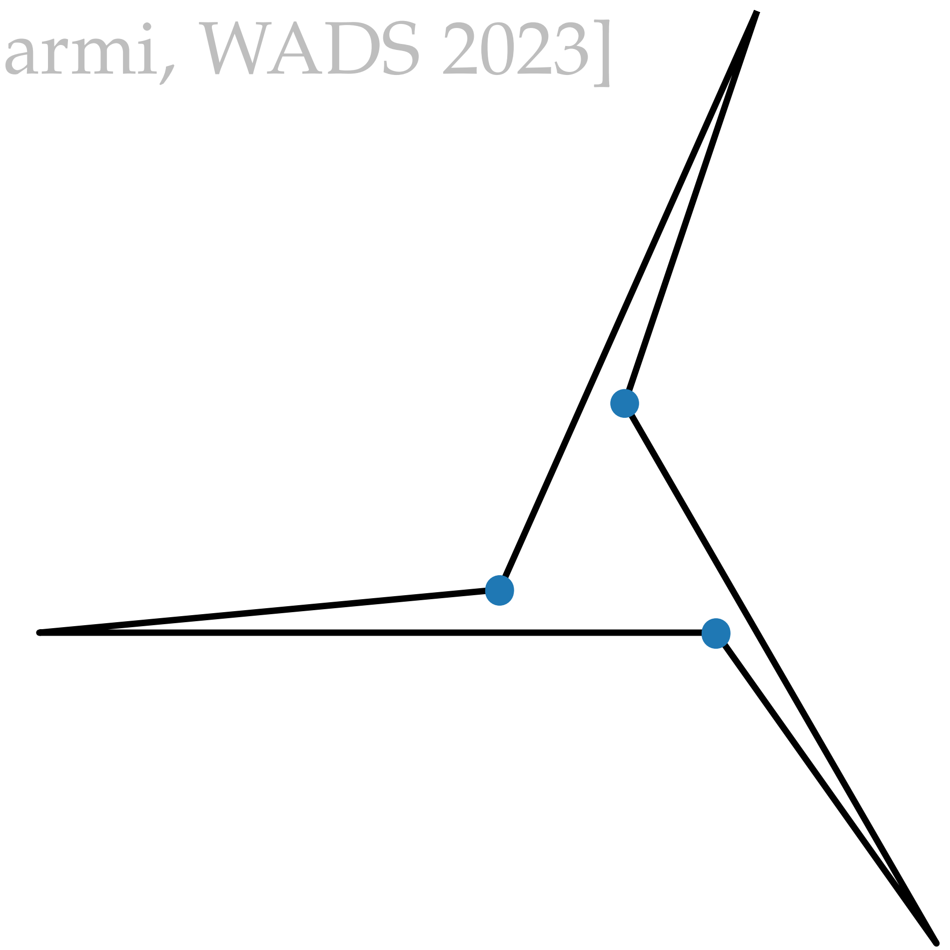

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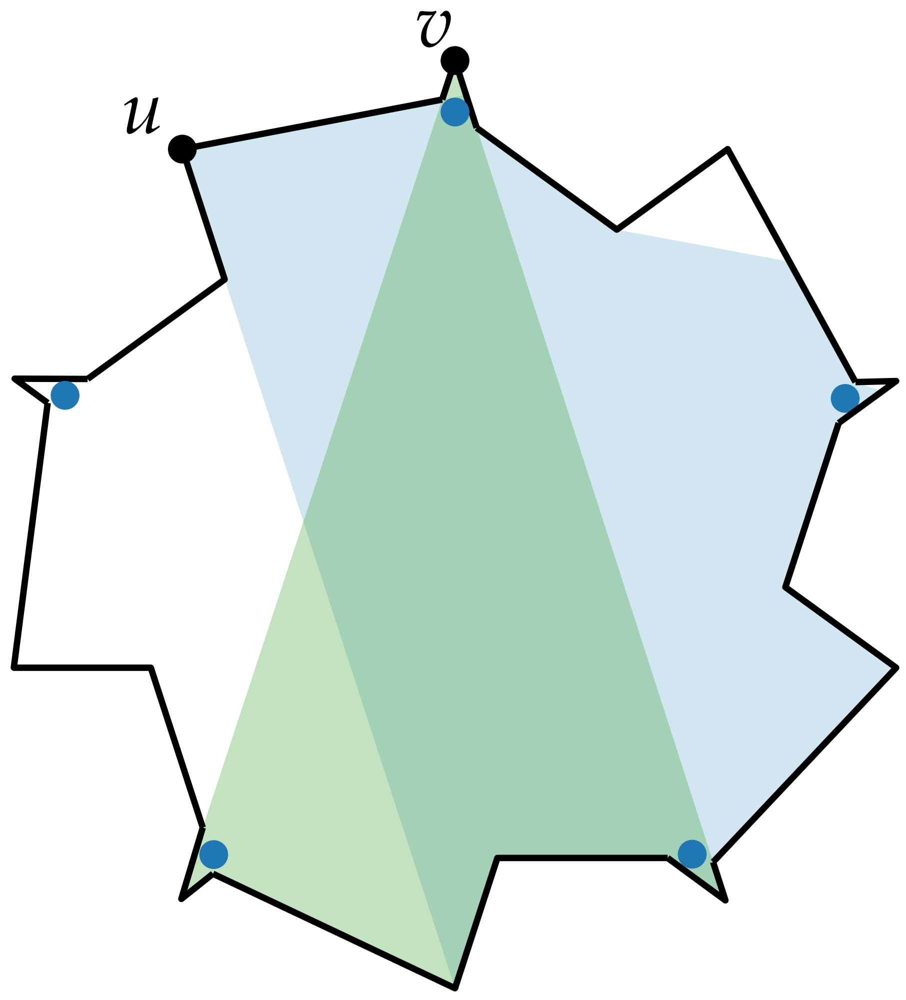

counterexample for $k = 3$

Counterexample for any k

Thm. For any $k \geq 2$ there exists a polygon P and a set of guards G such that P is k-guarded by G but there is no 2-coloring of G.

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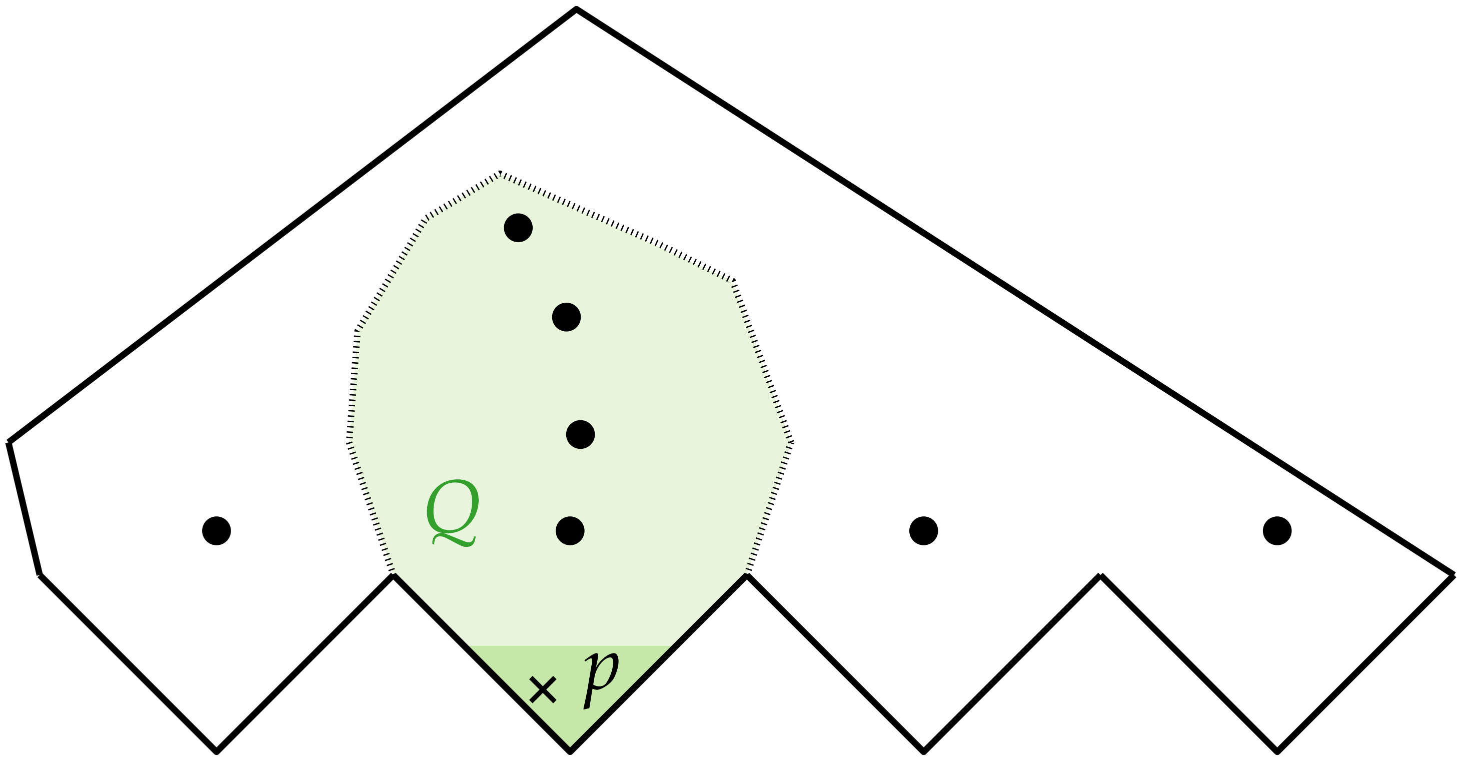

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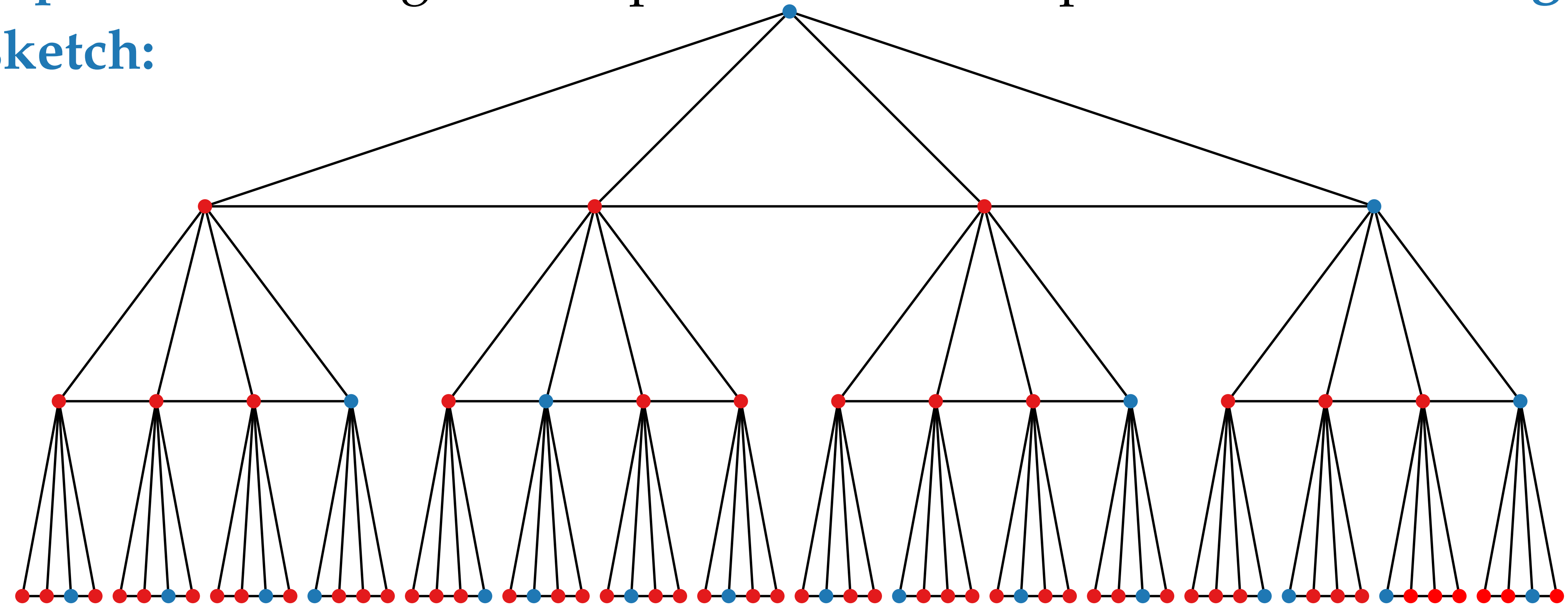

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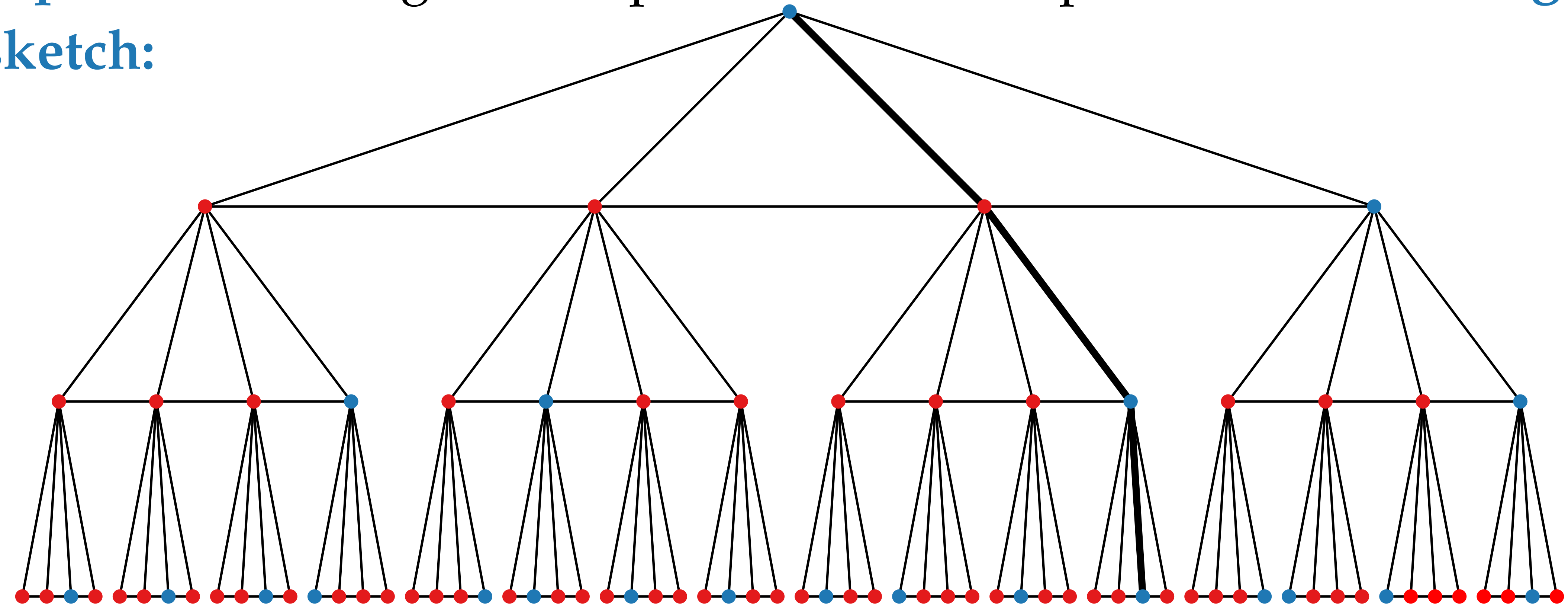

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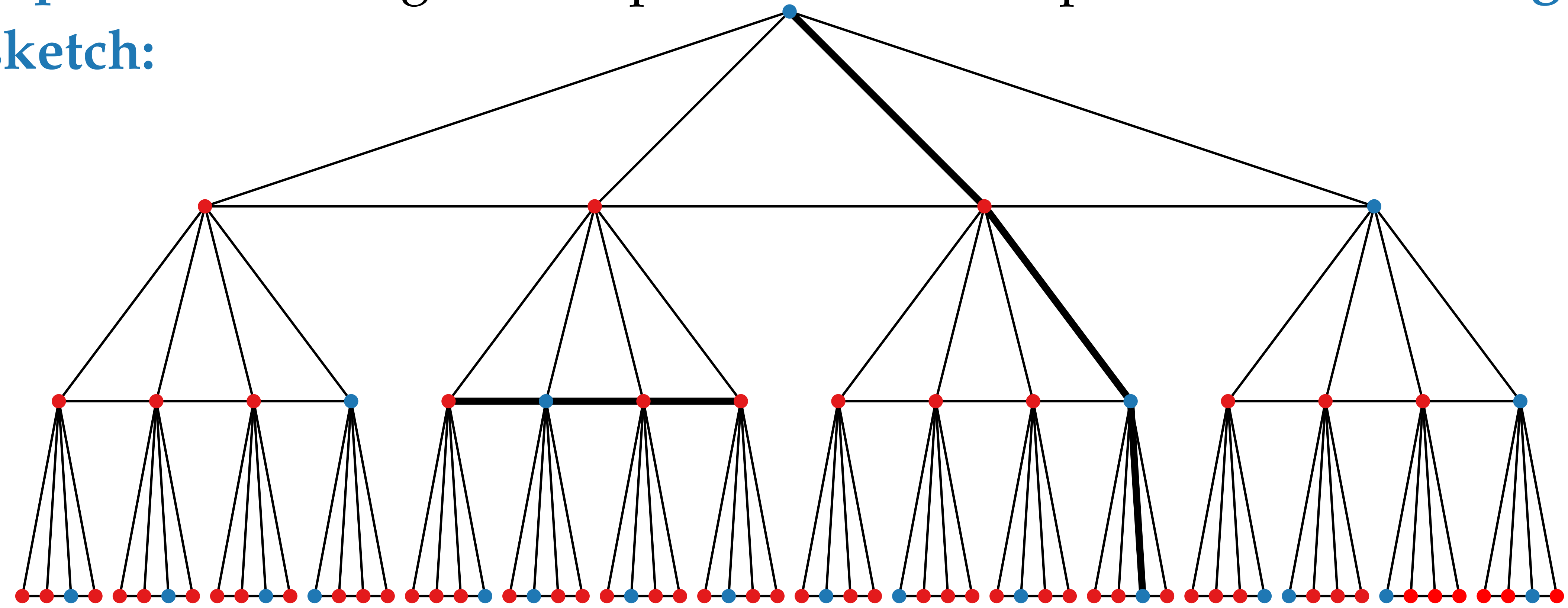

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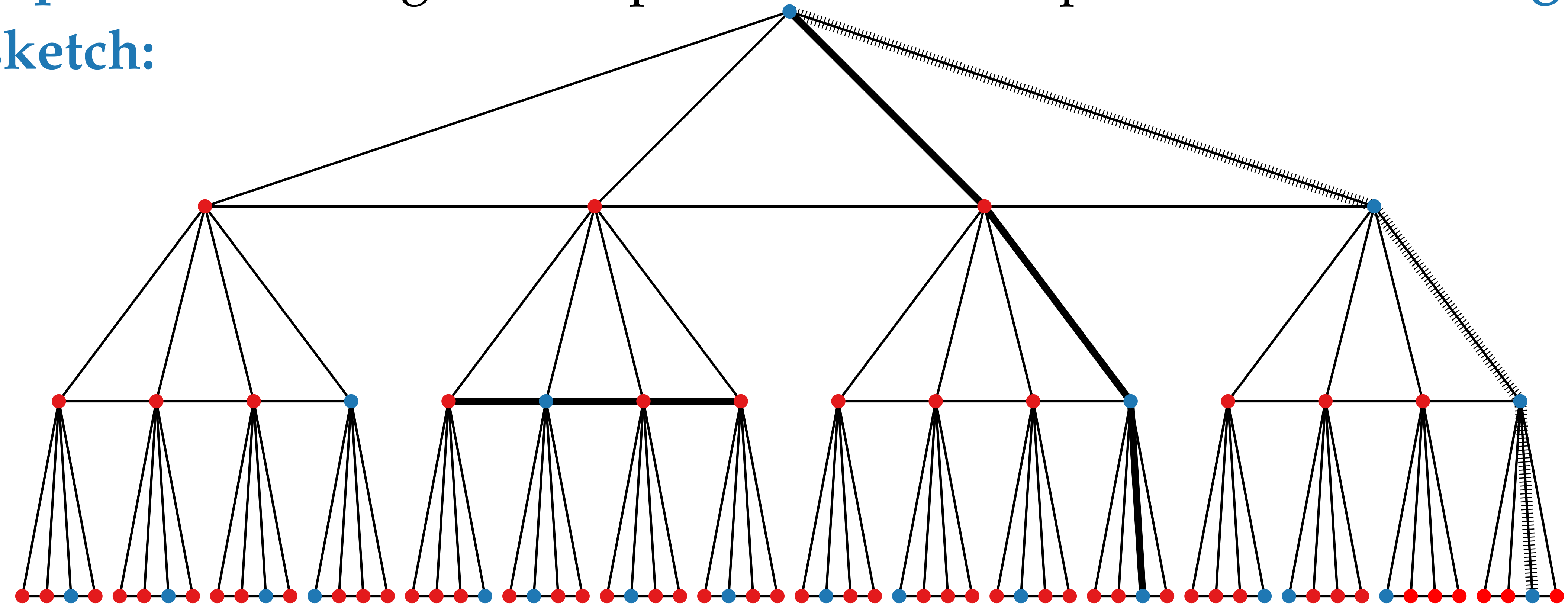

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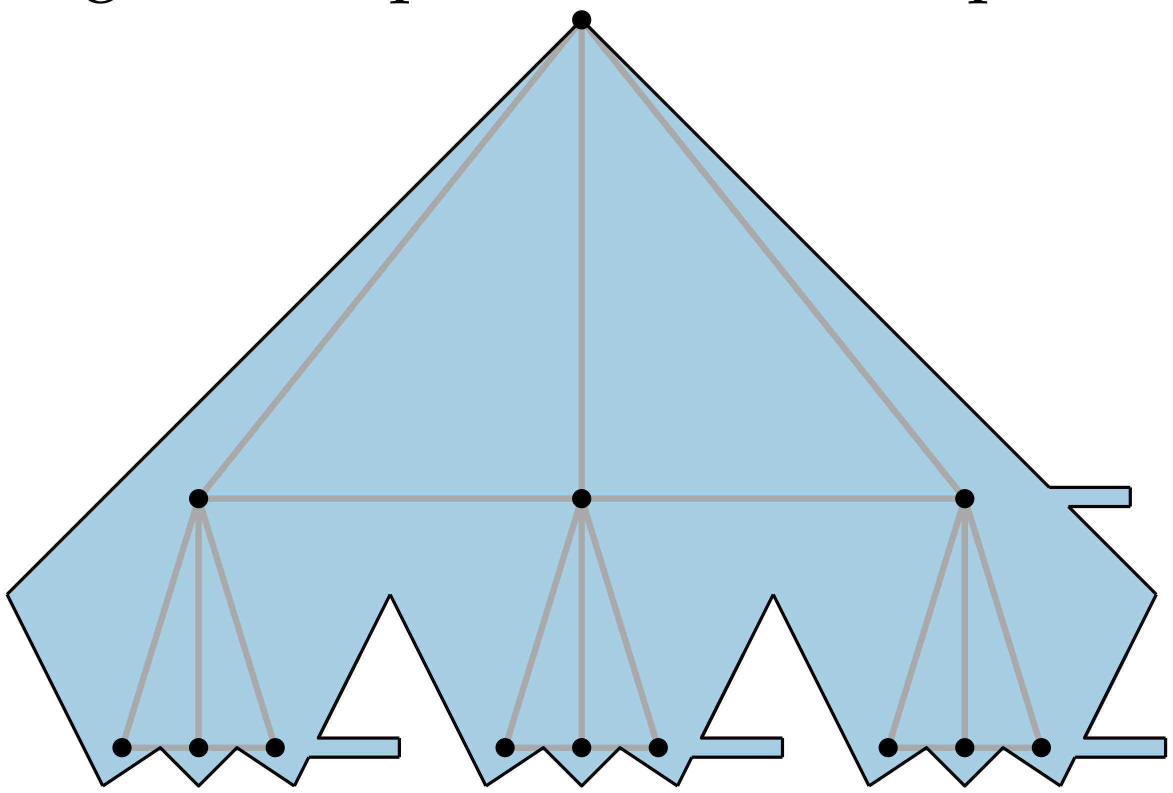
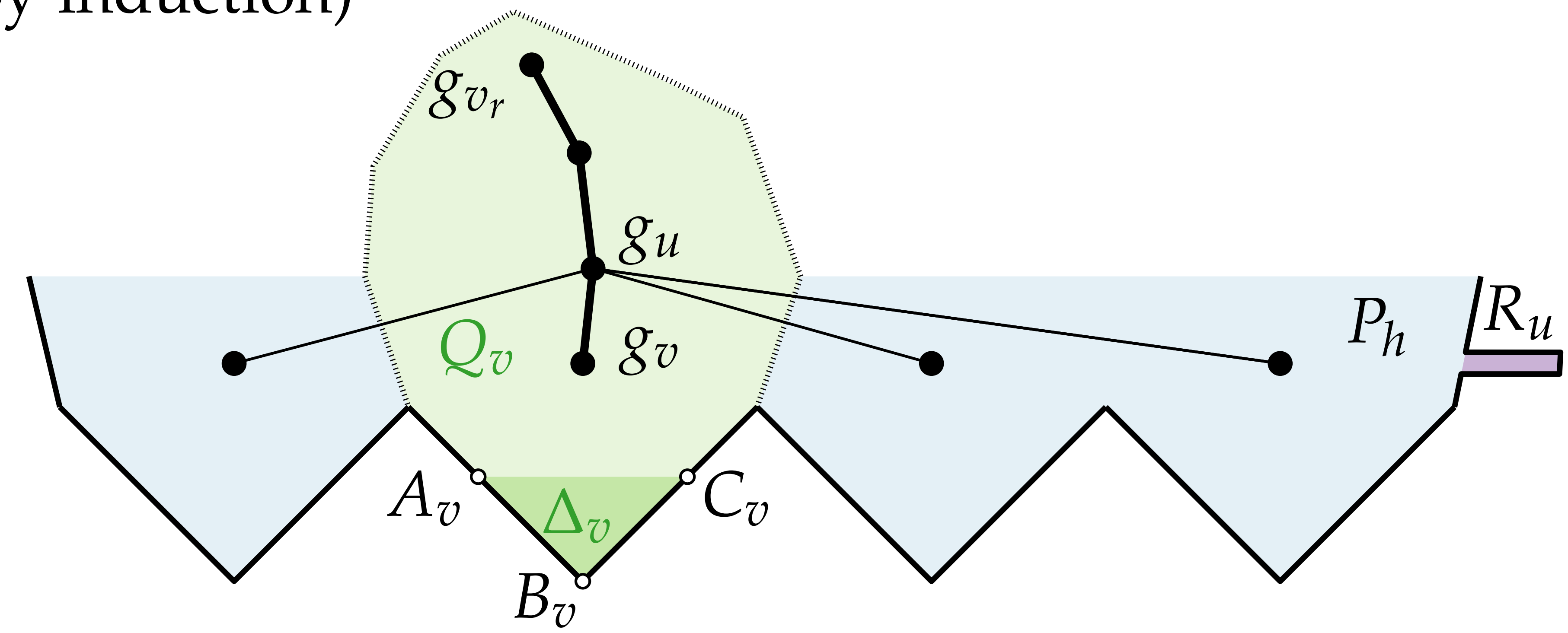


example of P for $k = 3$

Theorem proof sketch

Thm. There exists a polygon P and a set of guards G such that P is k-guarded by G but there is no 2-coloring of G.

Proof sketch: (by induction)



Invariants:

- $\forall$ root-to-leaf path $g_{v_r} g_v$ uniquely guards a convex region Q_v with witness Δ_v;
- $\forall$ internal node u the children of g_u uniquely guard a trapezoidal region R_u.

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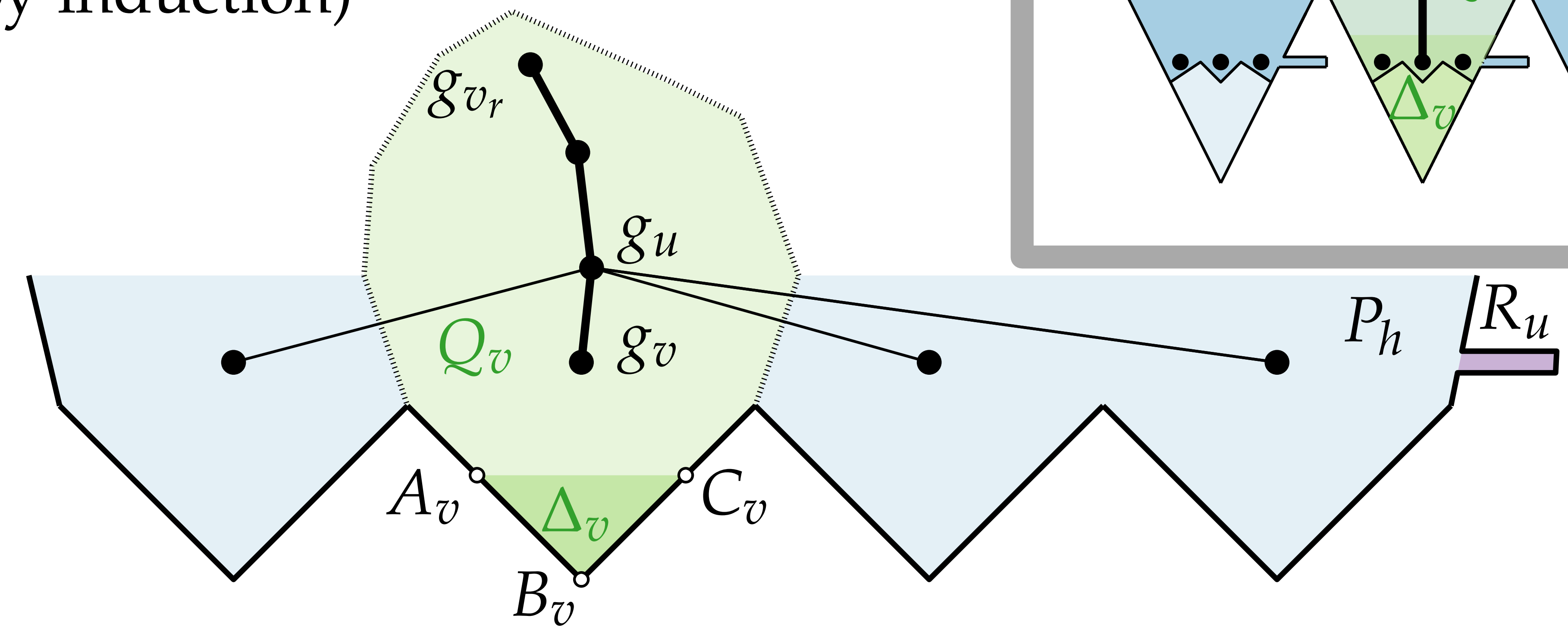

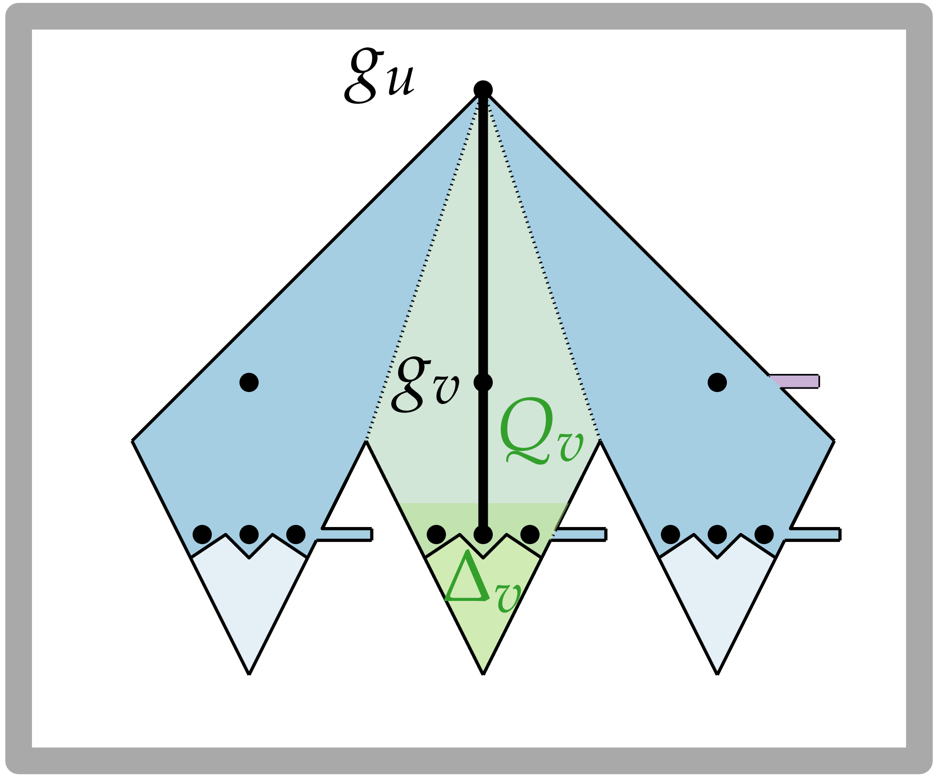

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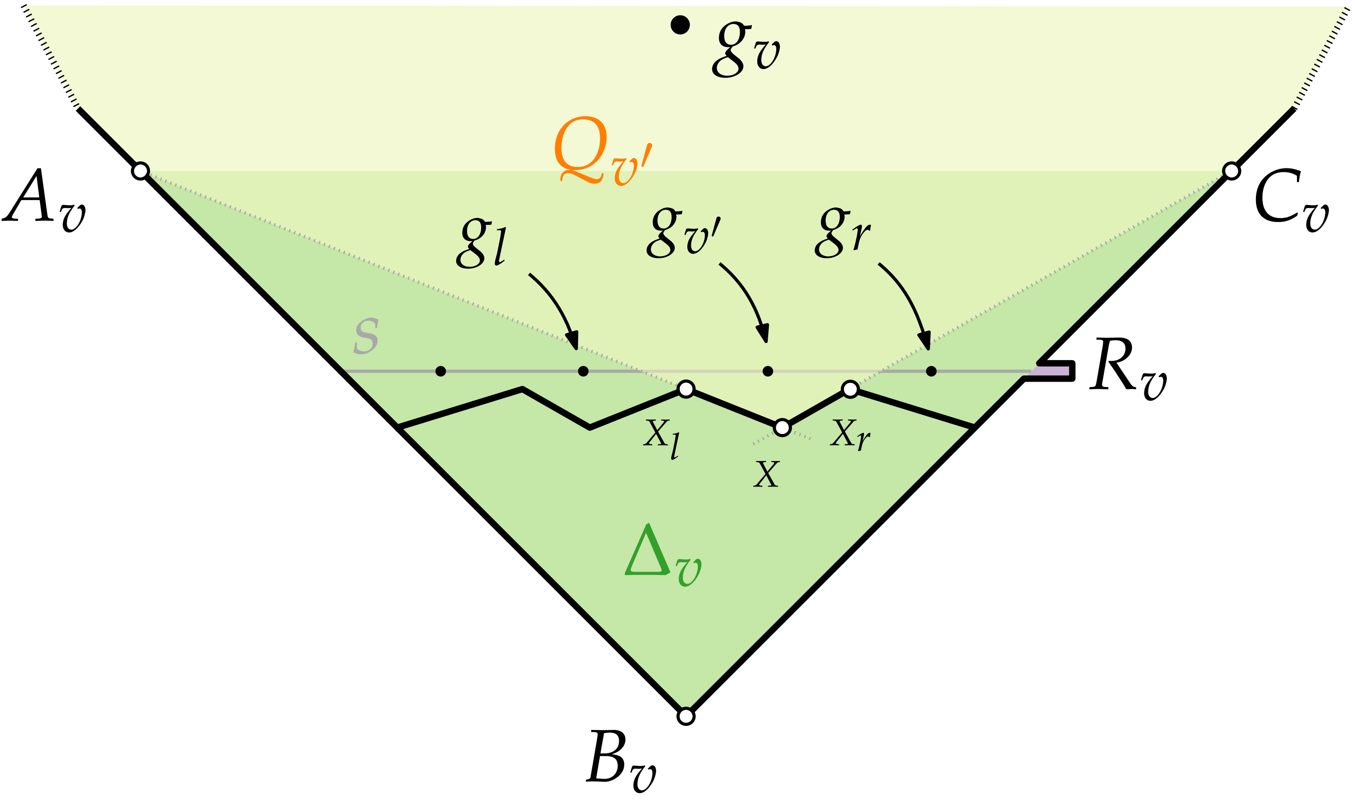

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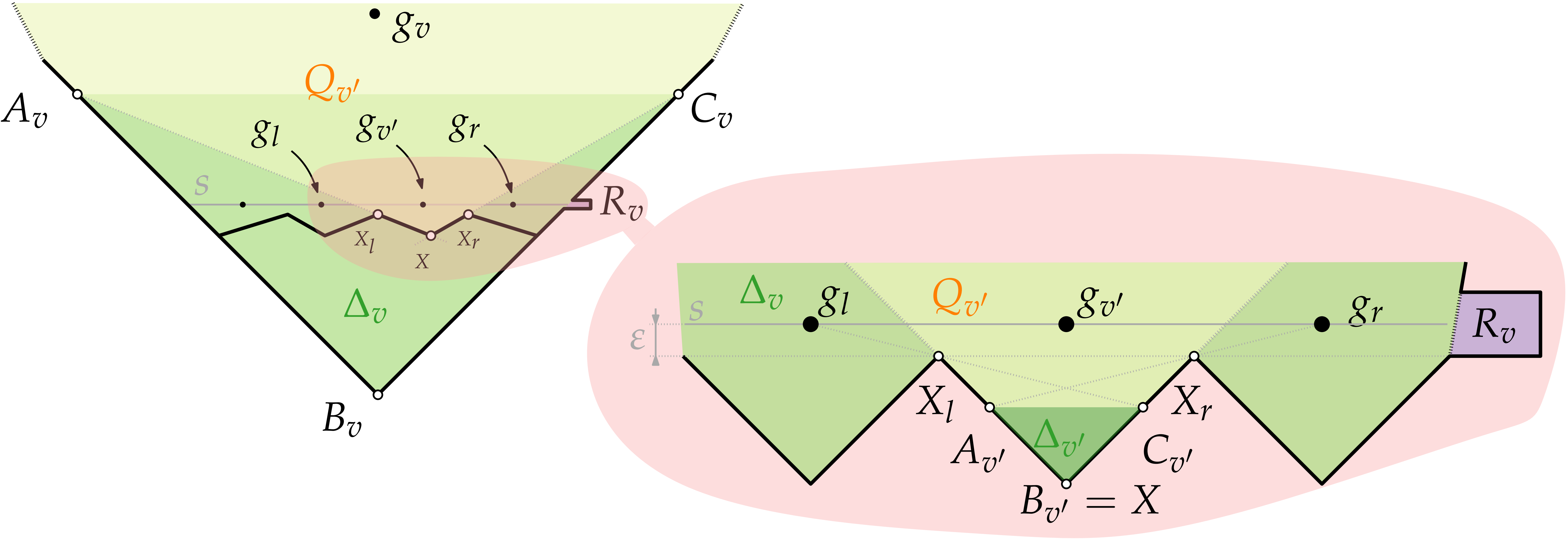

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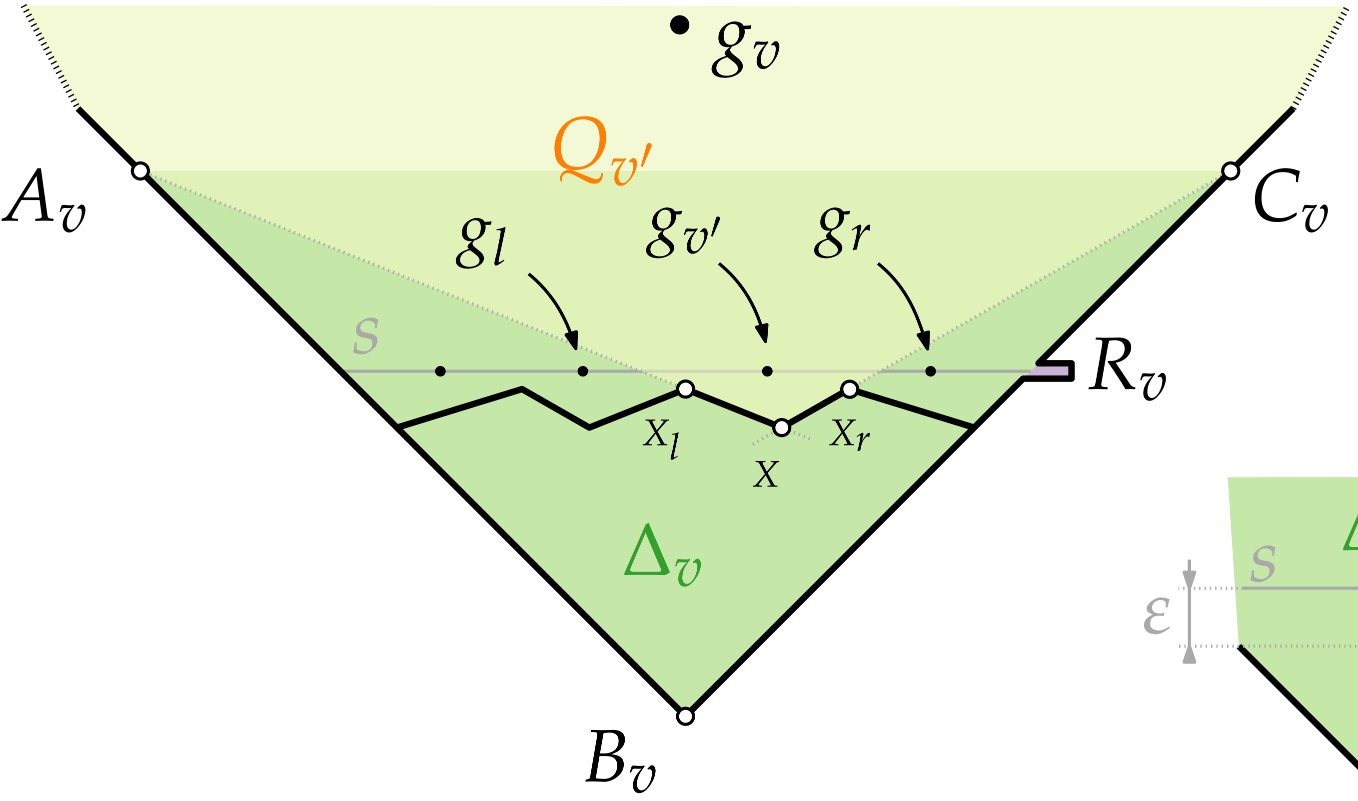

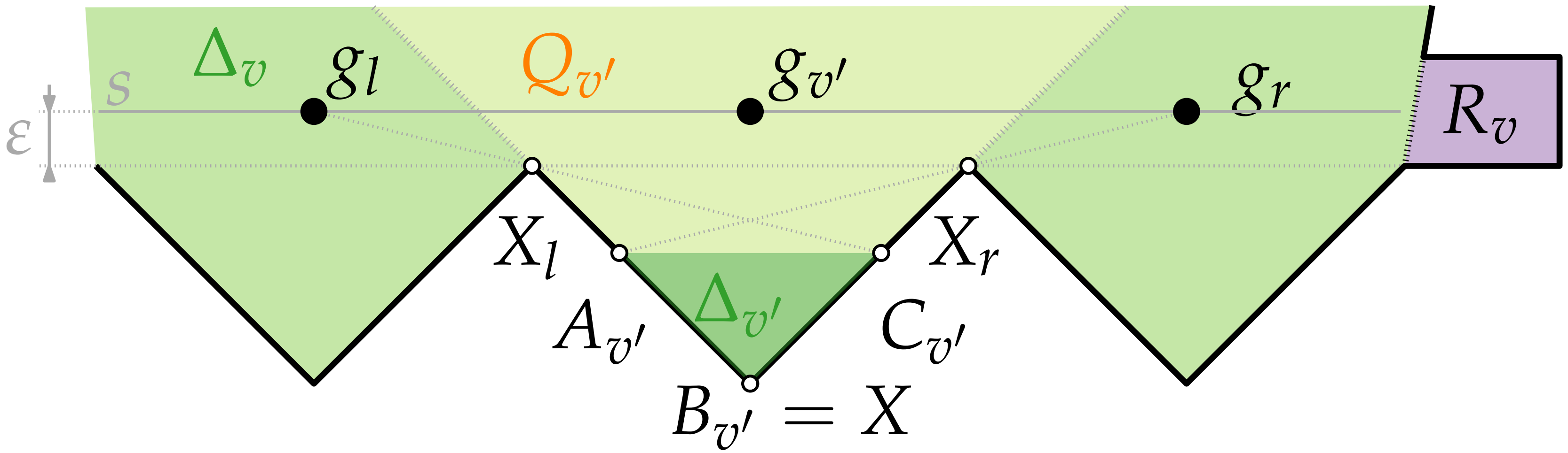

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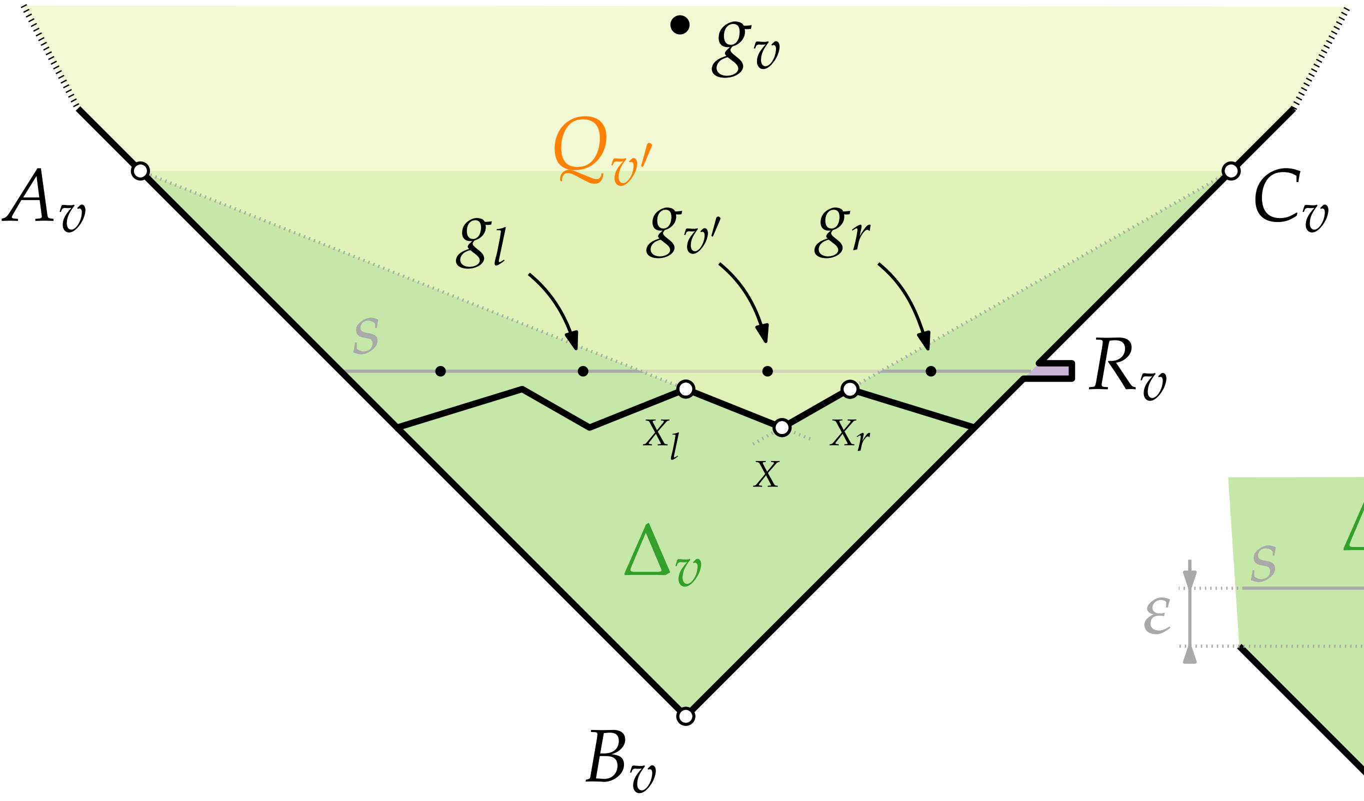

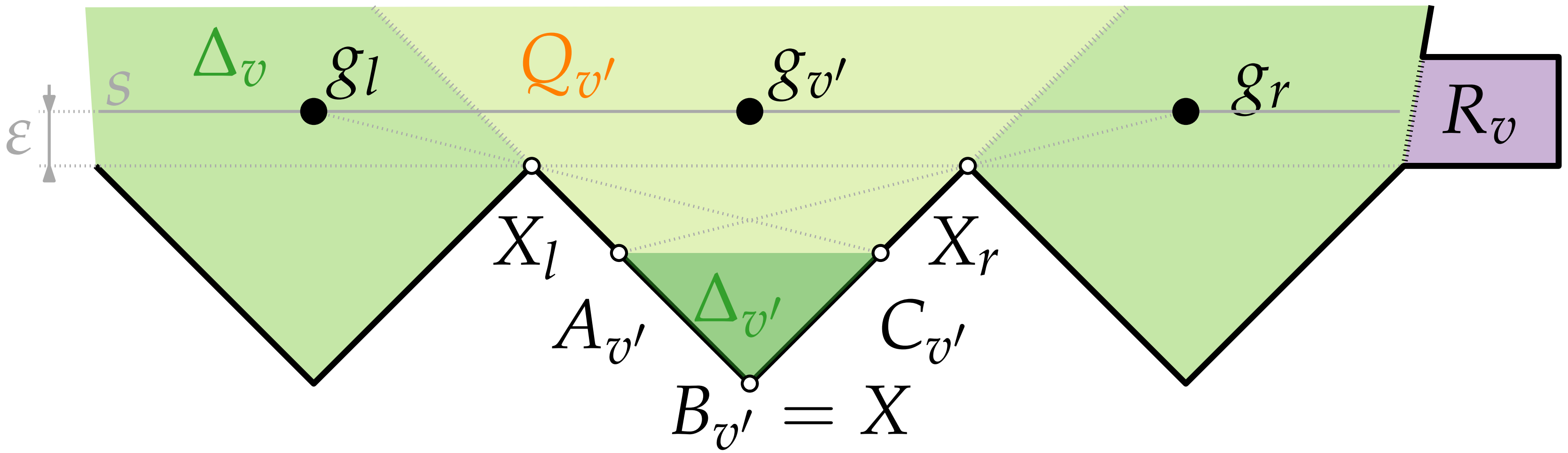

Invariants are maintained:

- $\forall$ root-to-leaf path $g_{v_r} g_{v'}$ uniquely guards a convex region $Q_{v'}$ with witness $\Delta_{v'}$;
- the set of children of g_v uniquely guards a trapezoidal region R_v. $\square$

Open questions

Our construction has exponential ratio of the lengths of the longest edge and the shortest edge.

Question 1 Is there a polygon P that is k-guarded by a set of guards G that is not 2-colorable for which the ratio of the lengths of the longest edge and the shortest edge is polynomial in k?

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Our construction for P_k has $\Theta(k^k)$ vertices.

Question 2 Can we show that P_k always needs $\omega(k)$ vertices?