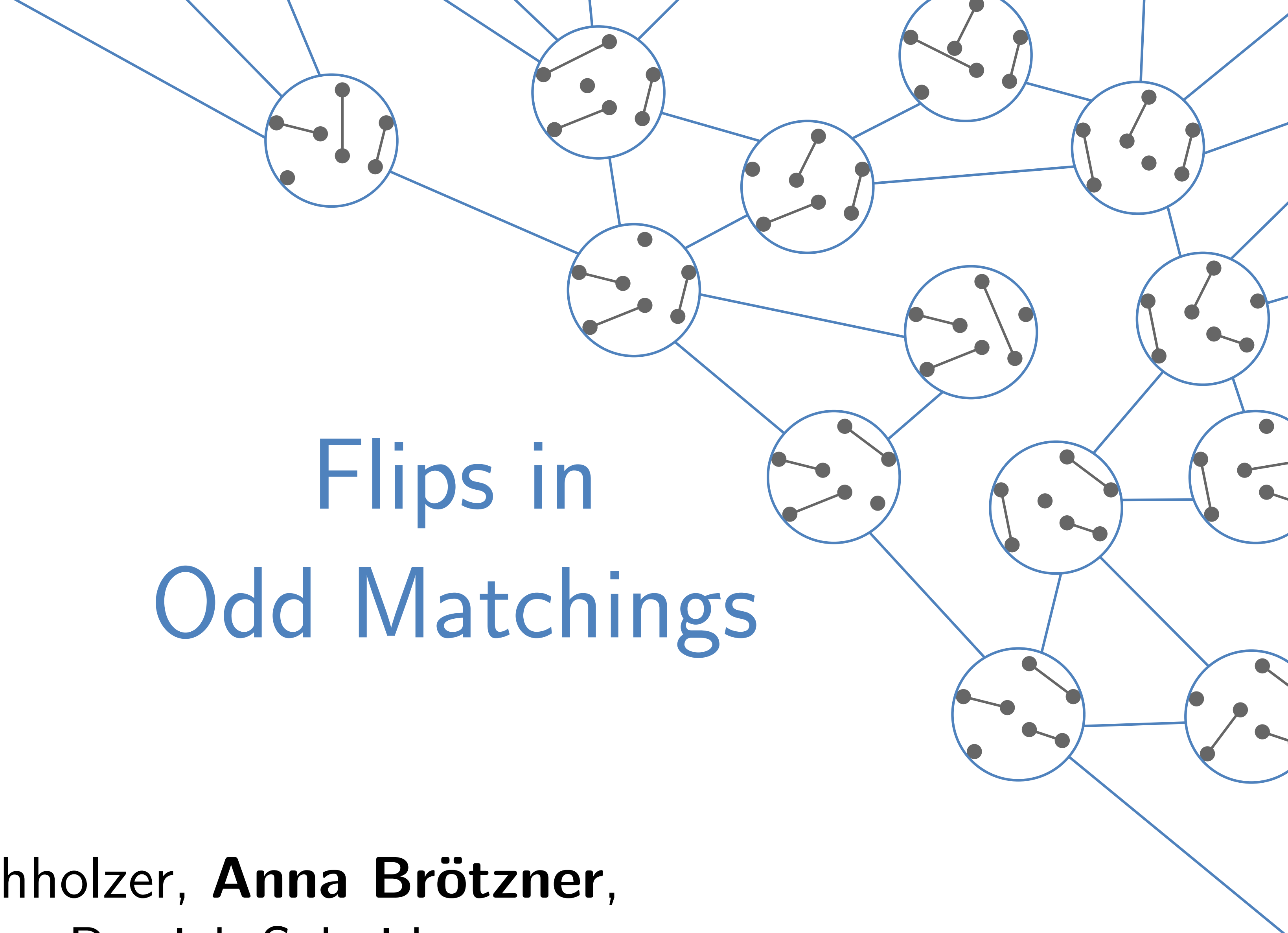

Flips in Odd Matchings

Oswin Aichholzer, **Anna Brötzner**,
Daniel Perz, Patrick Schnider

Edge Flips

Given a point set P in general position, let $\mathcal{F}$ be a family of plane straight-line drawings on P.

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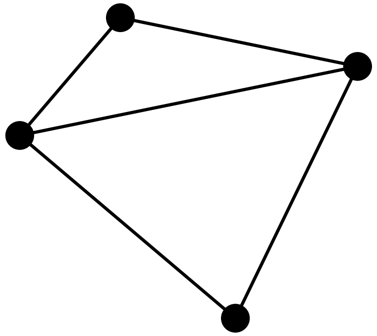
Triangulations



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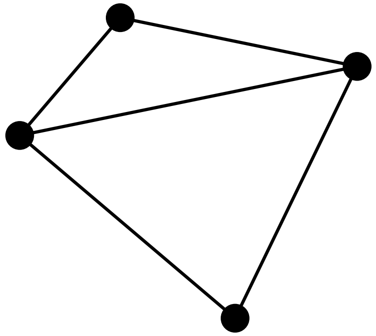
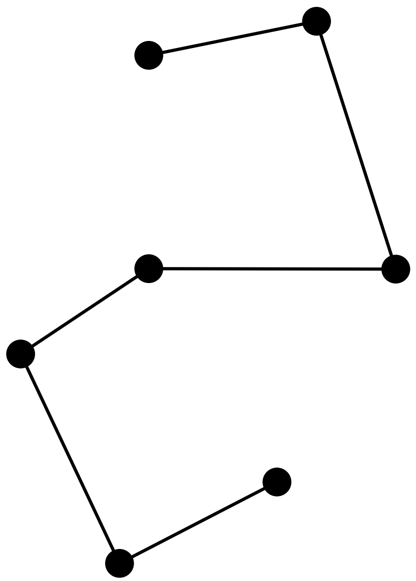


Plane spanning paths



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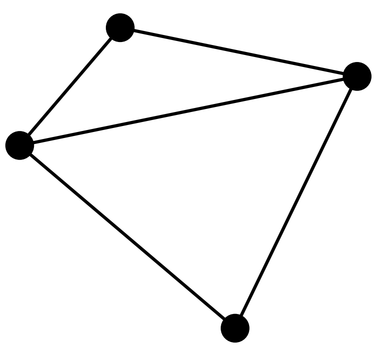
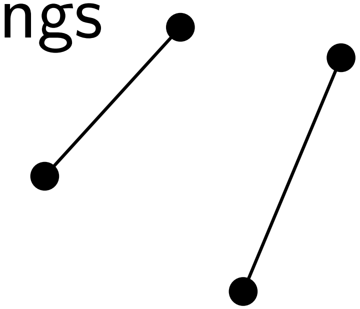


Plane perfect matchings



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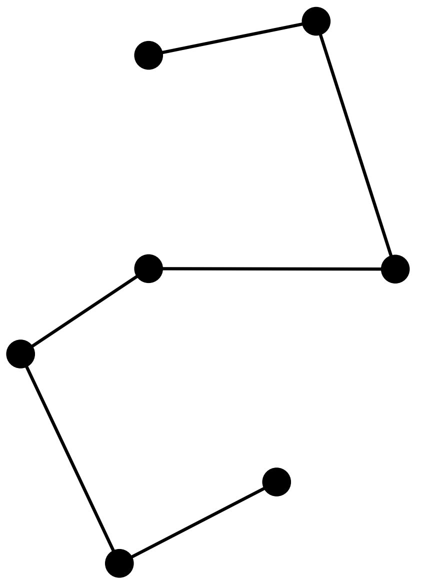

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Edge flip: replace a constant number of edges with other edges

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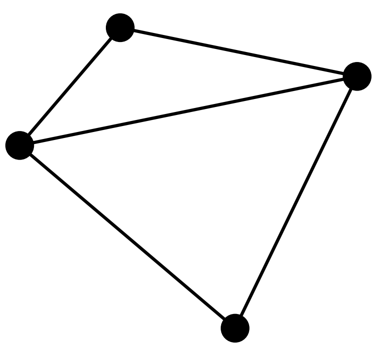

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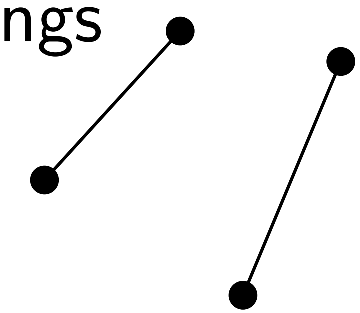

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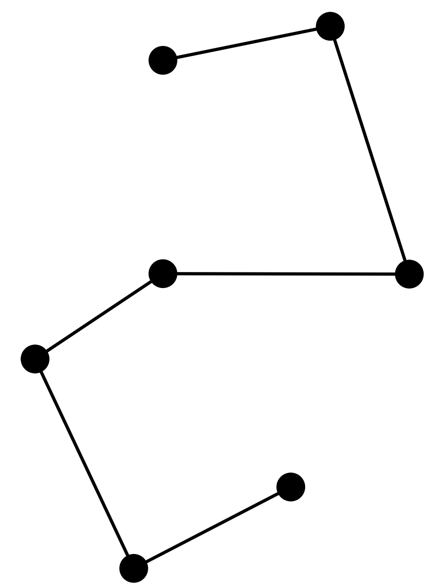

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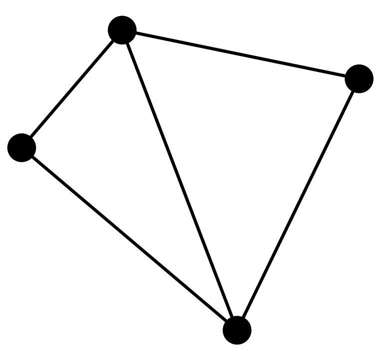

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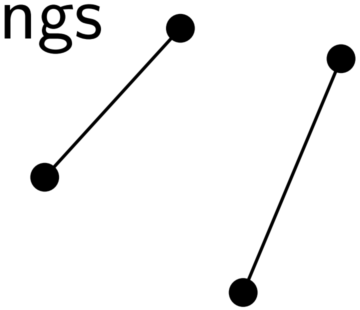

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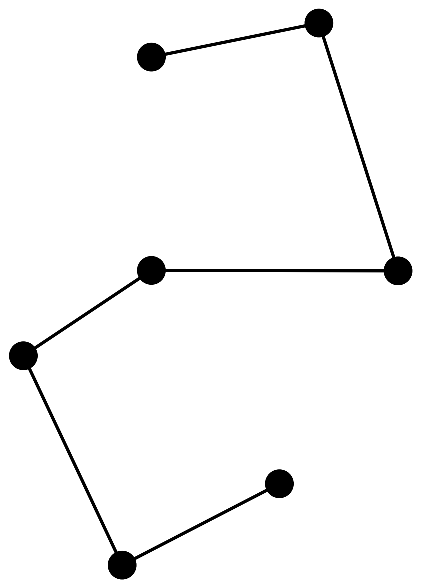

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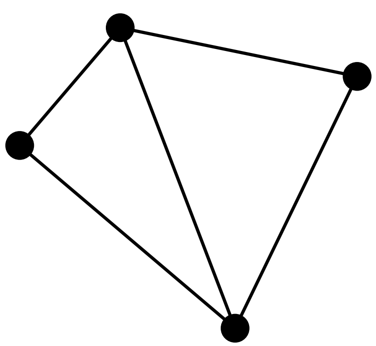

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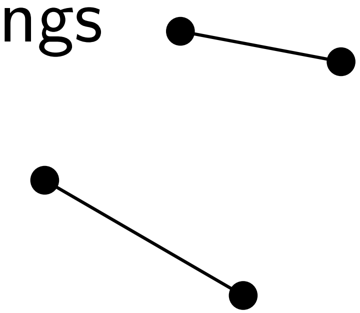

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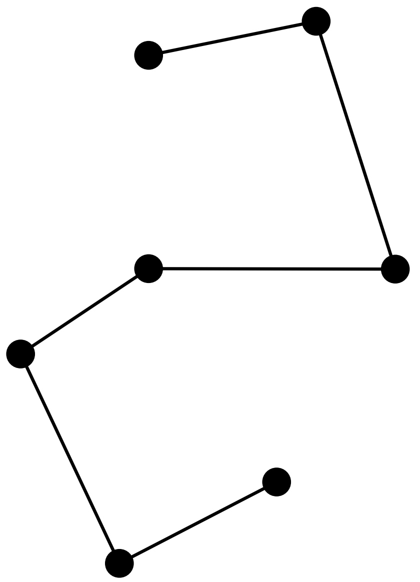

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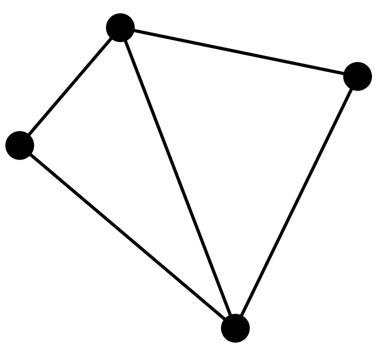

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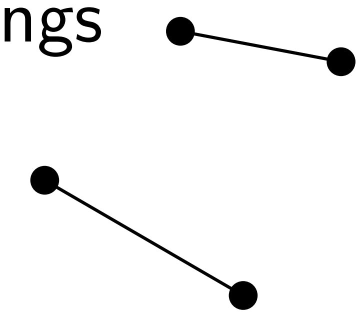

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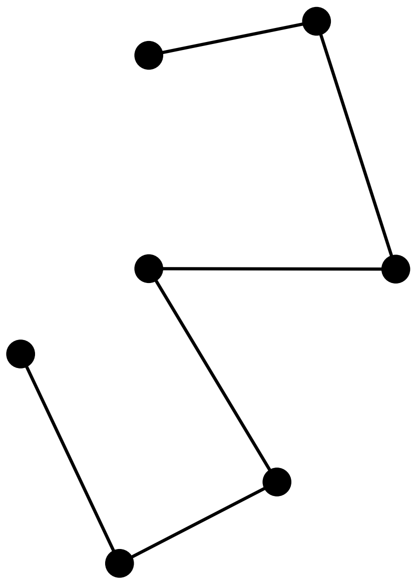

Flip Graph $G_{\mathcal{F}}$

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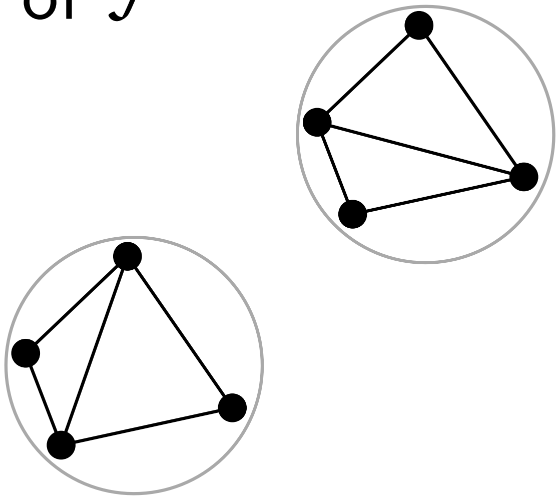
- Every vertex corresponds to a member of $\mathcal{F}$



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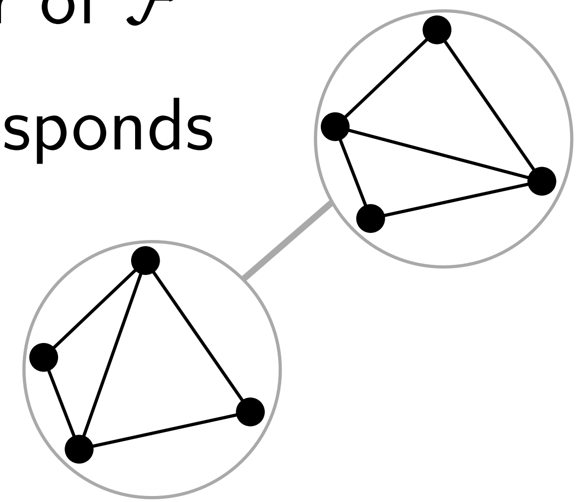

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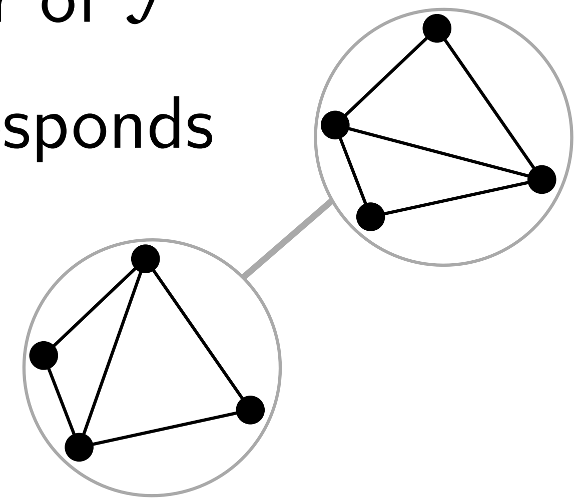

Is the flip graph connected?

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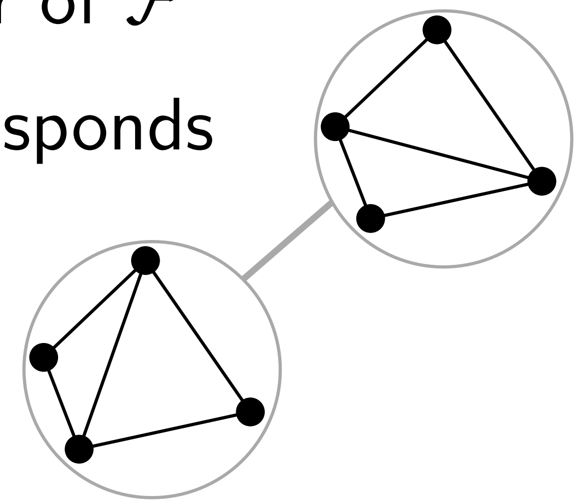

Is the flip graph connected?

True for:
- Triangulations
- Plane spanning trees
- Plane spanning paths (on certain point sets)

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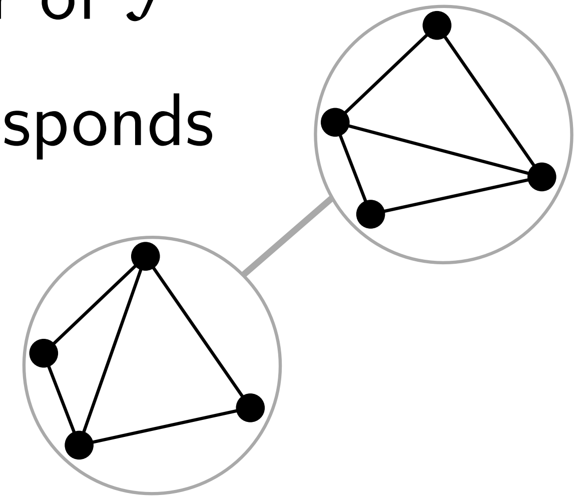

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How about plane perfect matchings?

Our Setting

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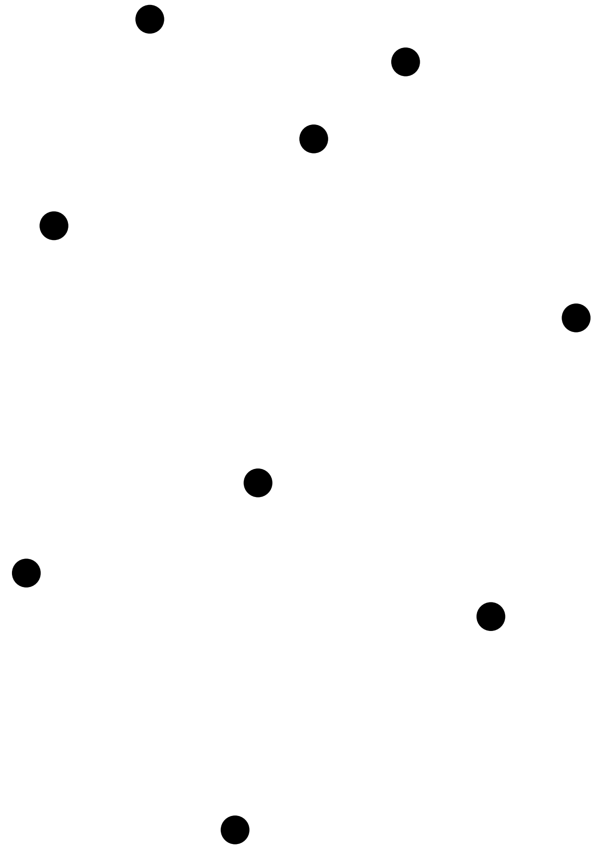
- Point set P with $2m+1$ points



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- Point set P with $2m + 1$ points
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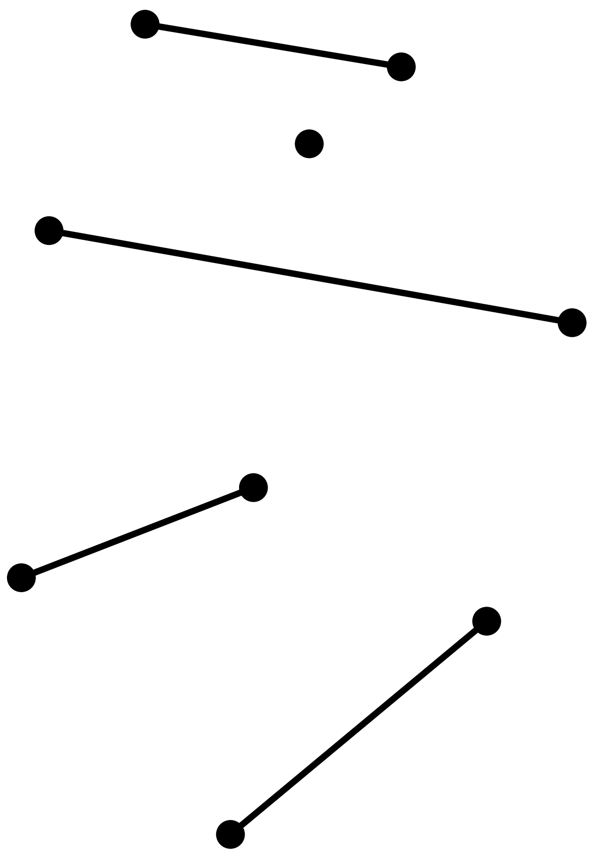

Our Setting

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- Plane almost perfect matching:
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- Edge flips for almost
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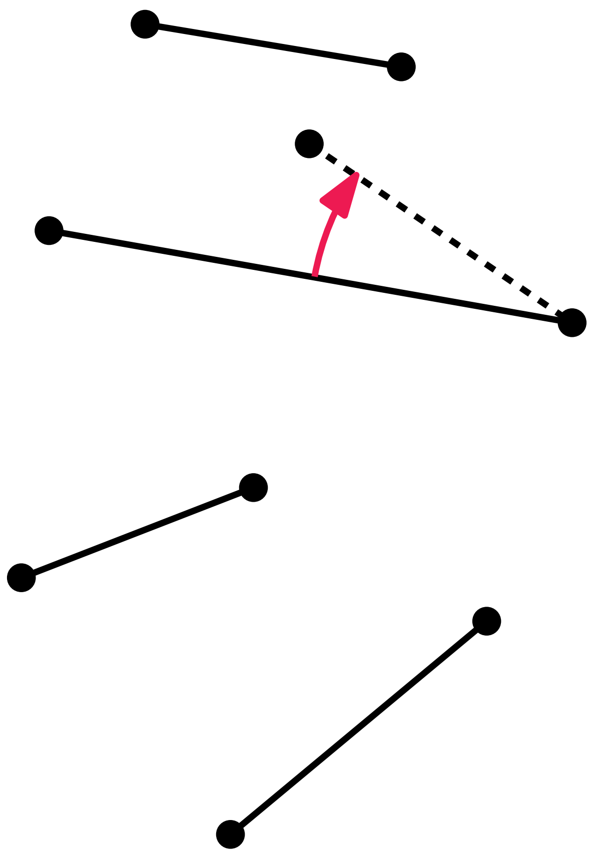
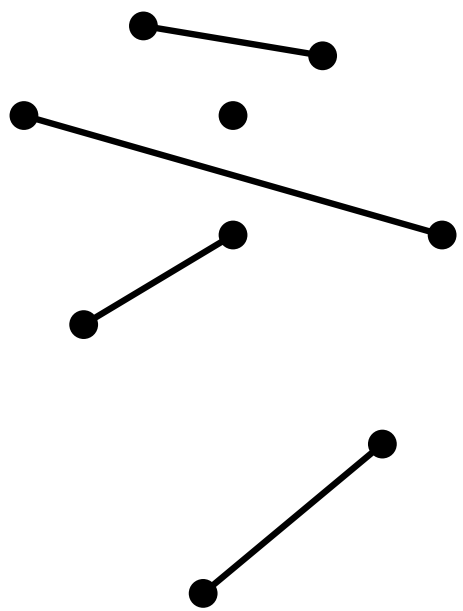
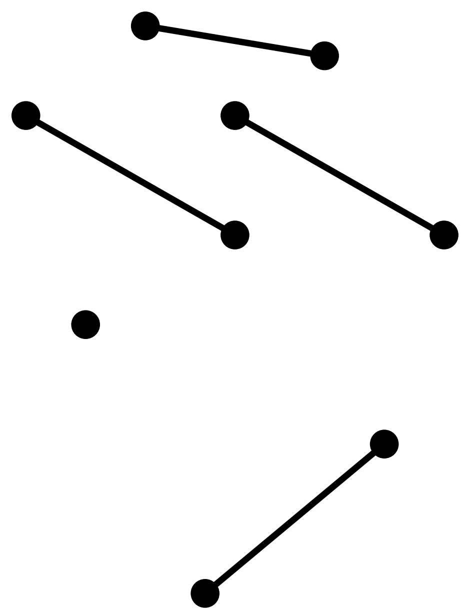


Problem

Given a point set and two plane almost perfect matchings M_1, M_2 on it. Is it always possible to transform M_1 into M_2 by a series of flips?





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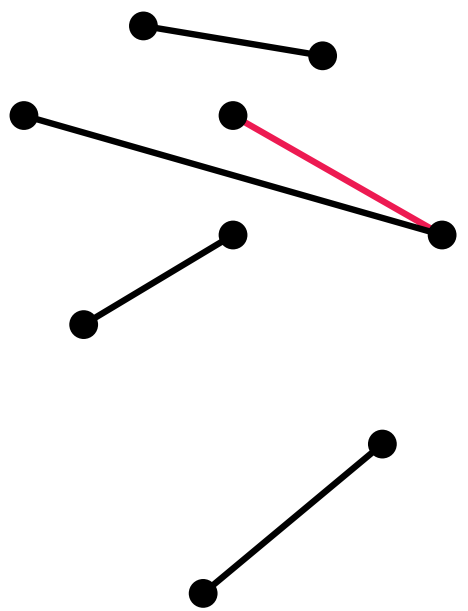

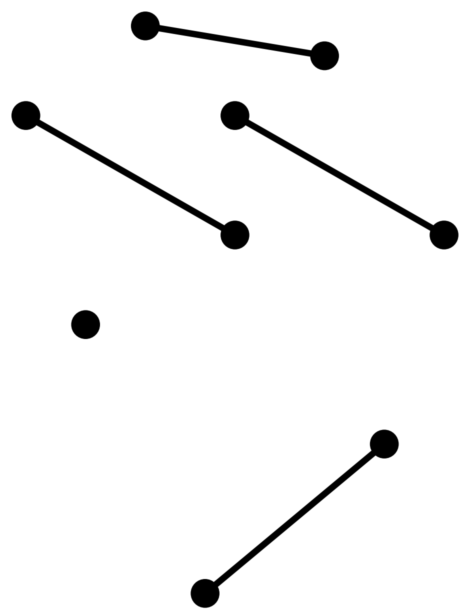

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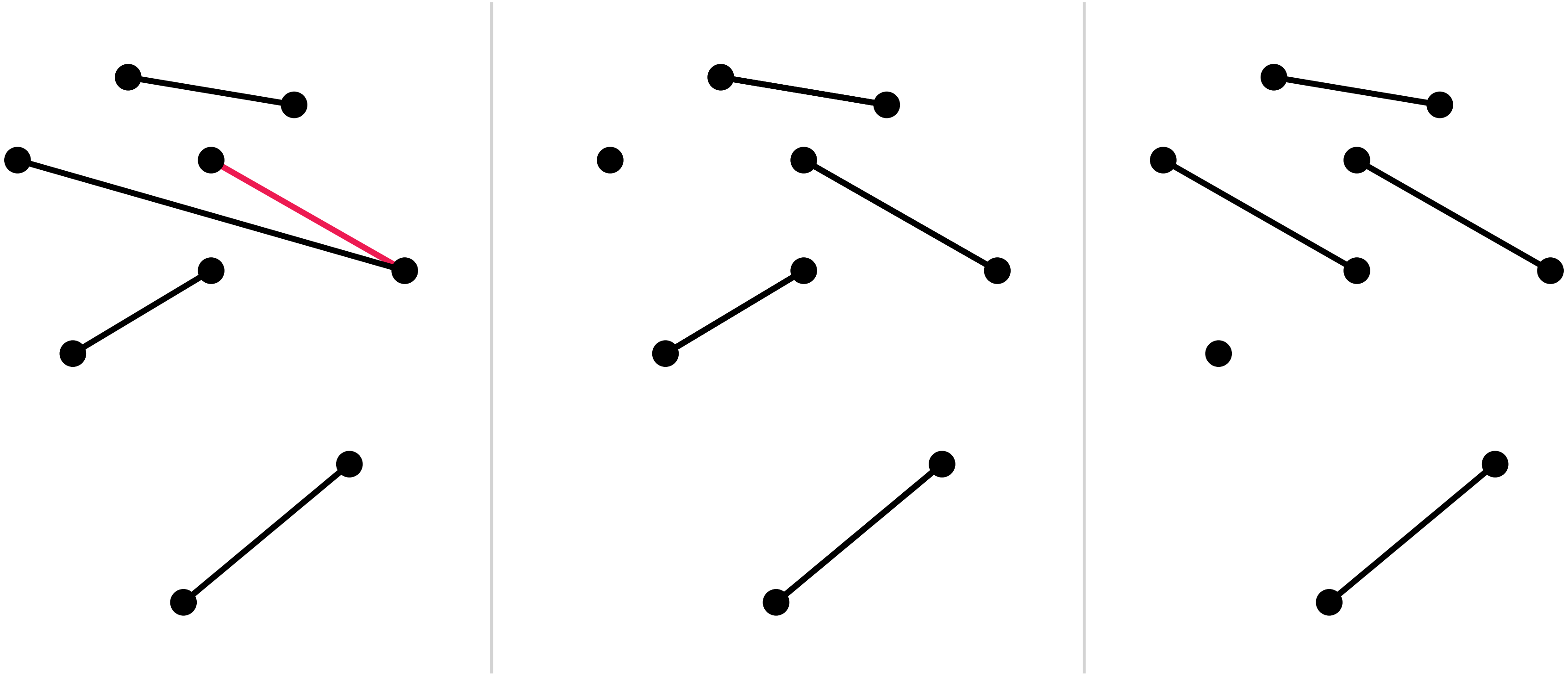

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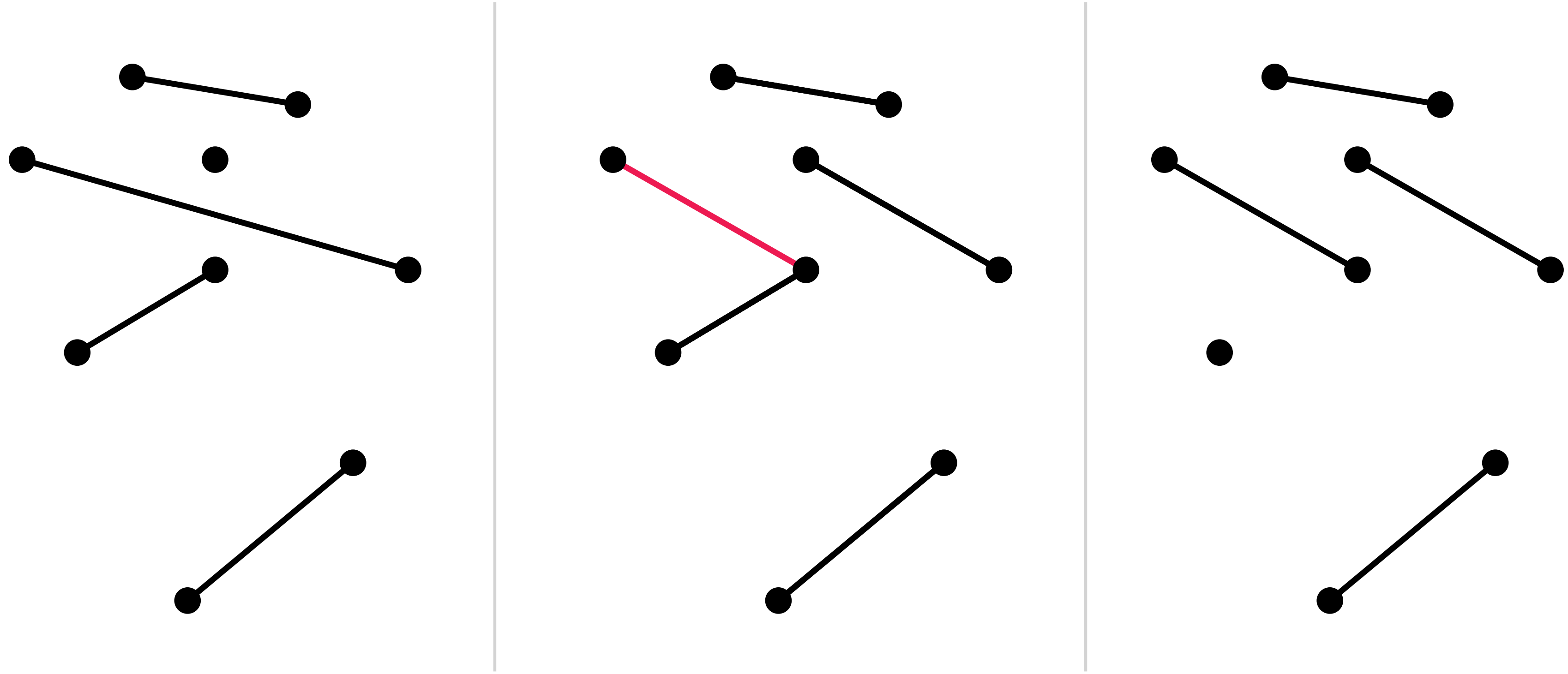

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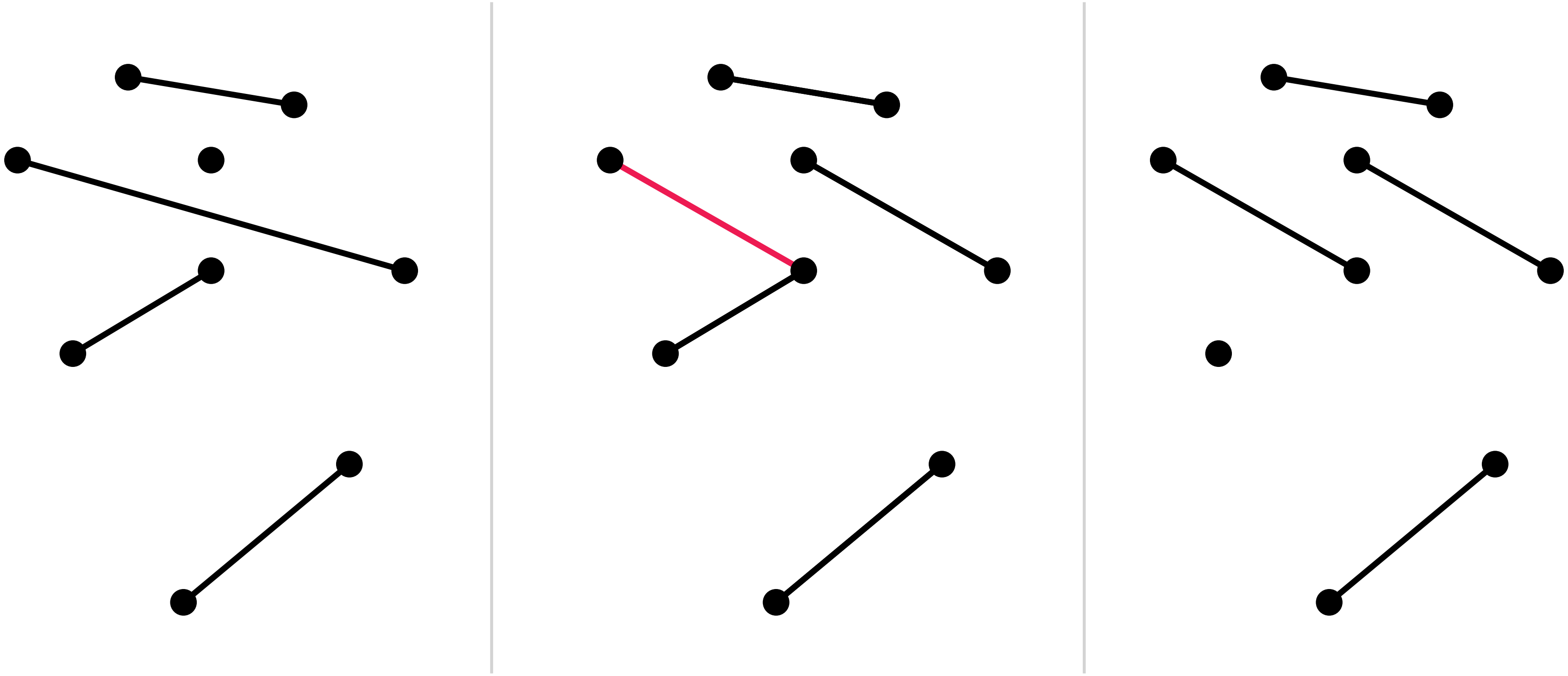

In other words: Is the flip graph connected?

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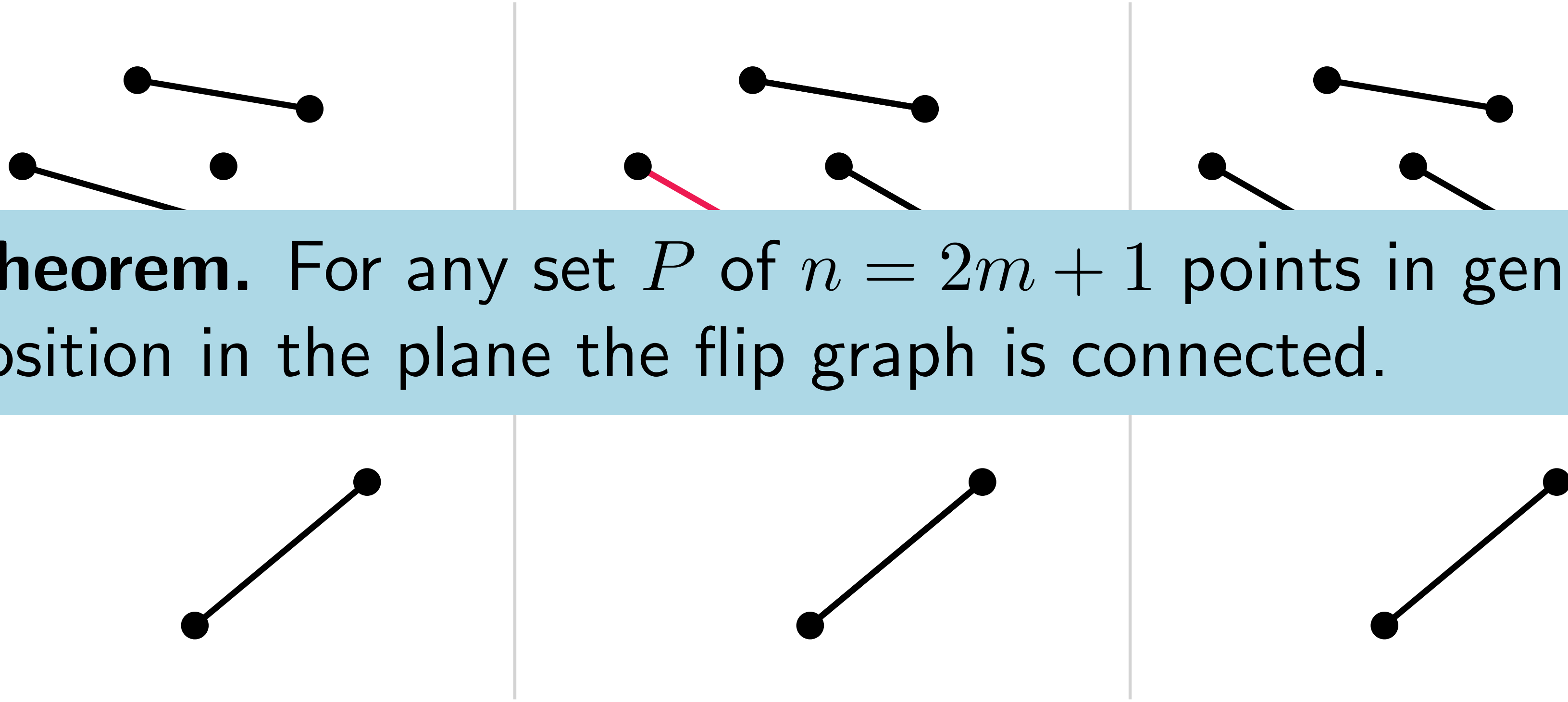
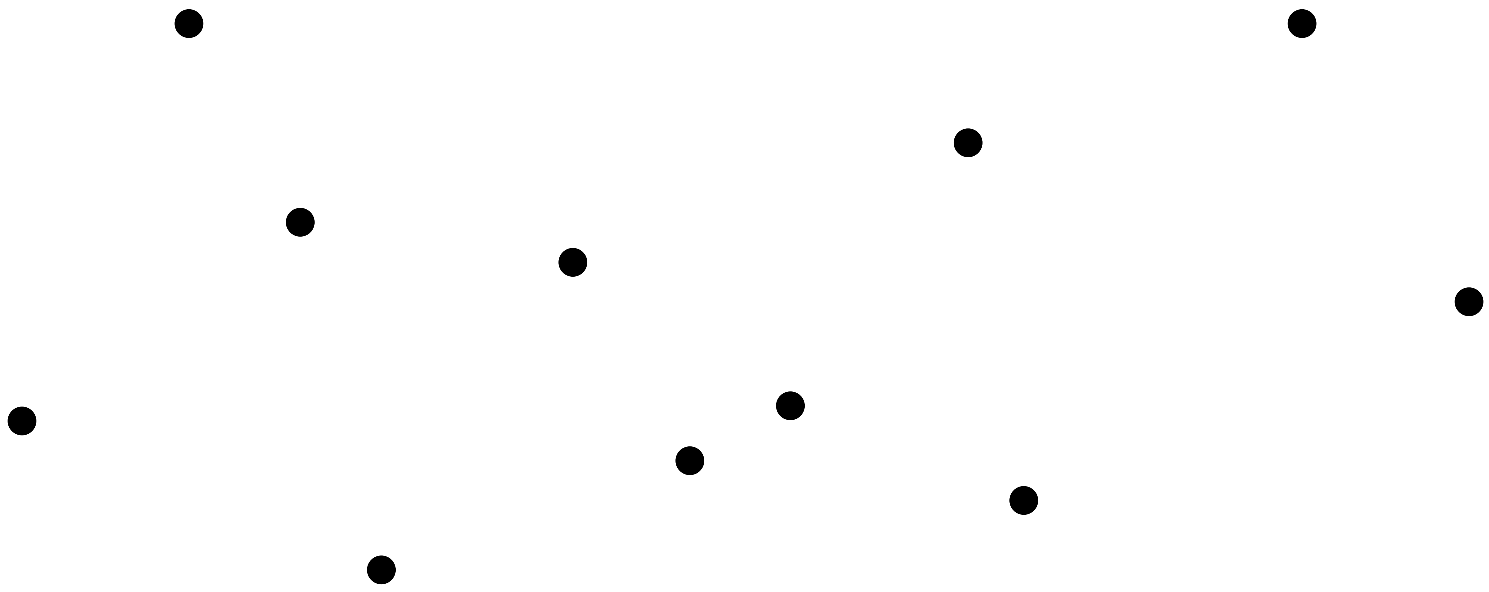


Theorem. For any set P of $n = 2m + 1$ points in general position in the plane the flip graph is connected.

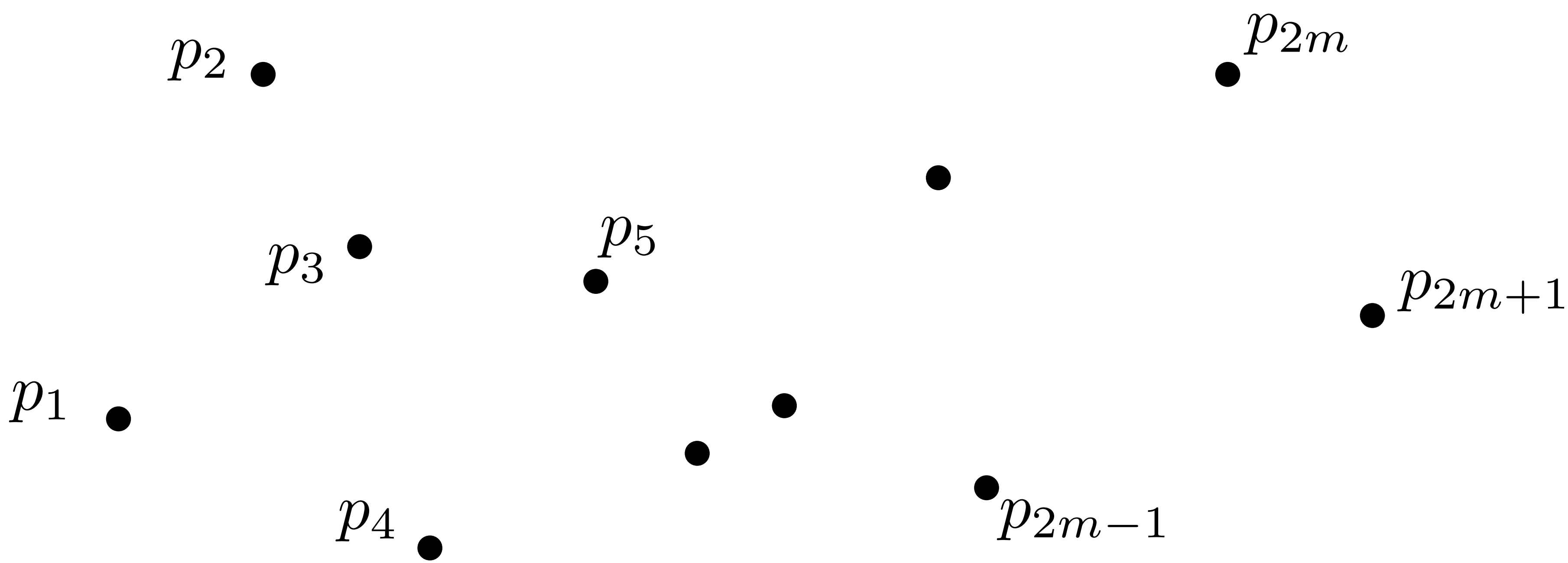
In other words: Is the flip graph connected?

Flip to Canonical Matching M_c



Flip to Canonical Matching M_c

- Sort the points by increasing x-coordinates



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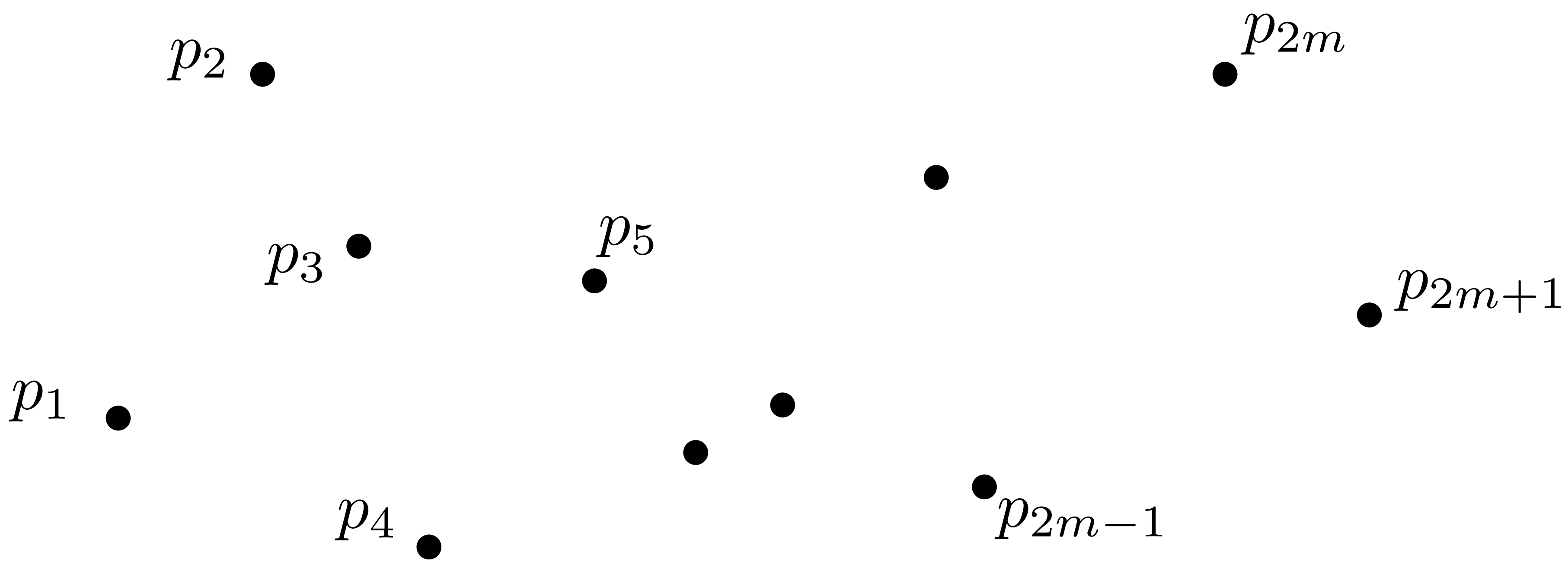

We want to flip any matching M to the canonical matching $M_c = \bigcup_{i=1}^{m} \{p_{2i-1}, p_{2i}\}$.

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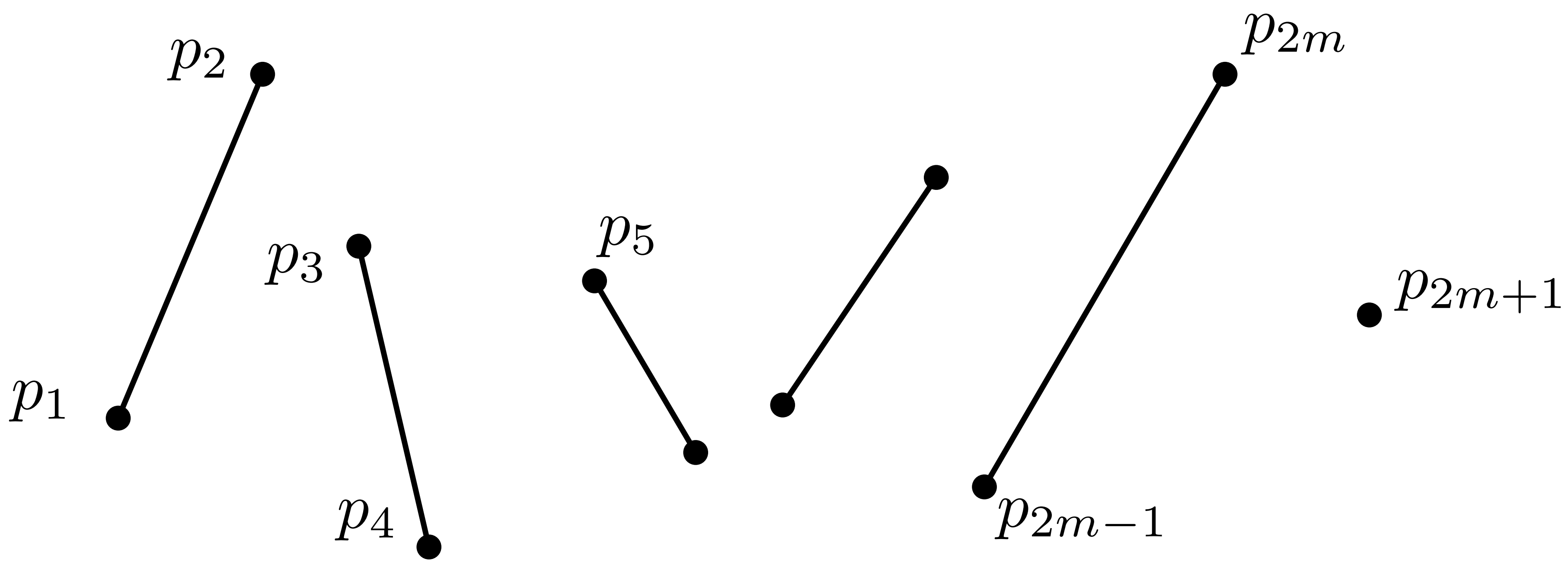

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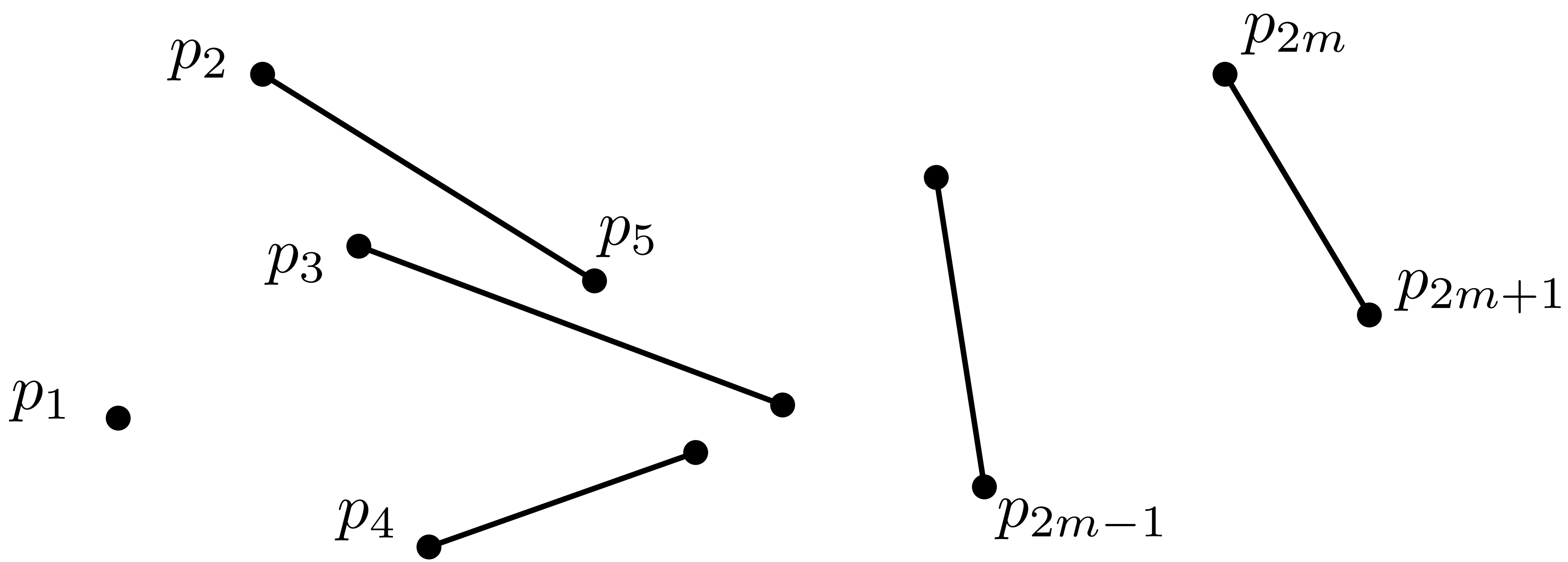

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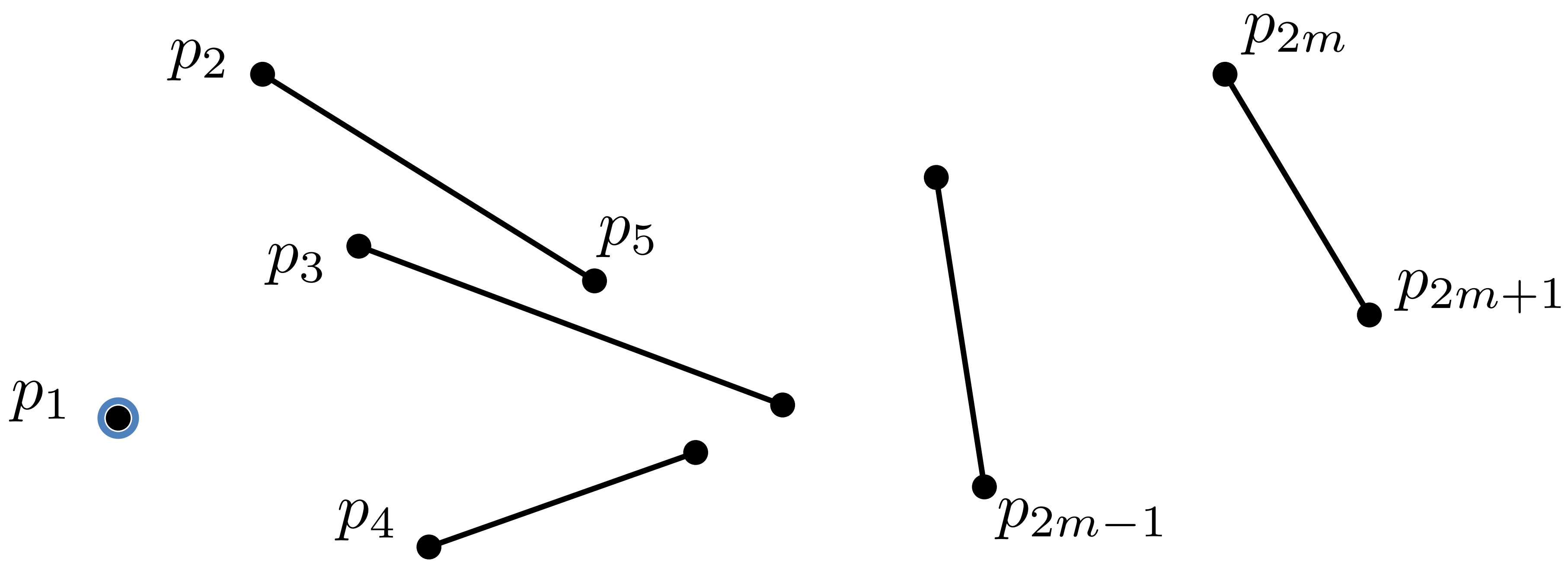

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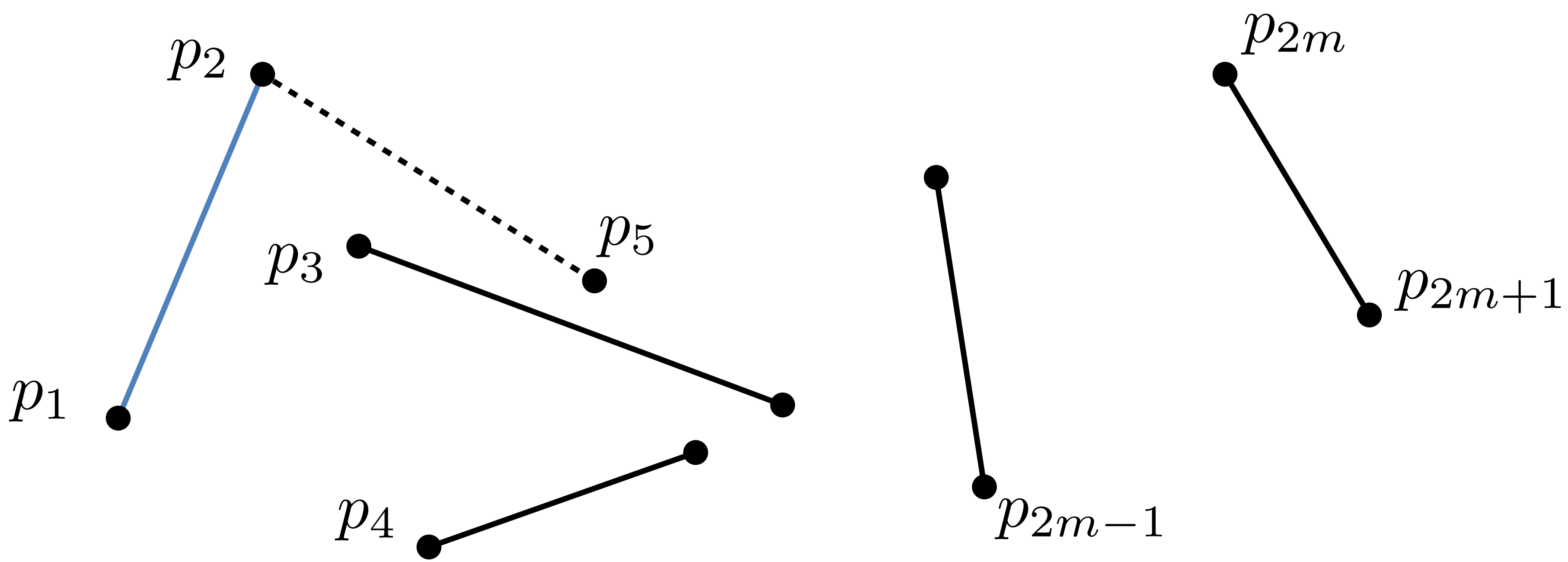

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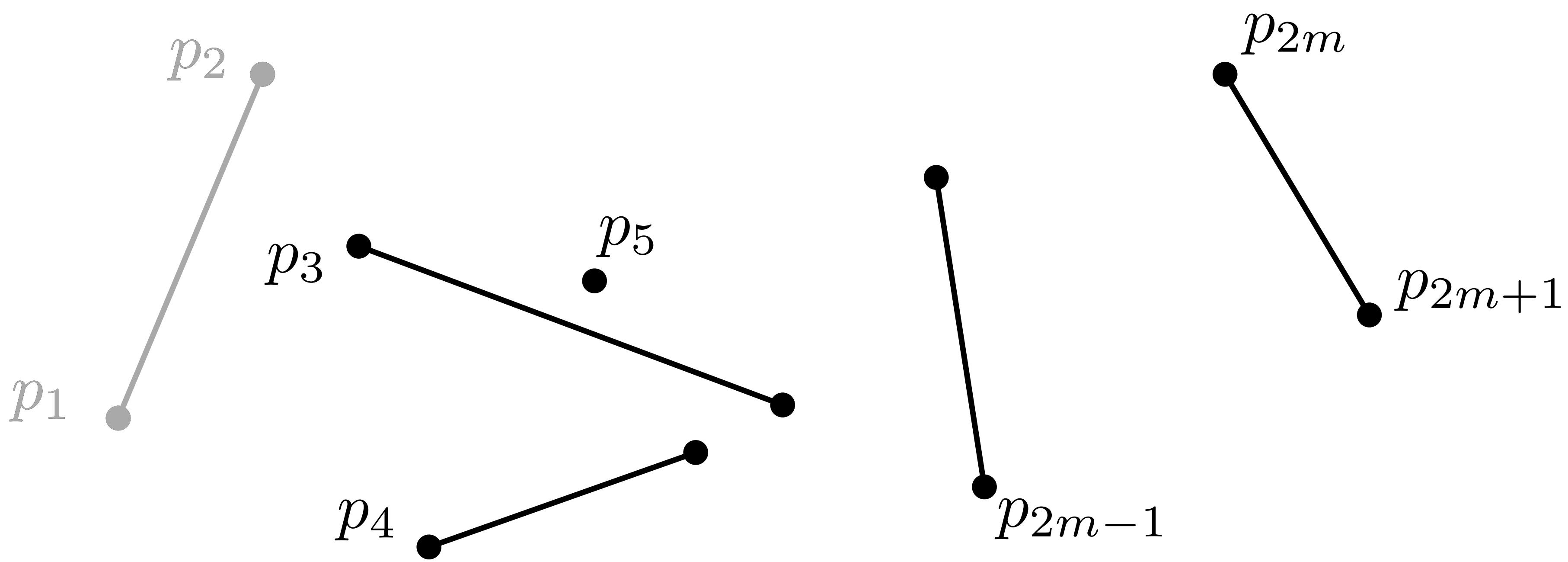

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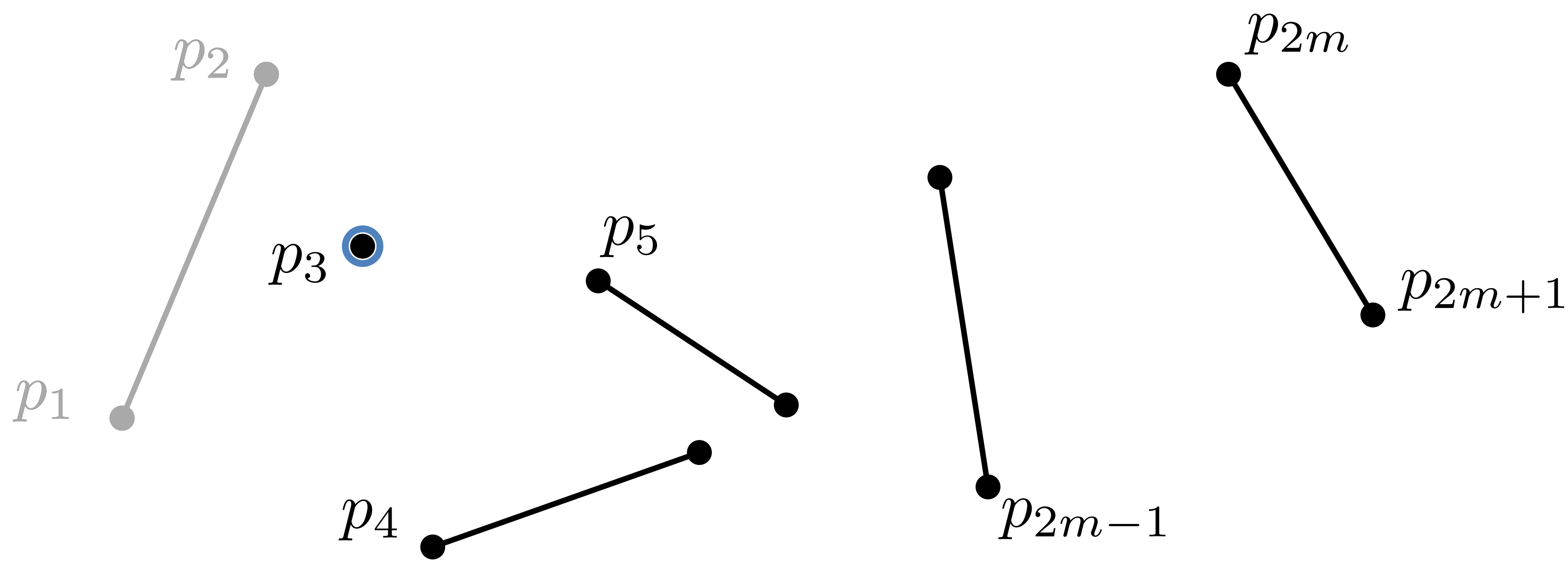

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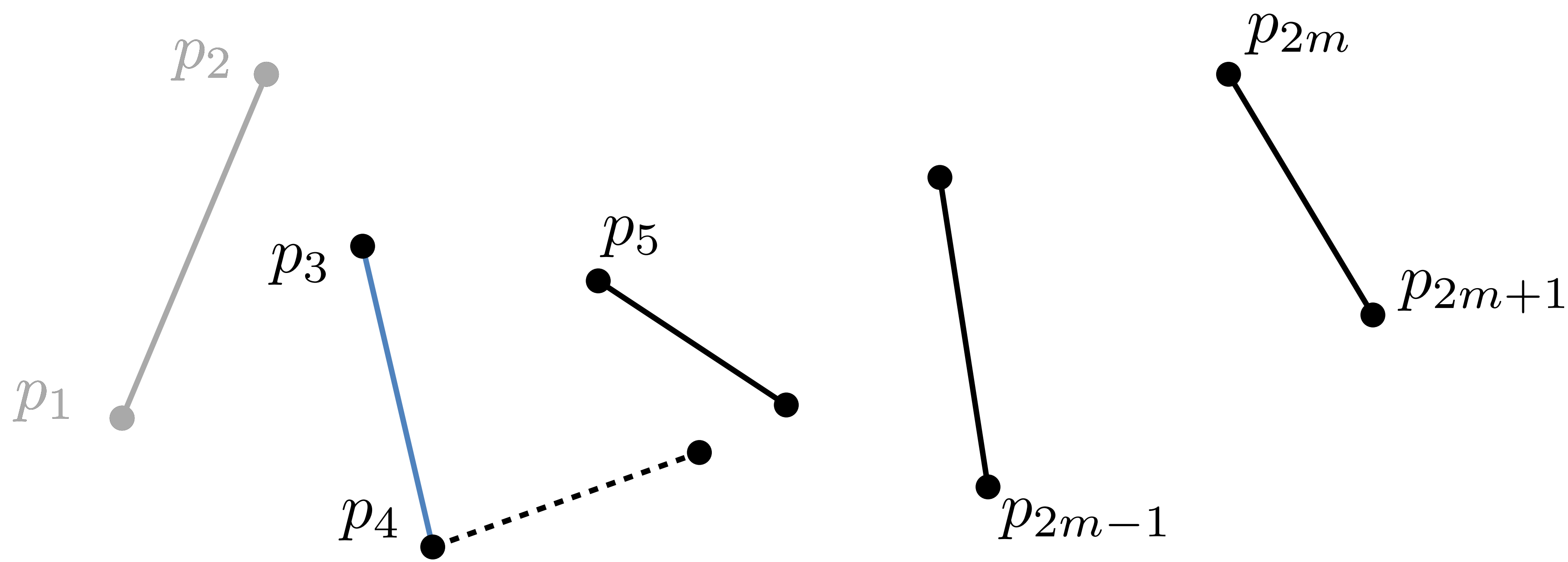

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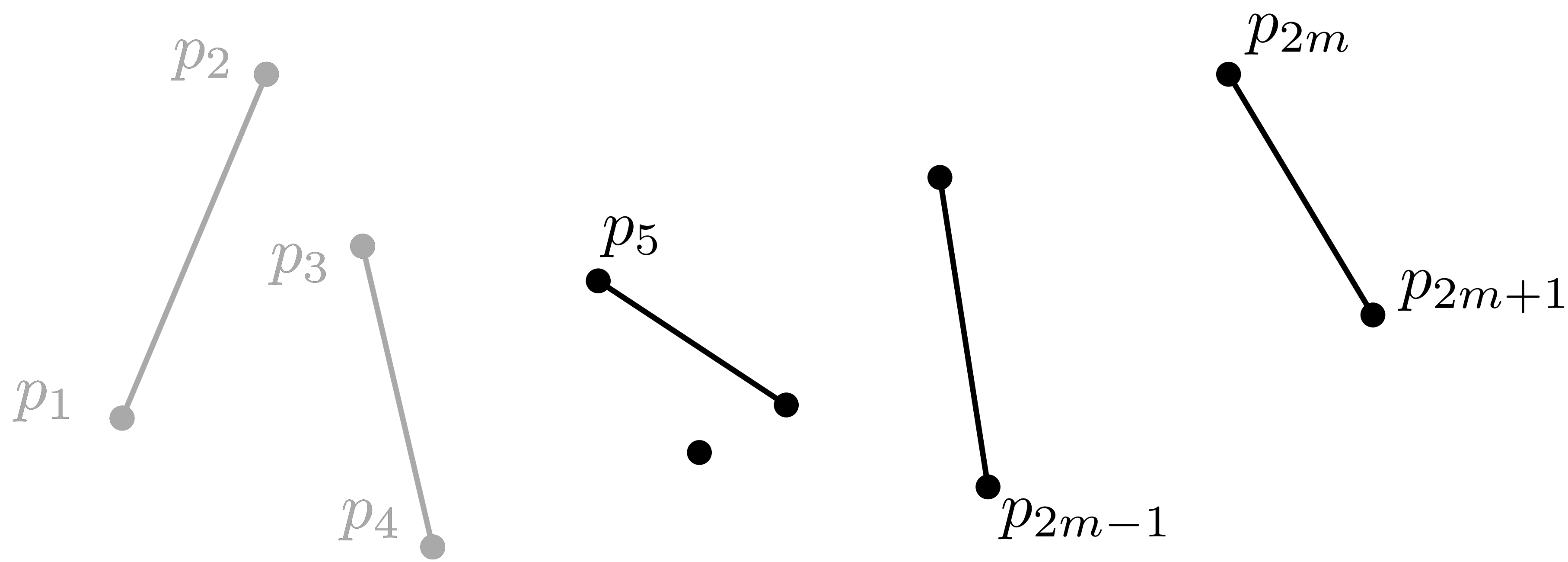

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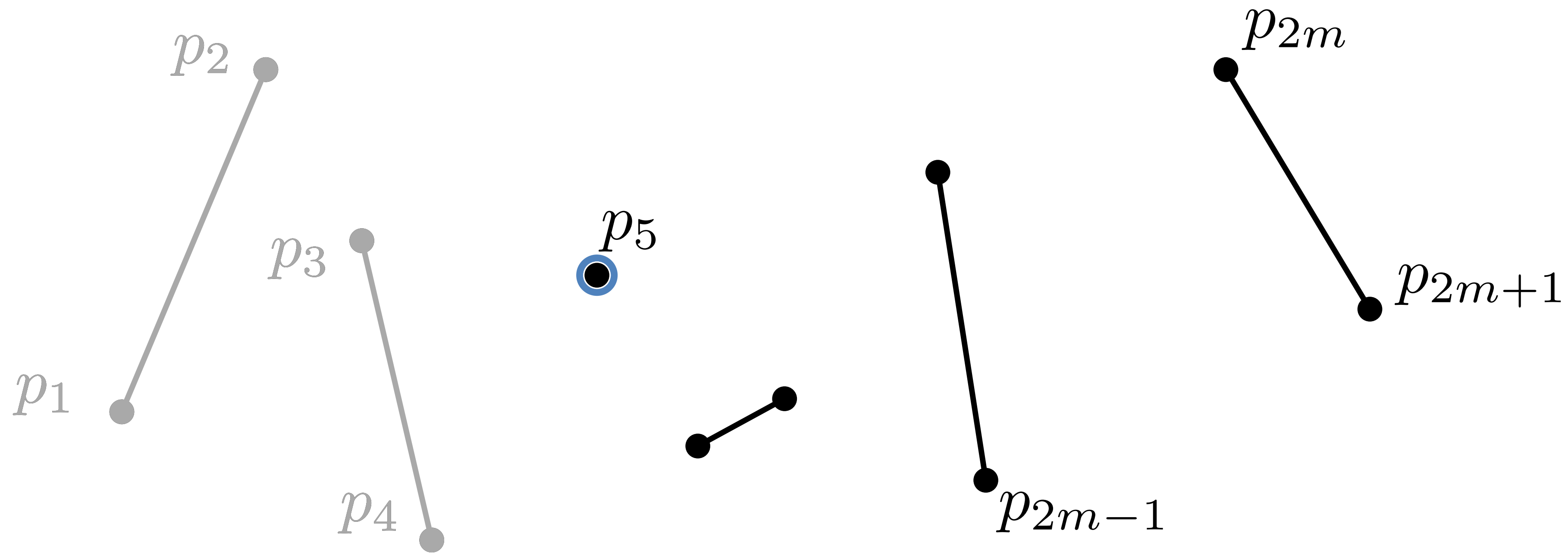

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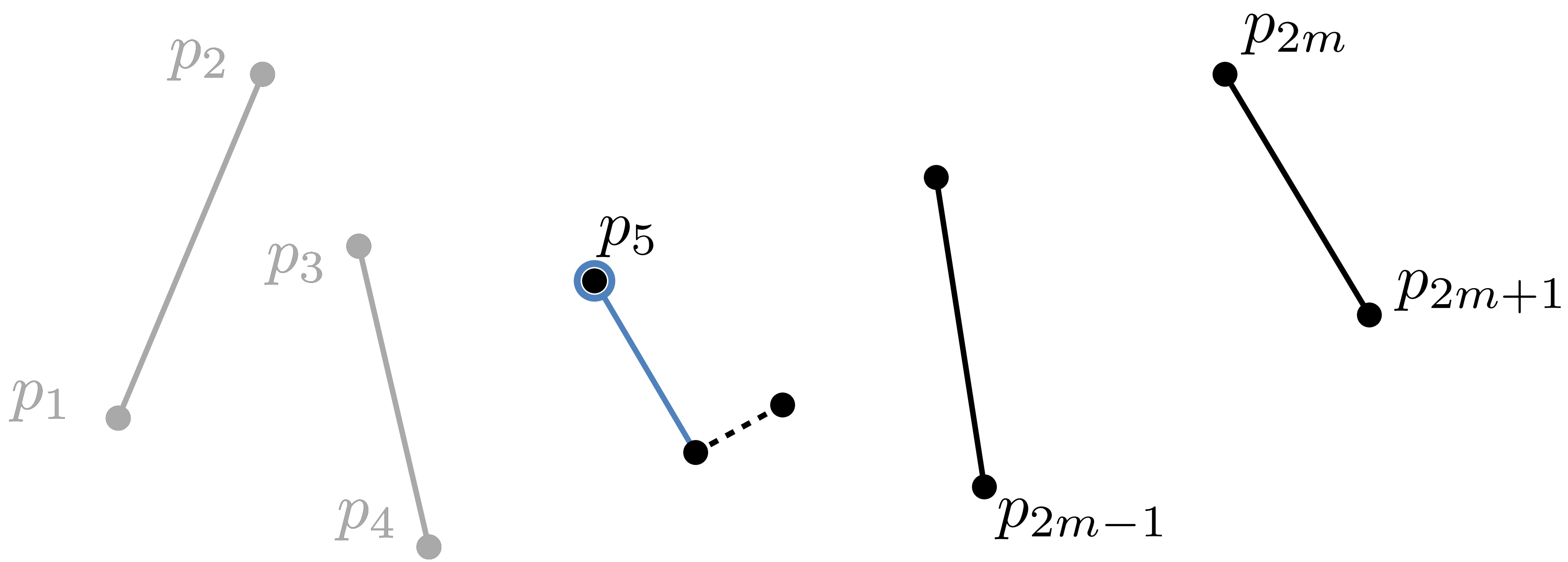

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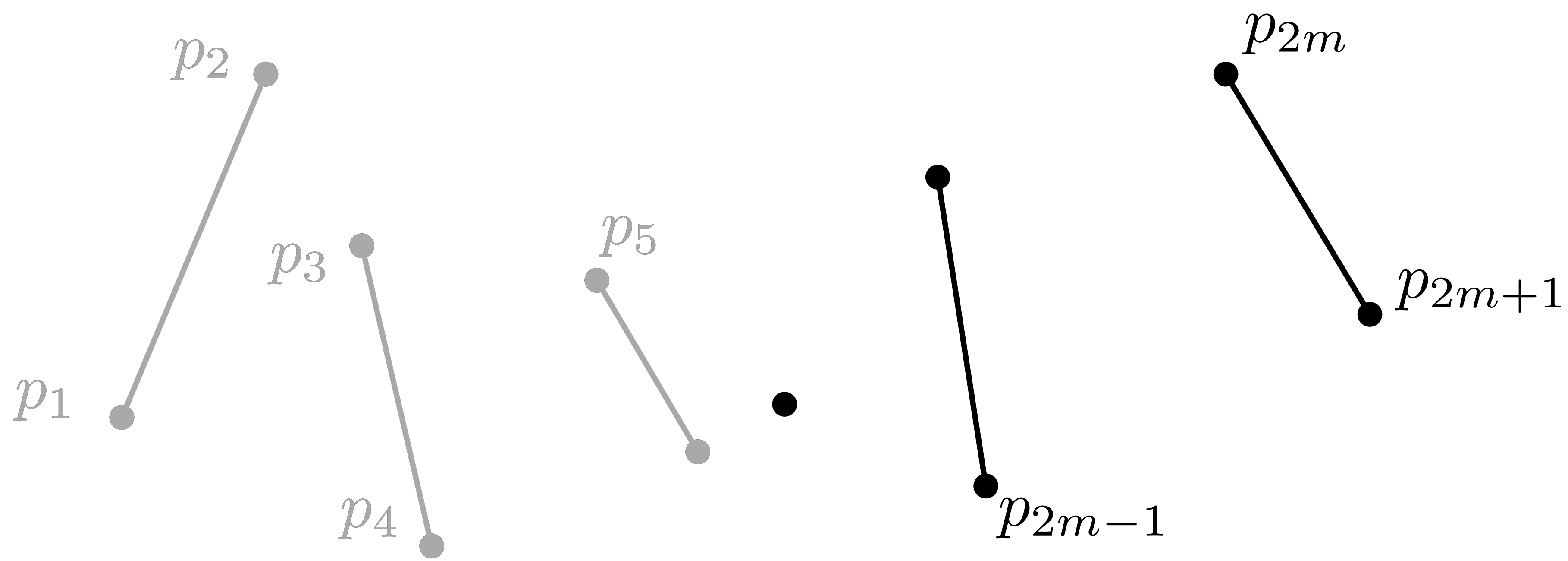

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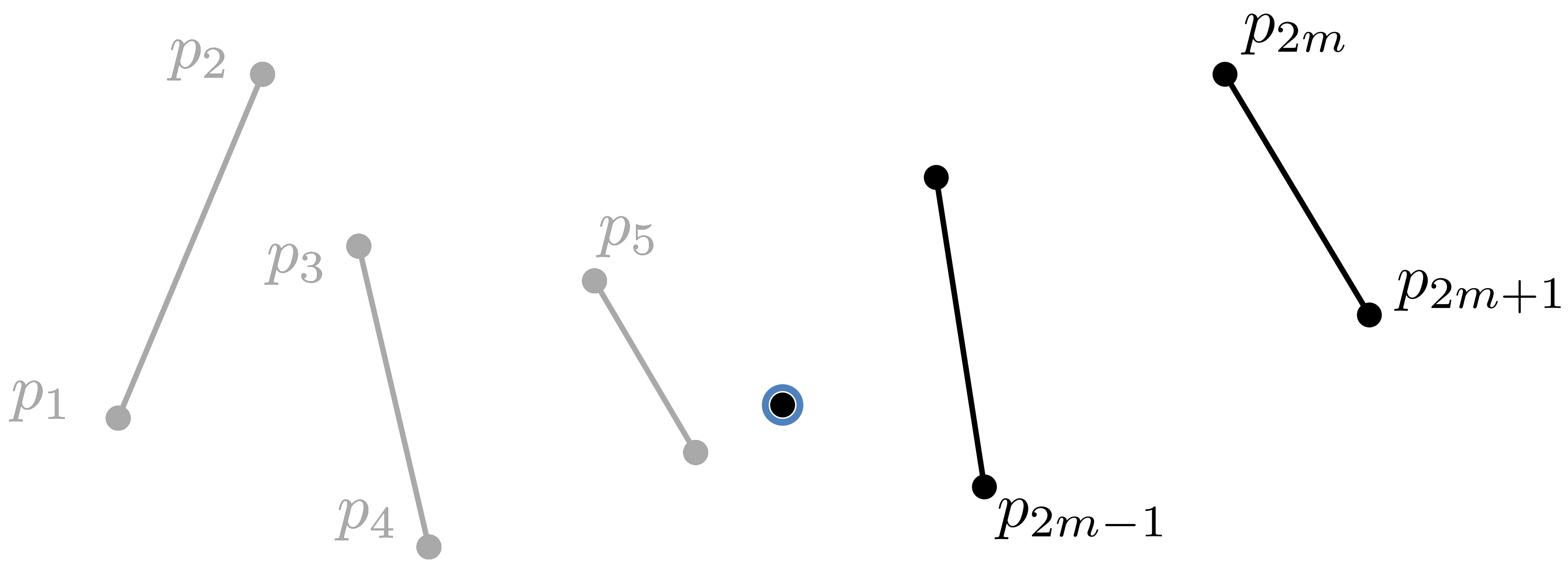

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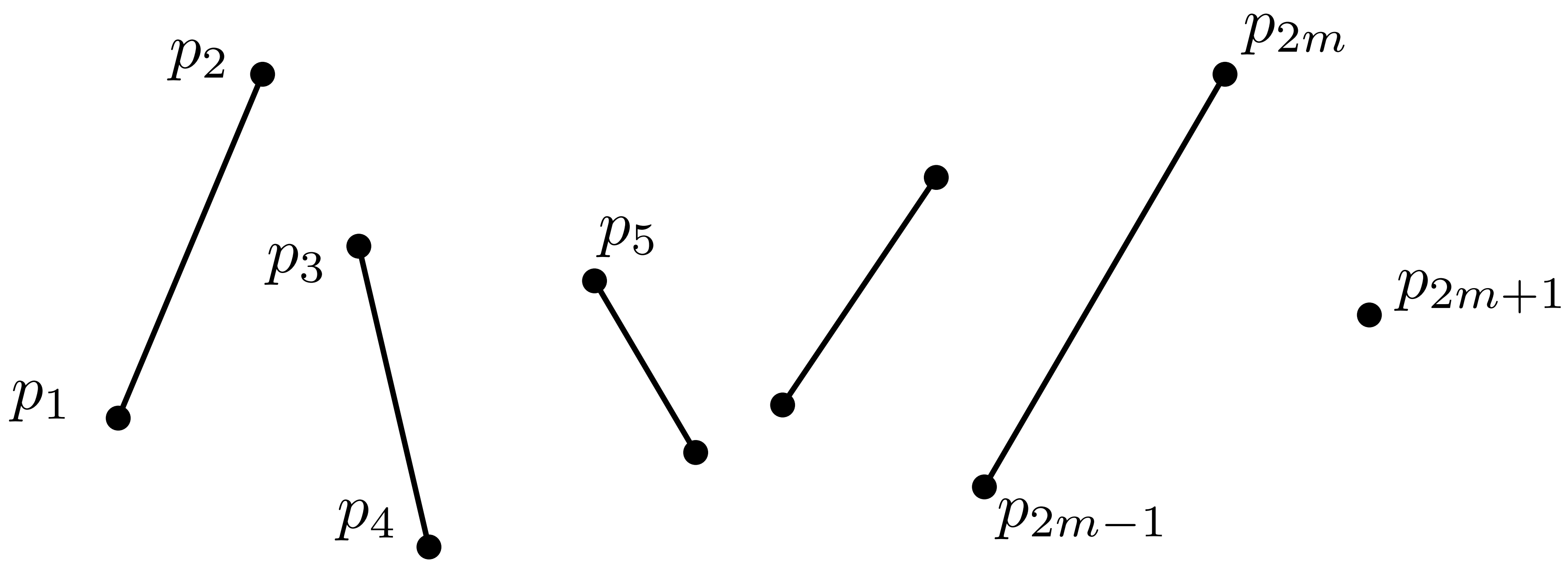

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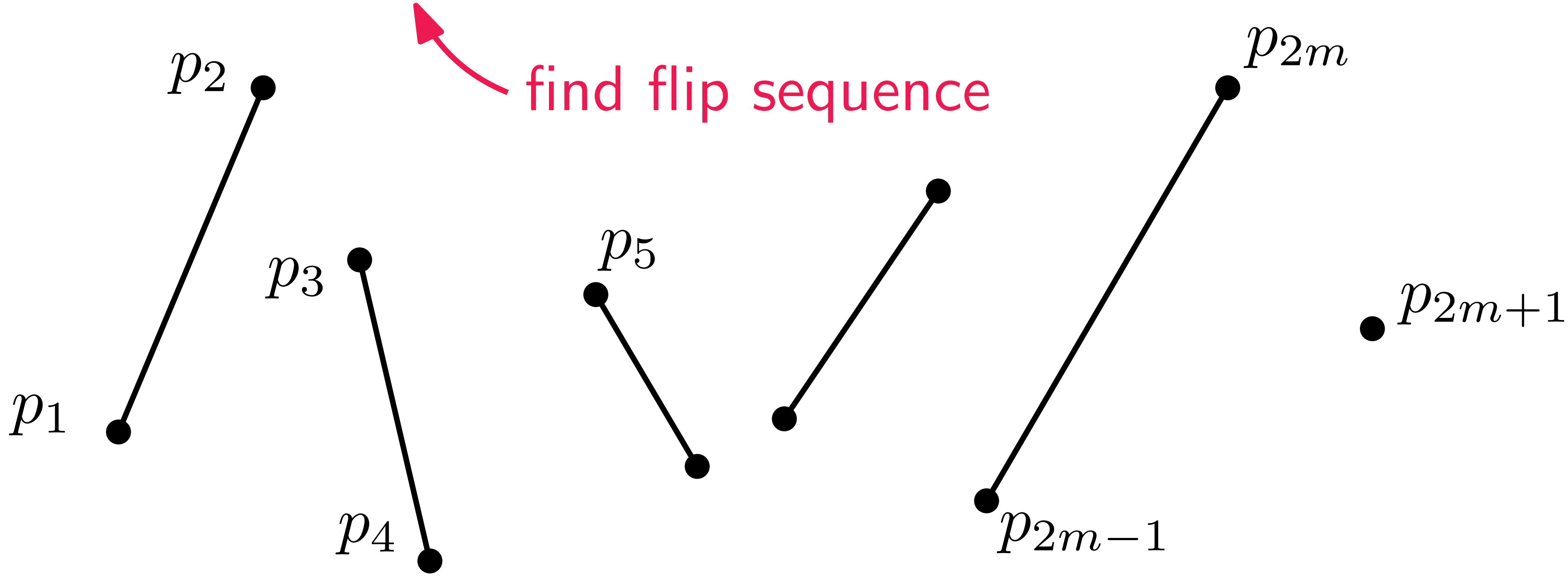

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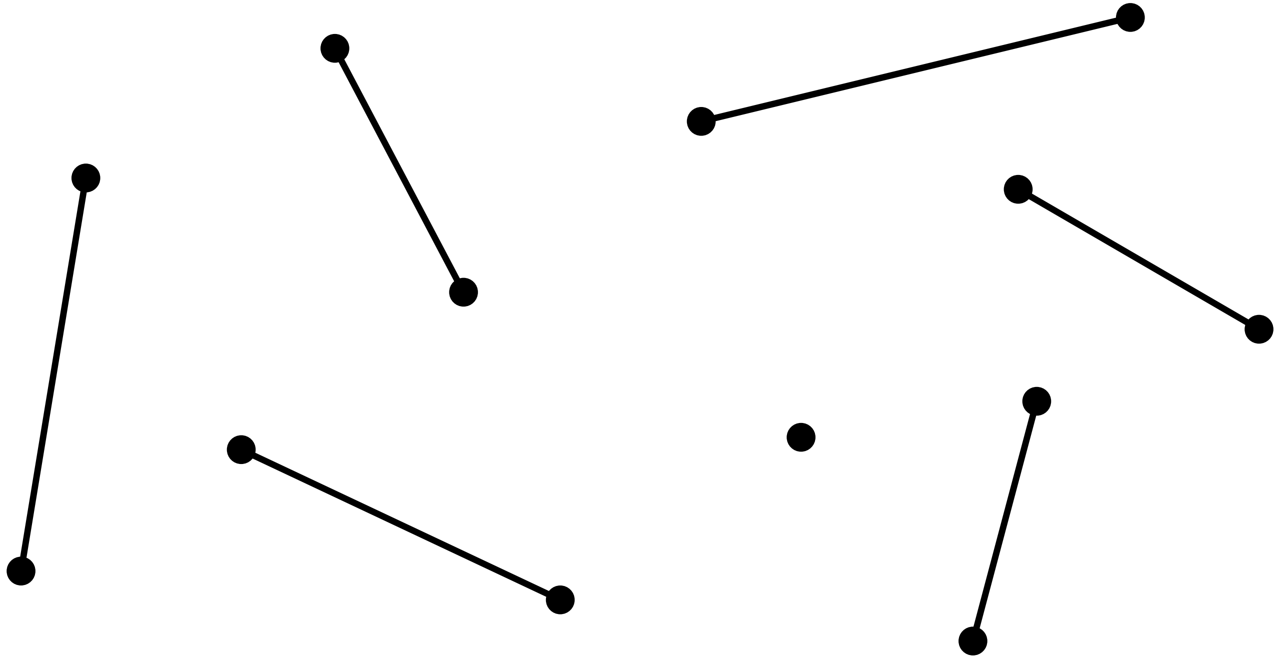
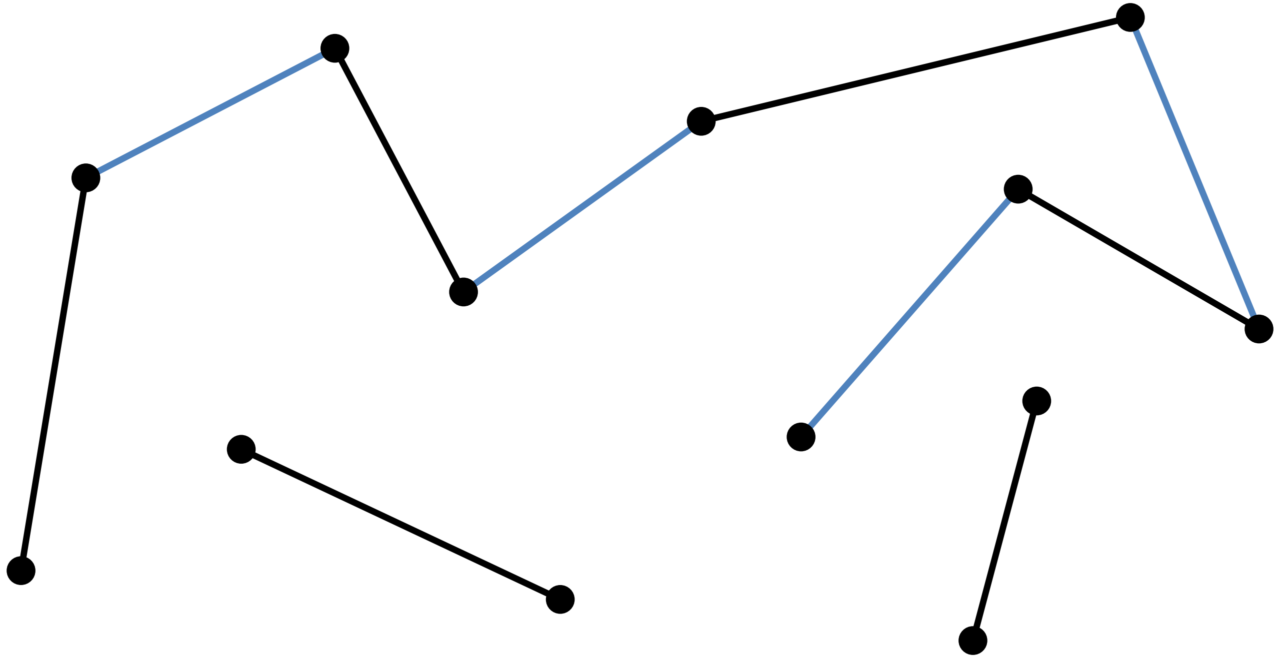
Flip Sequence



Flip Sequence

Observation: A plane alternating path gives rise to a flip sequence.



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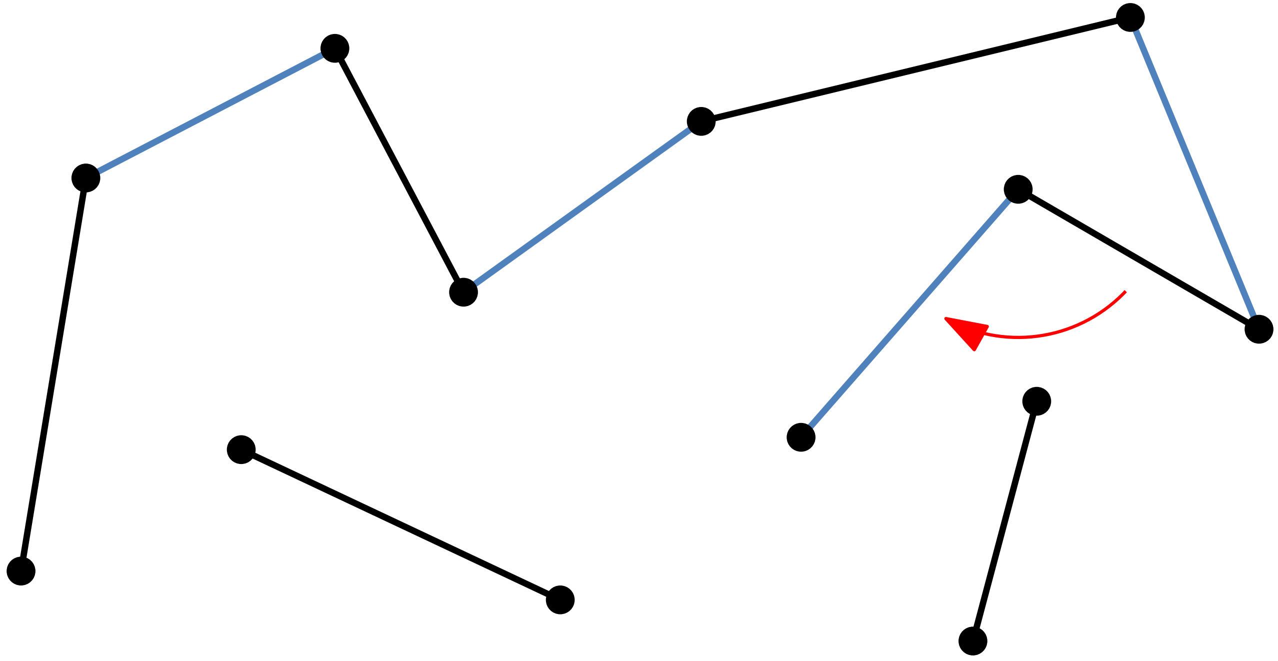

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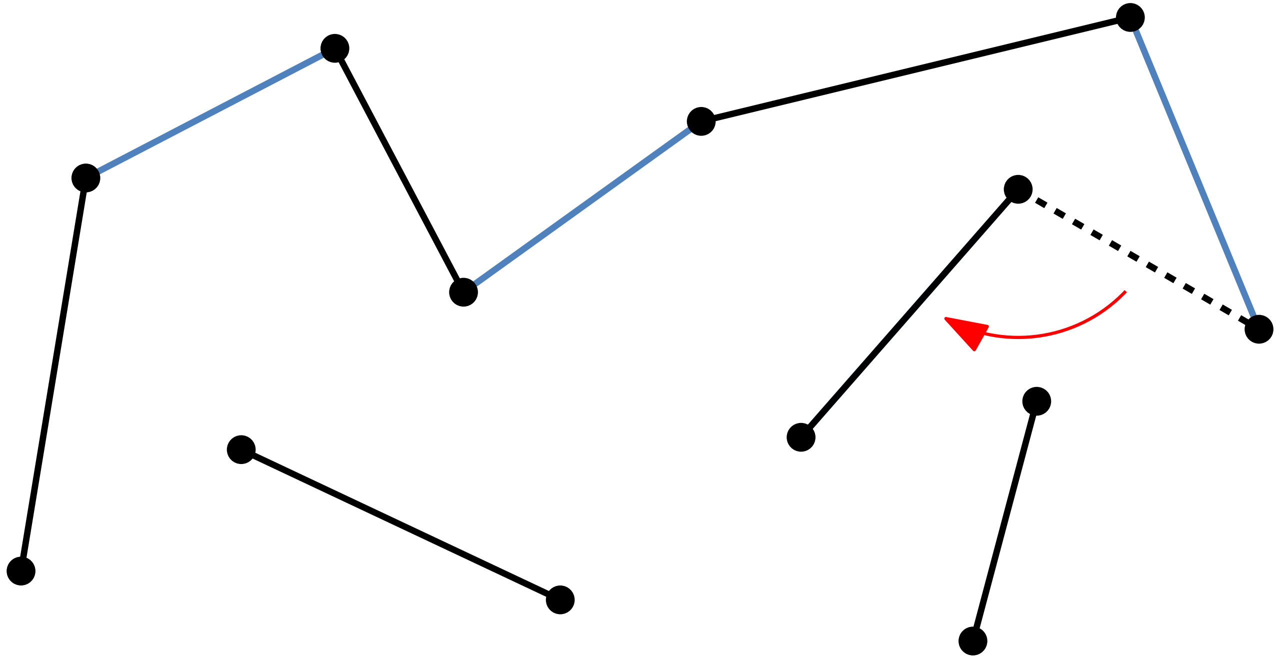

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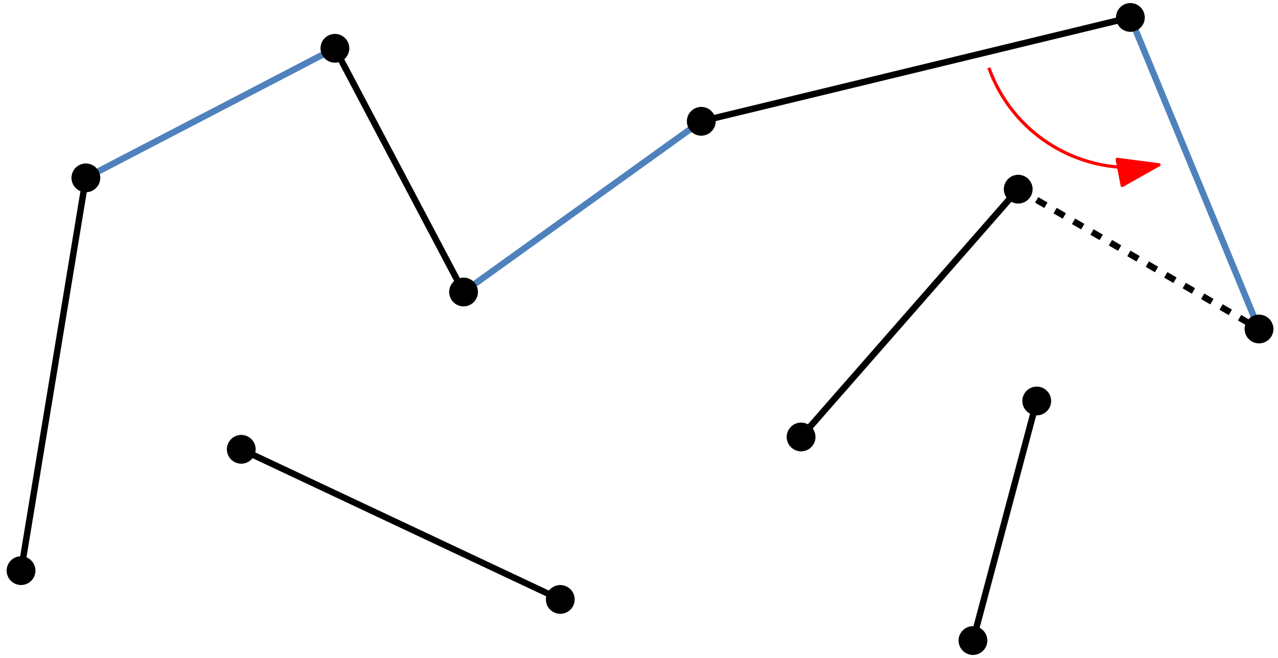

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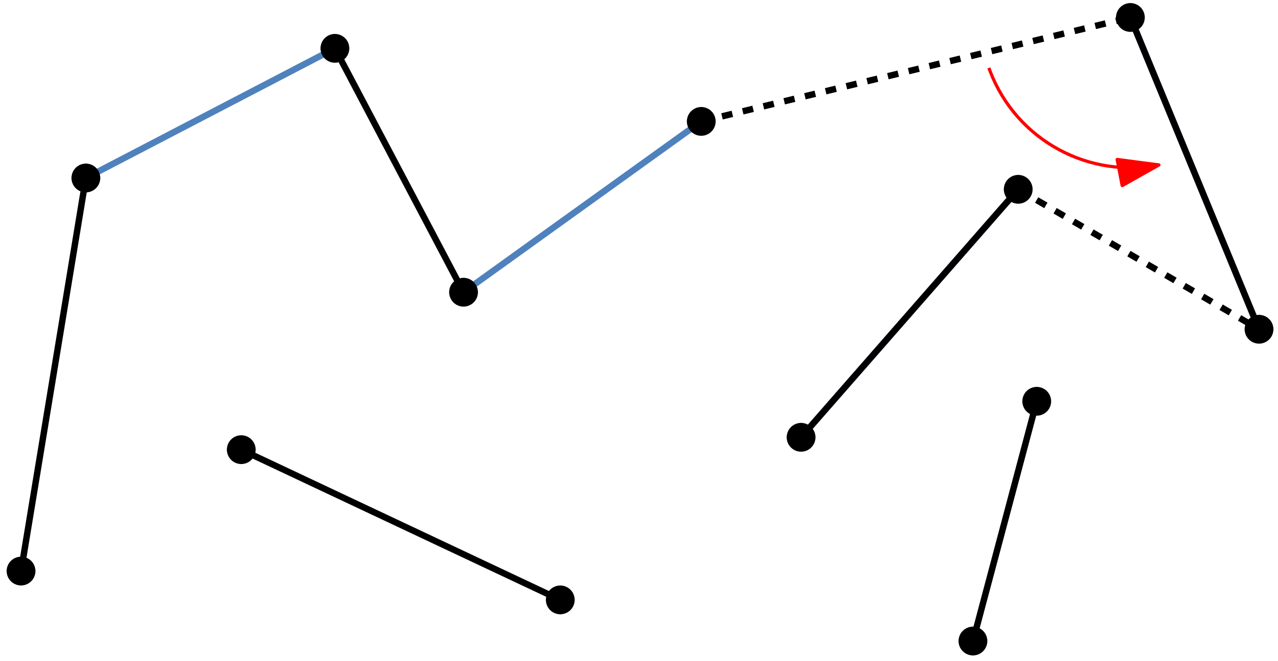

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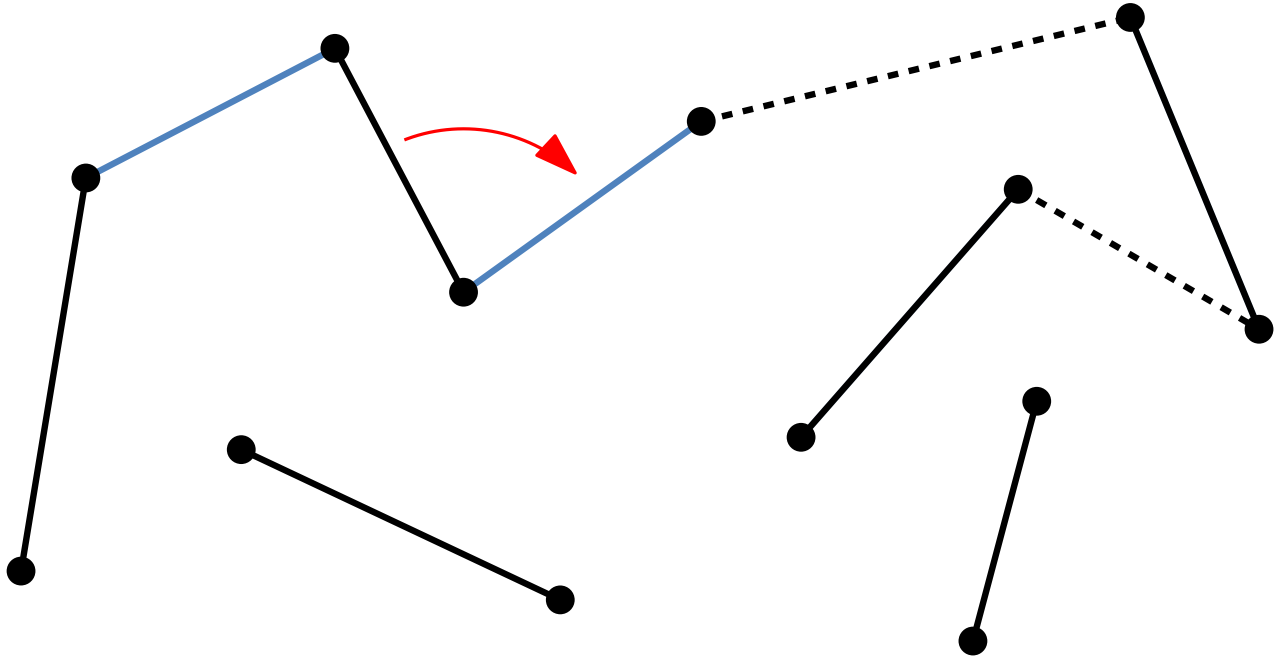

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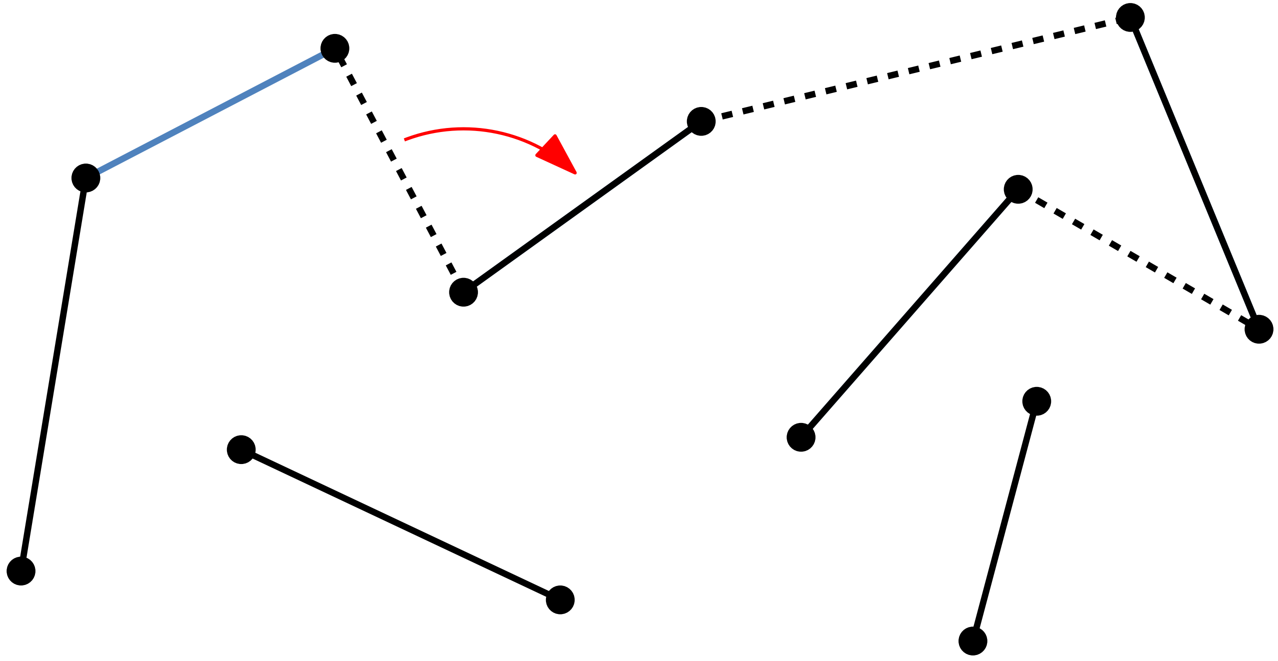

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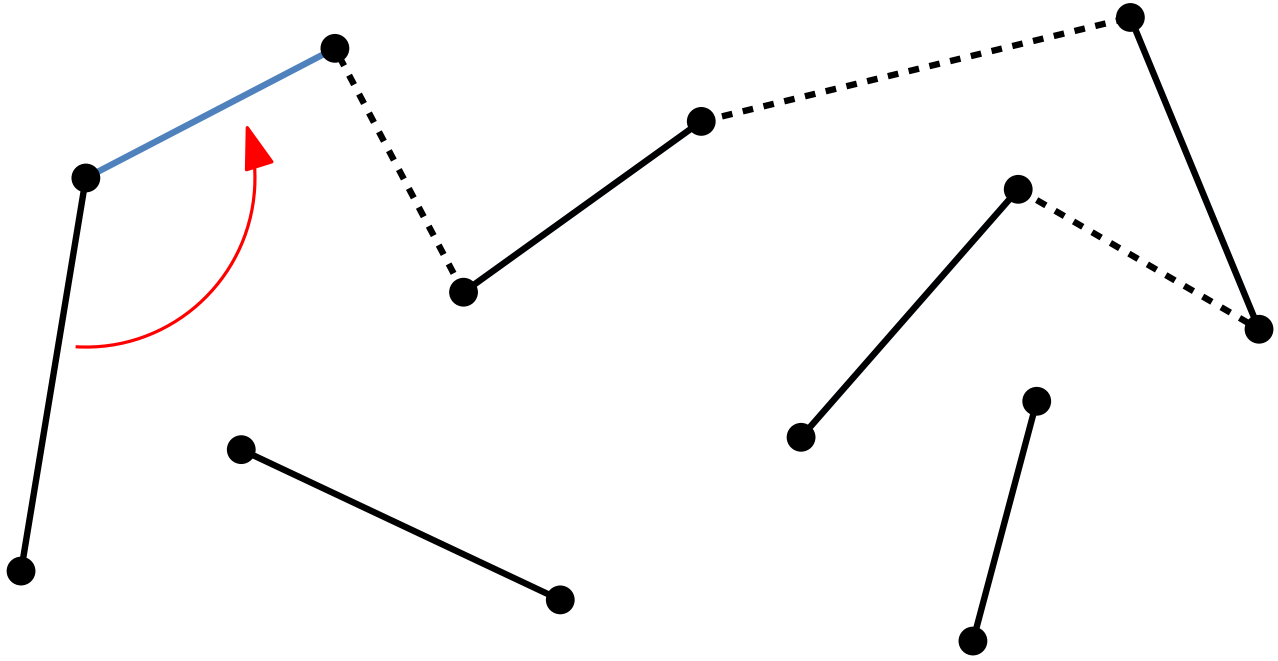

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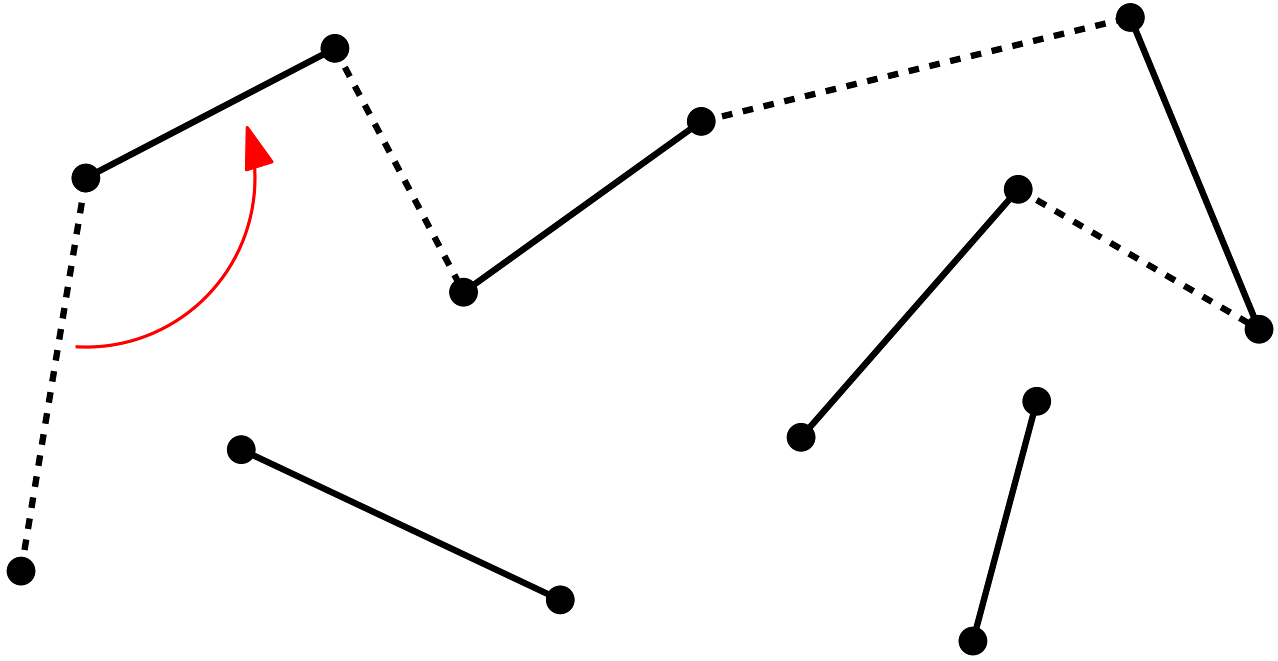

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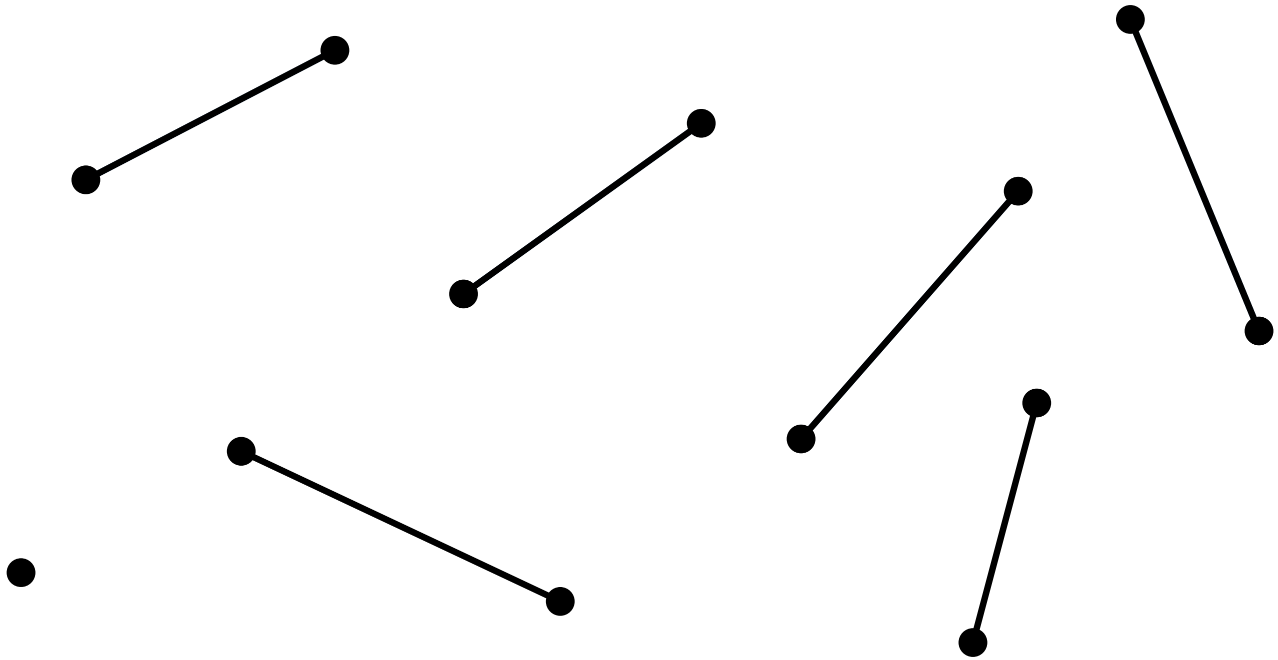

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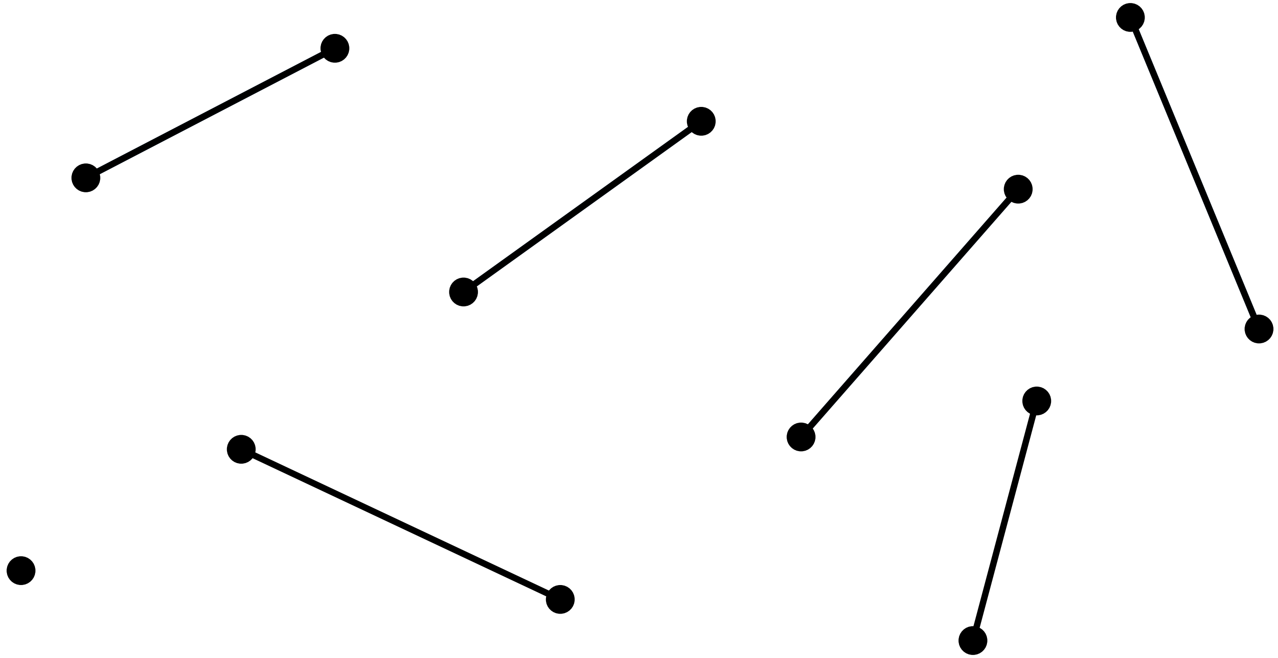

→ Find a plane alternating path between the unmatched point and the leftmost point

Detour: Segment Endpoint Visibility Graphs

Plane perfect matching $\widehat{=}$ segments in the plane

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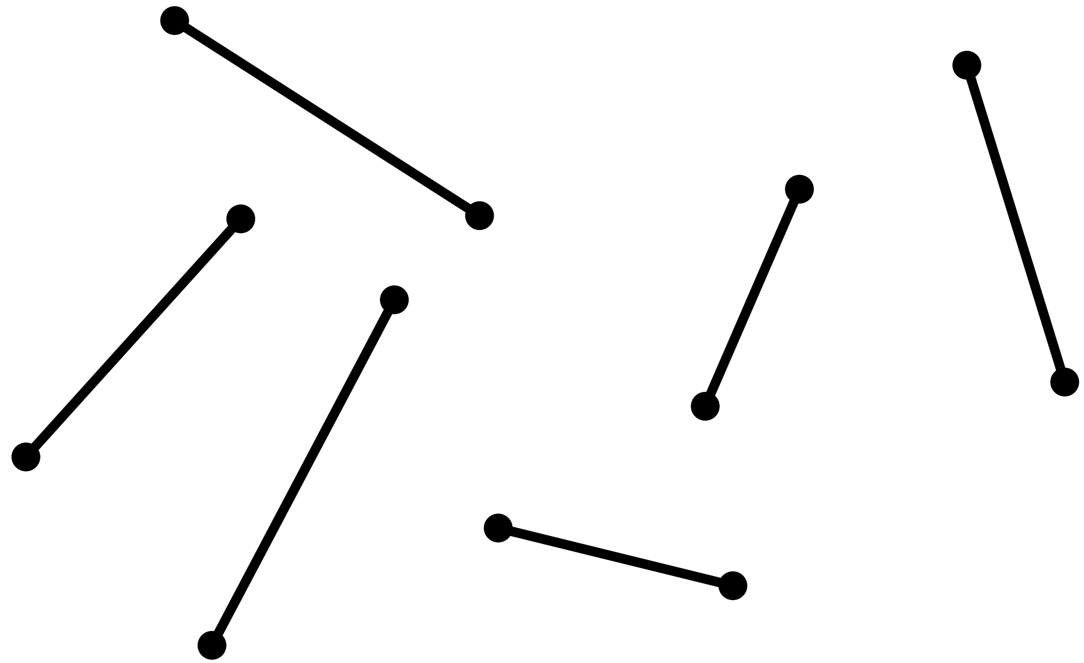

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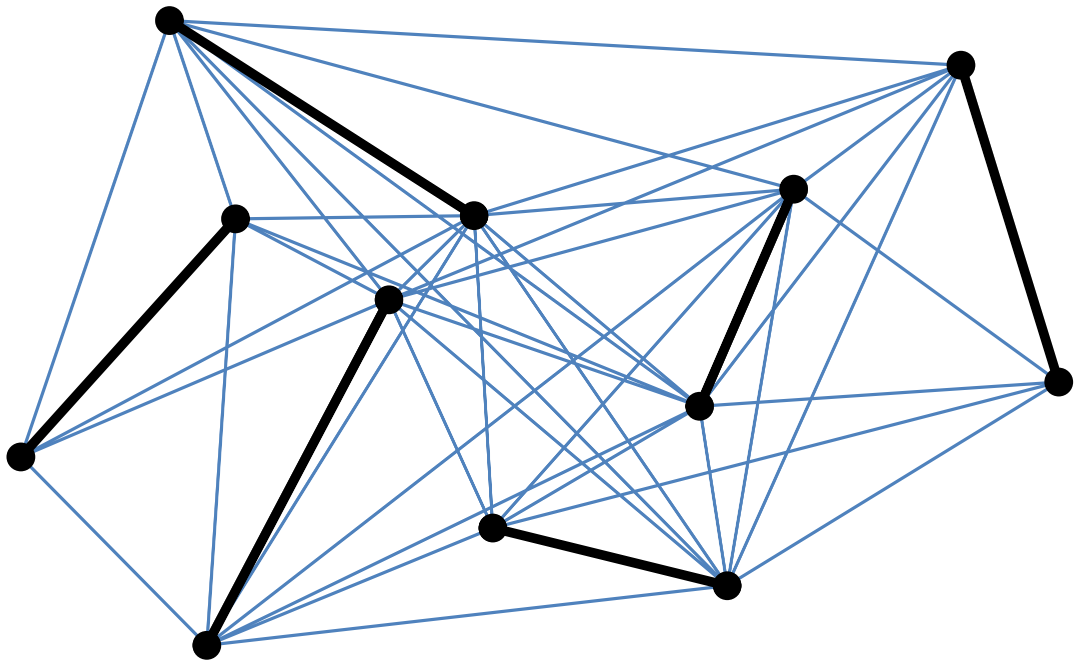

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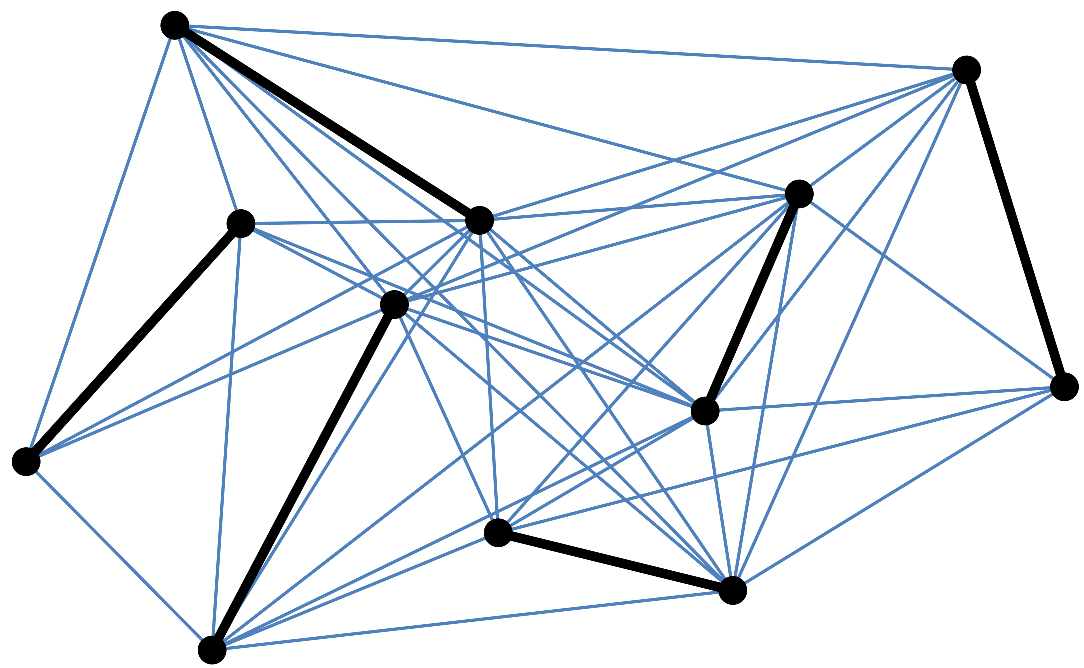

Theorem: Every segment endpoint visibility graph contains a plane Hamiltonian cycle. [Hoffmann, Tóth 2003]

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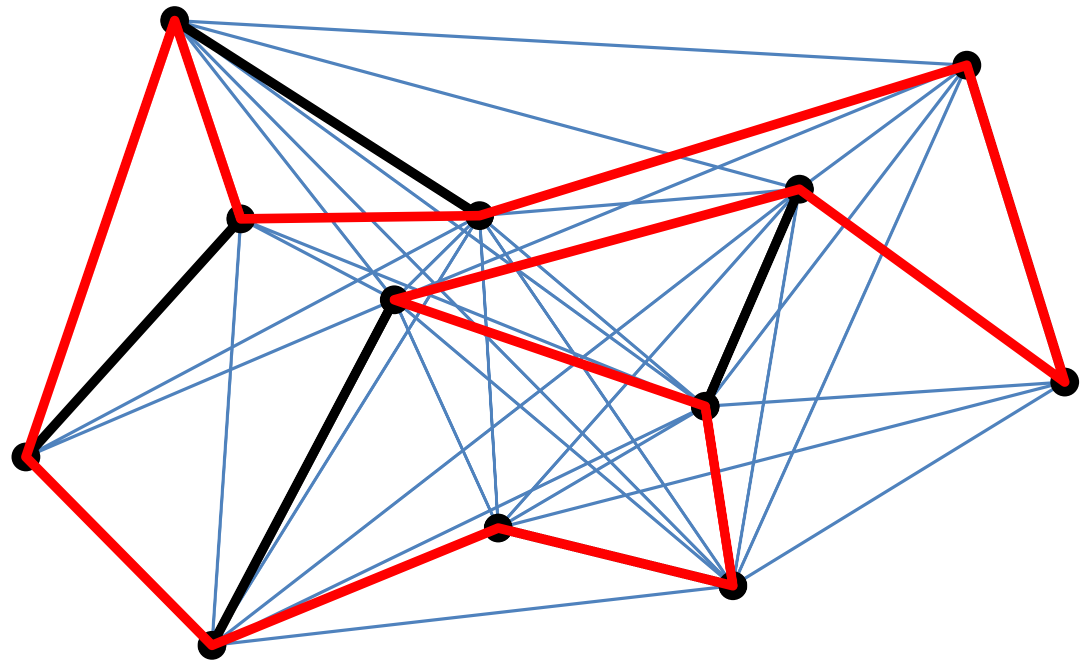

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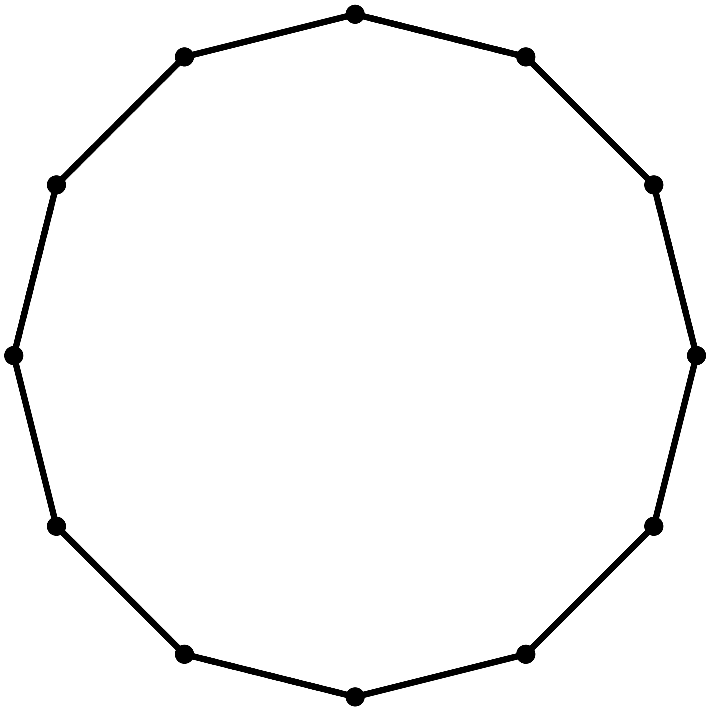

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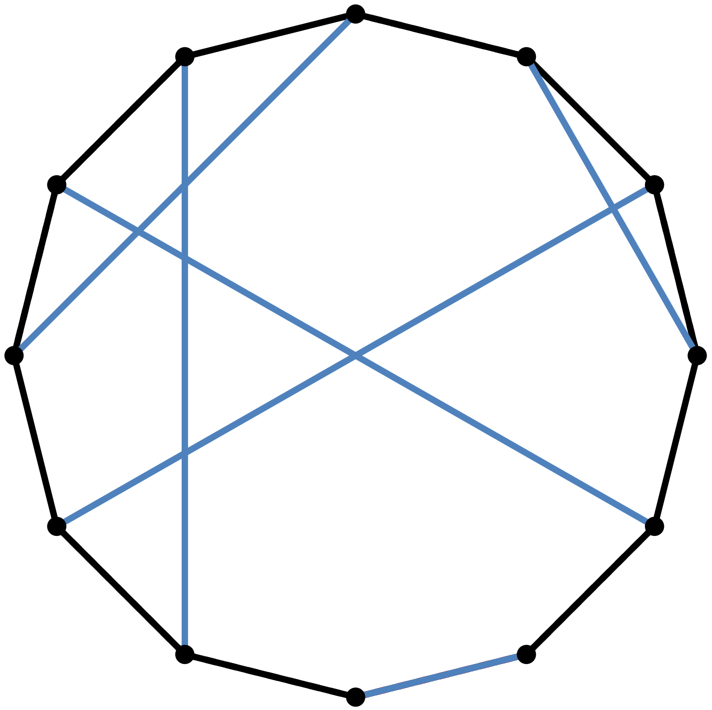

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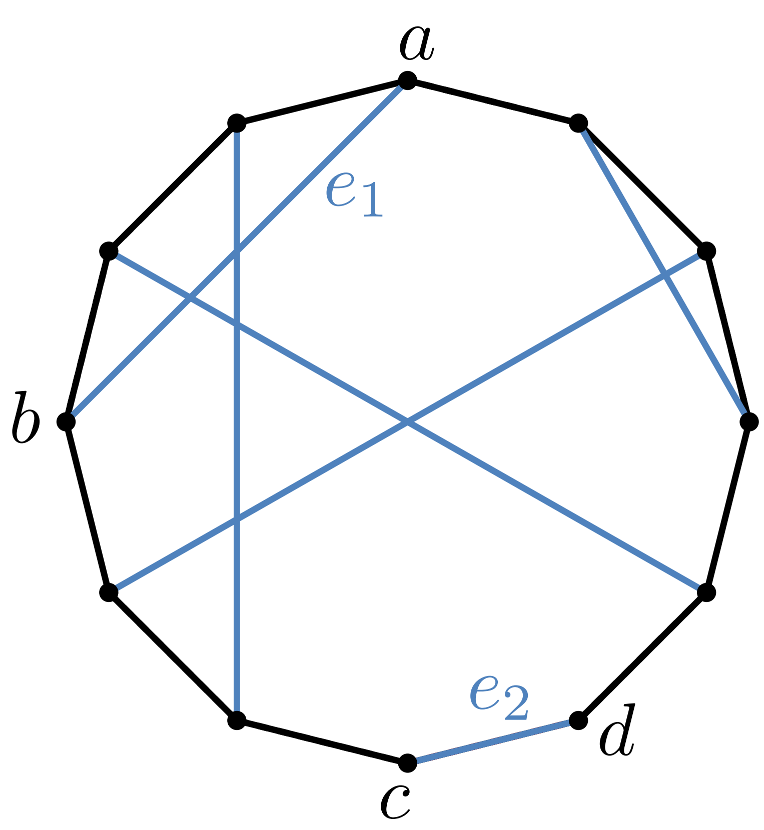

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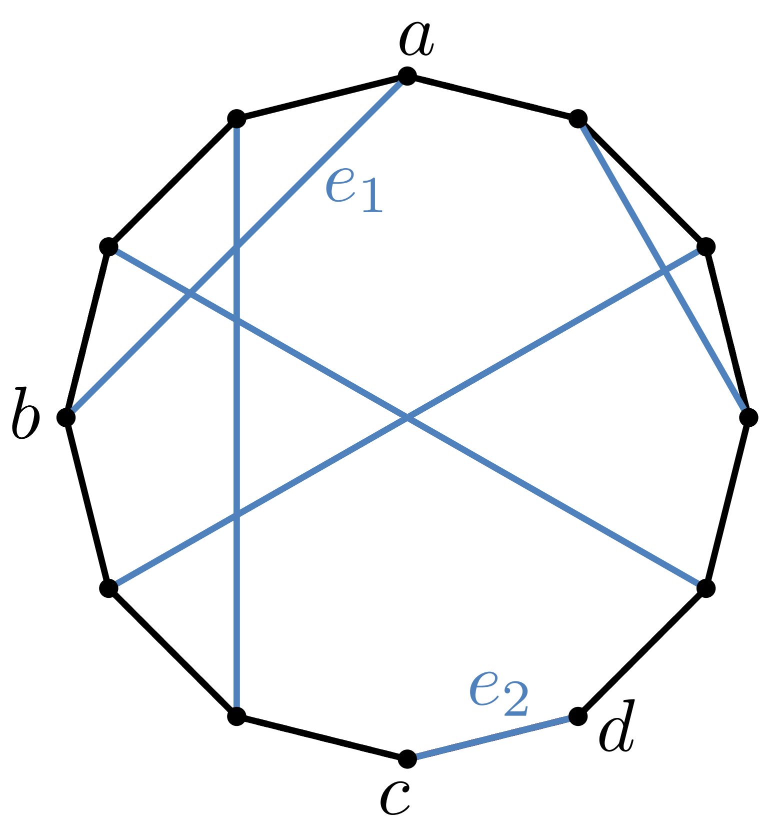

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Lemma 1: There exists an alternating path P that starts at vertex a and edge e_1 and ends at vertex c.

Finding a Plane Alternating Path

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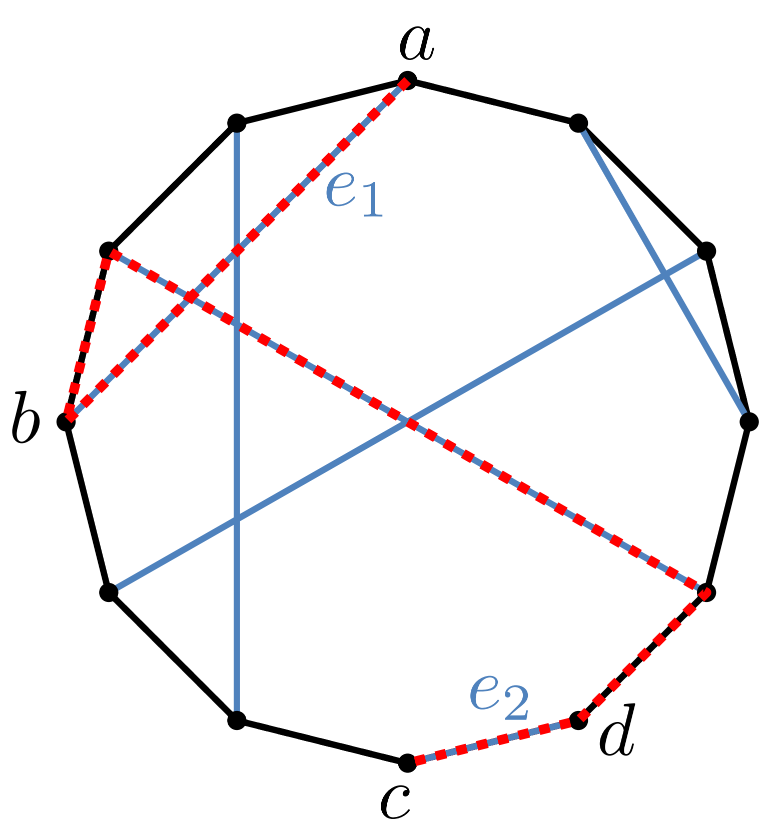

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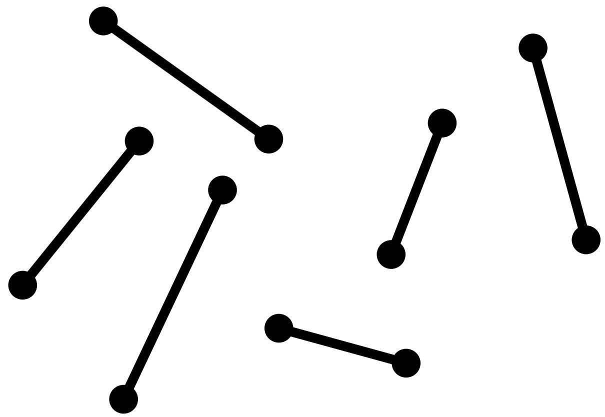

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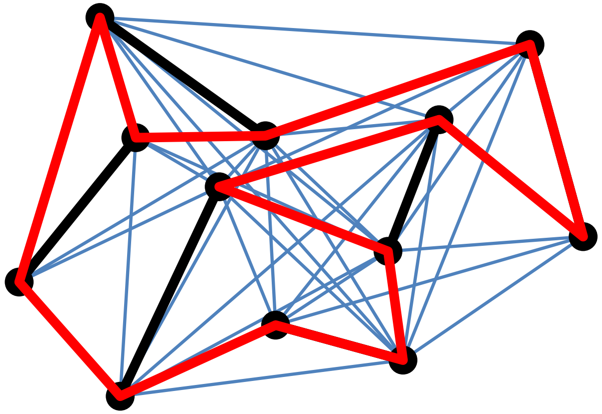

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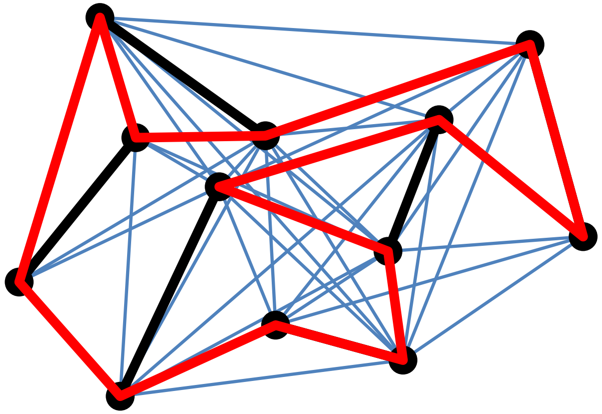

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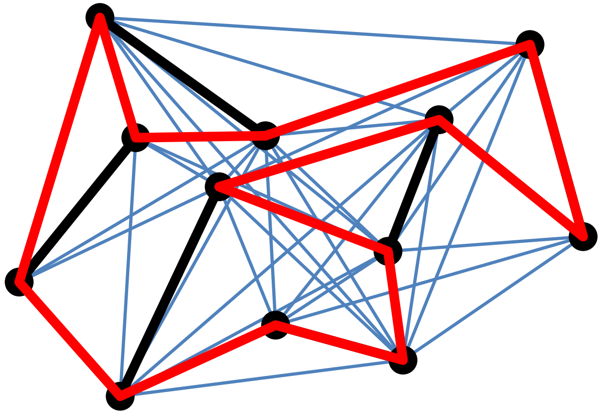

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Back to Our Setting

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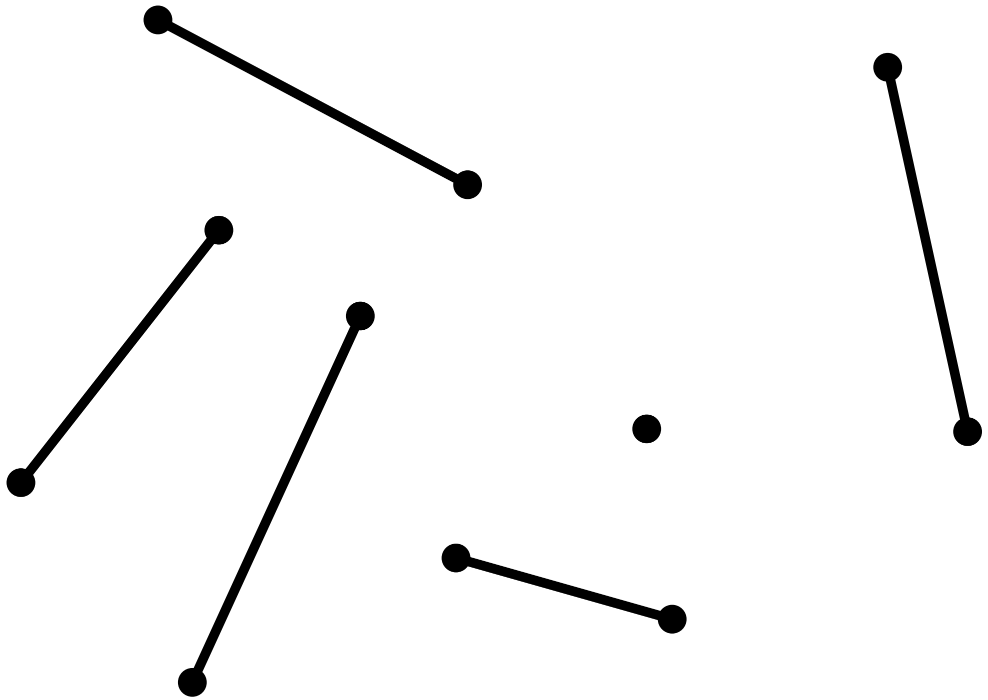

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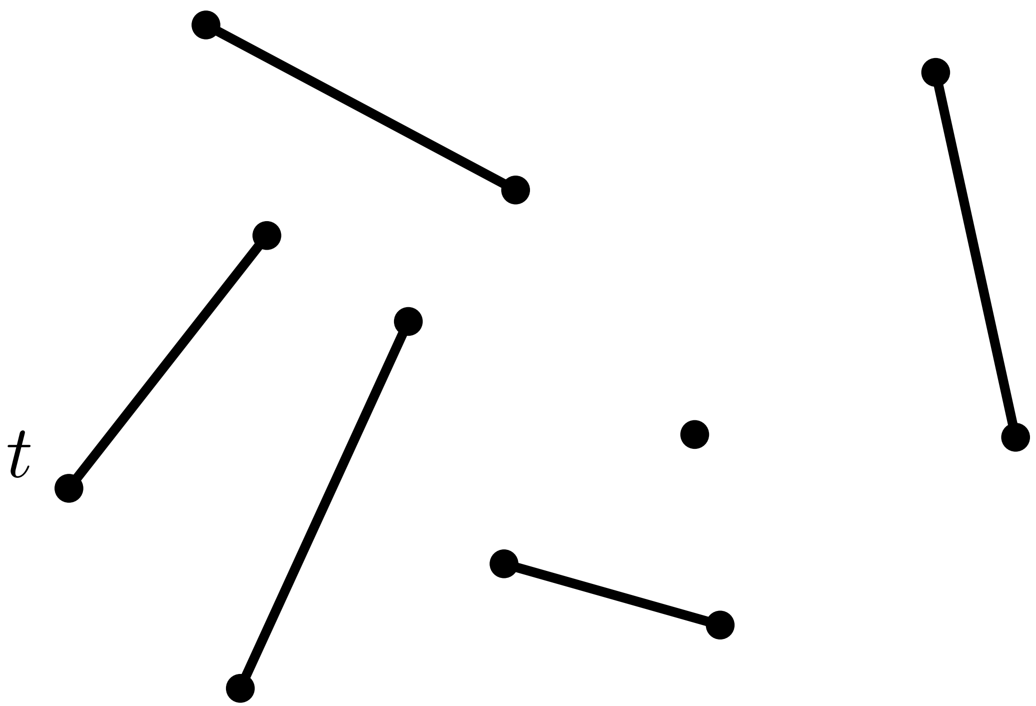
Lemma 2: Let t be a point on the convex hull of P. There exists a sequence of flips to a matching where t is unmatched.



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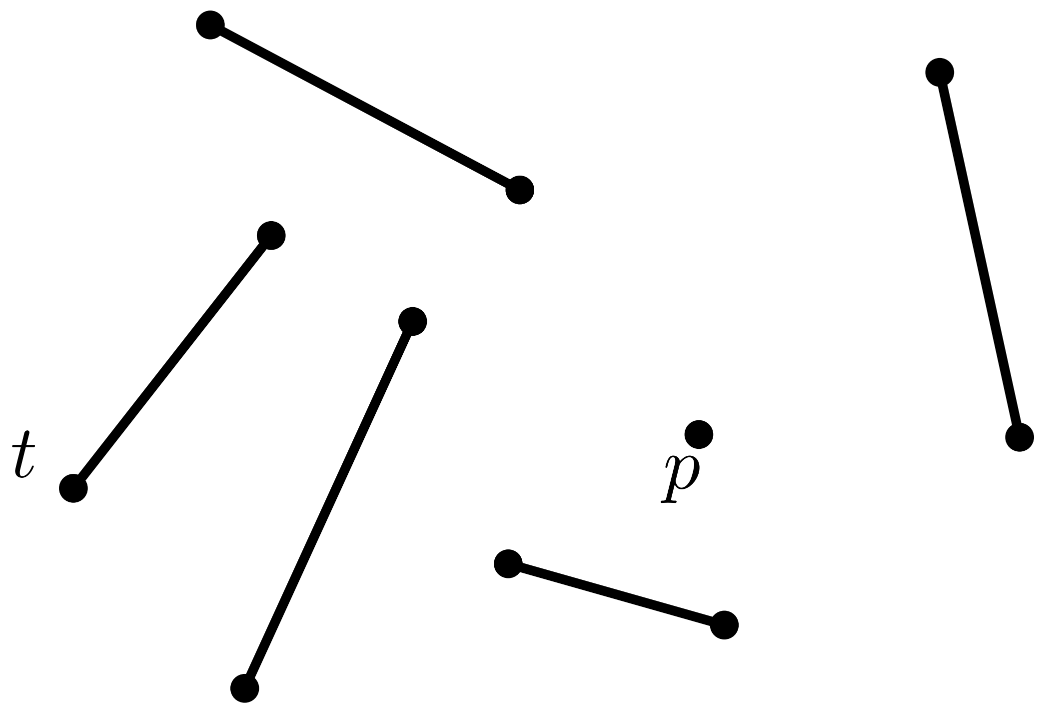

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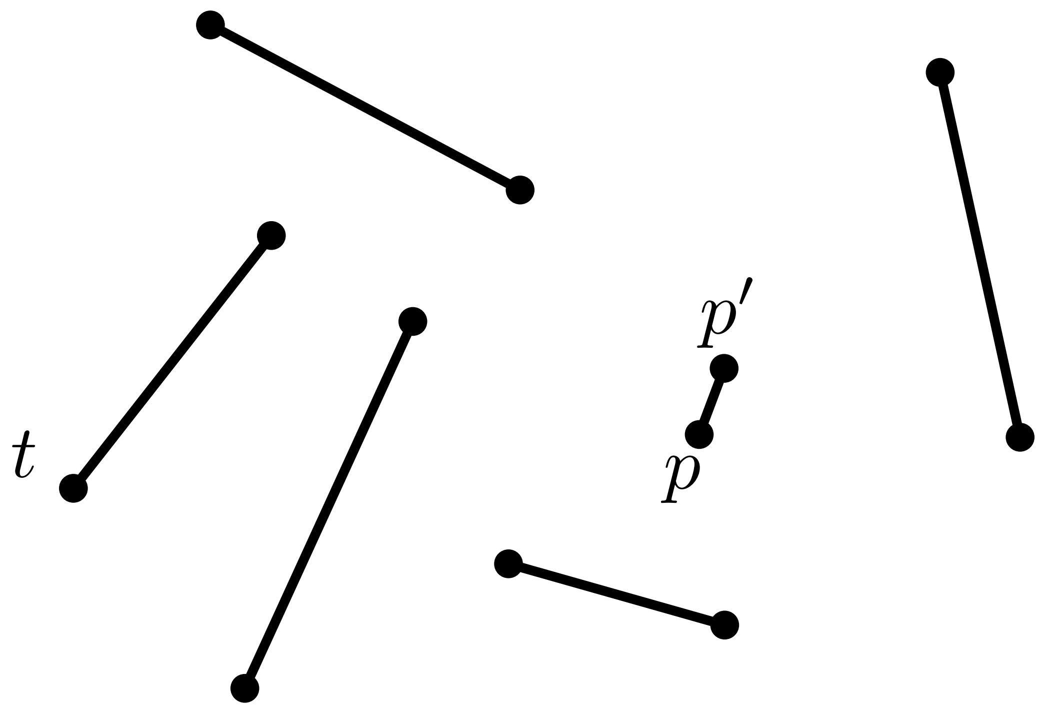

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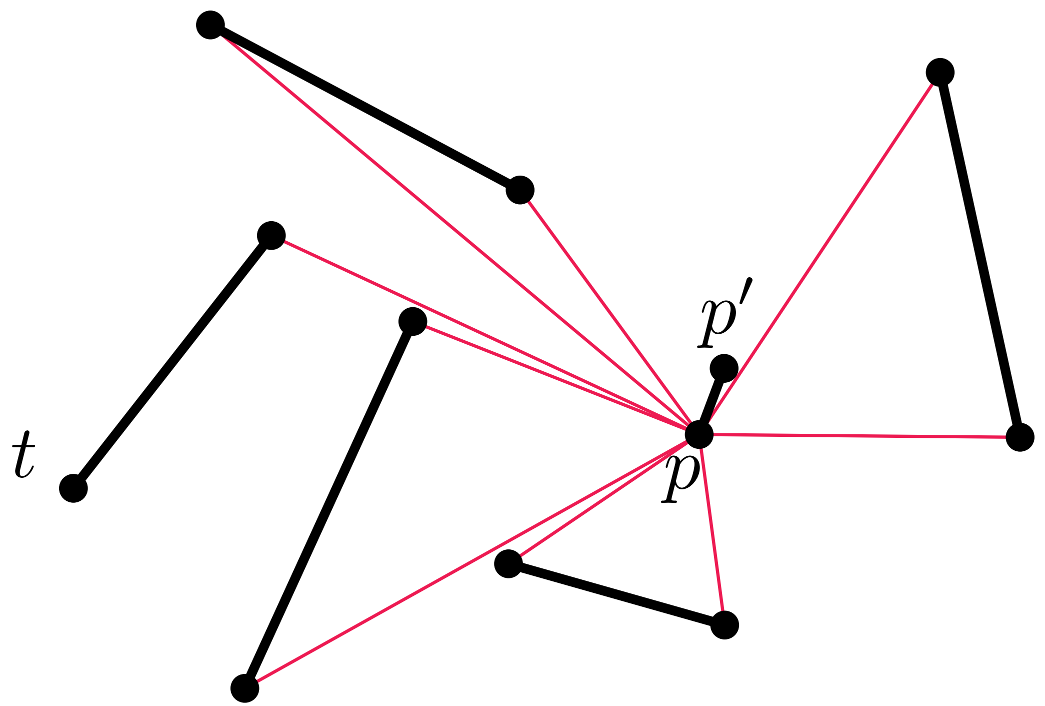

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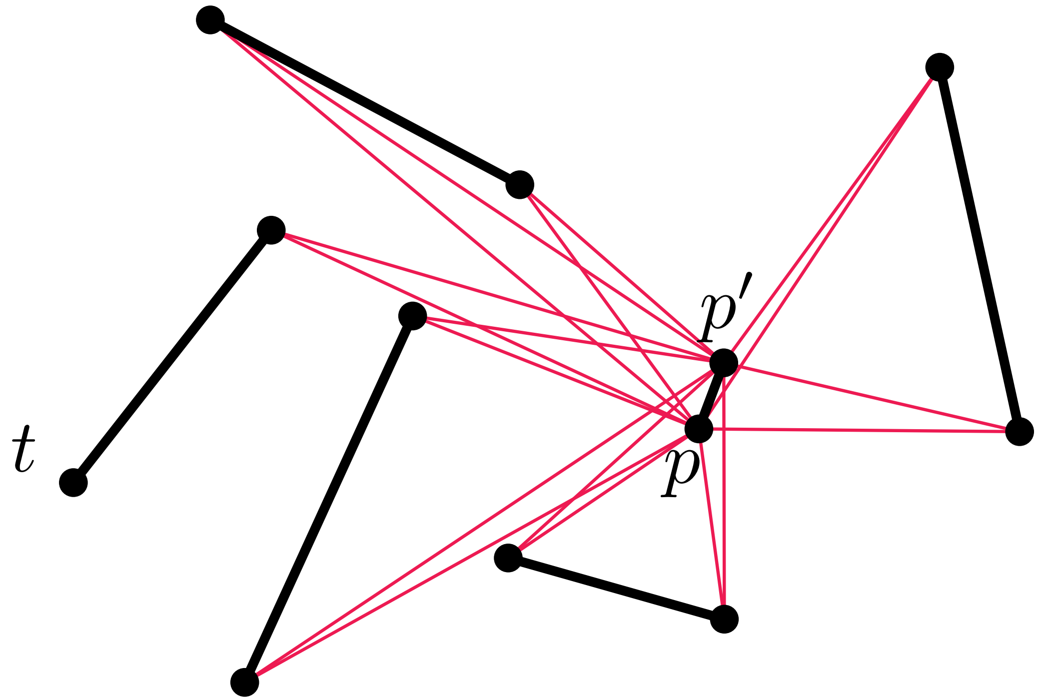

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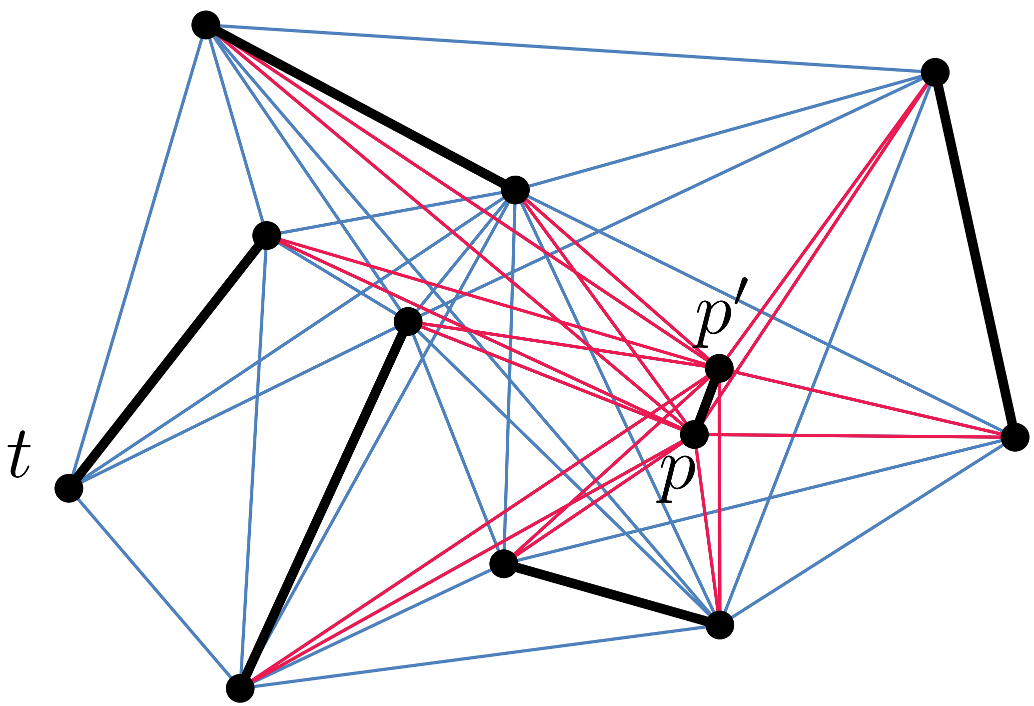

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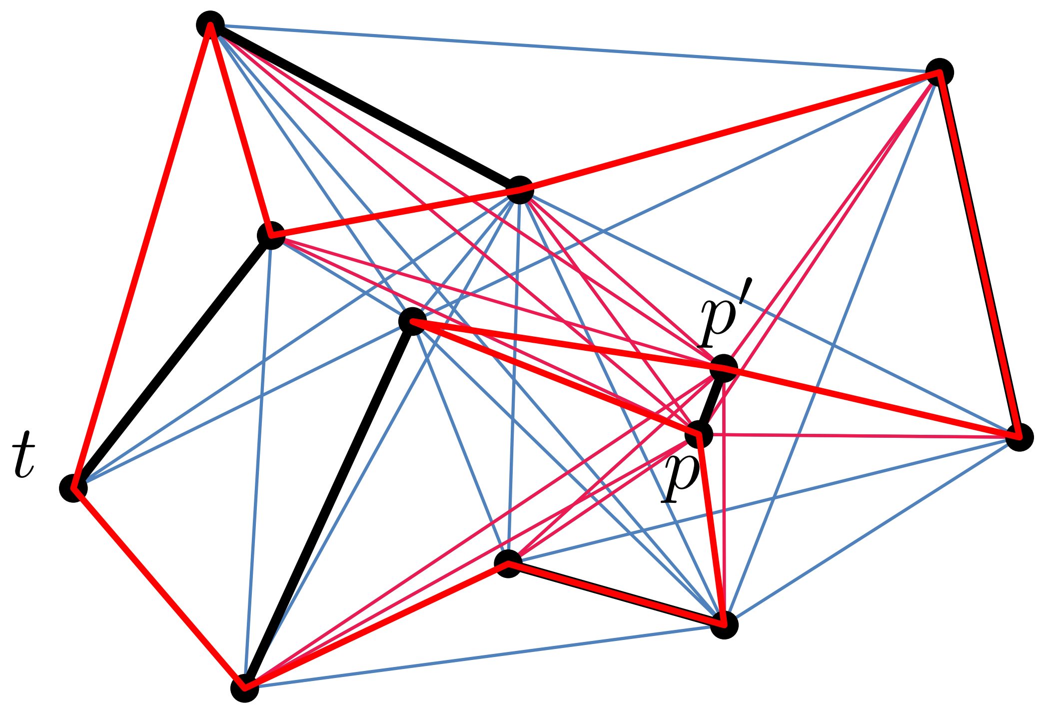

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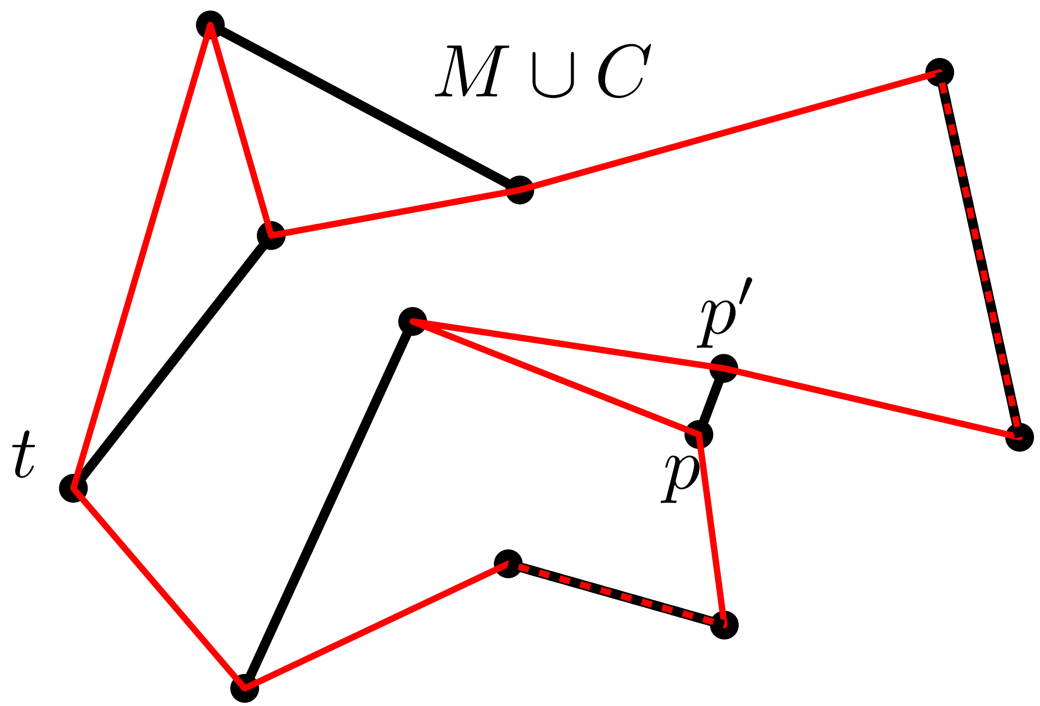

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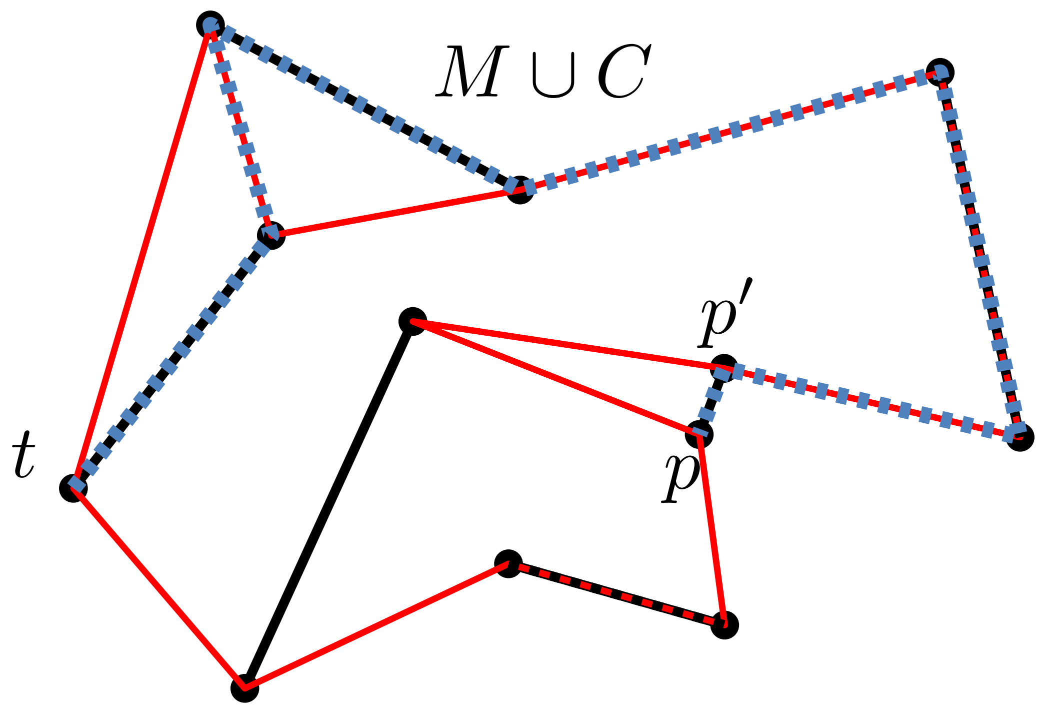

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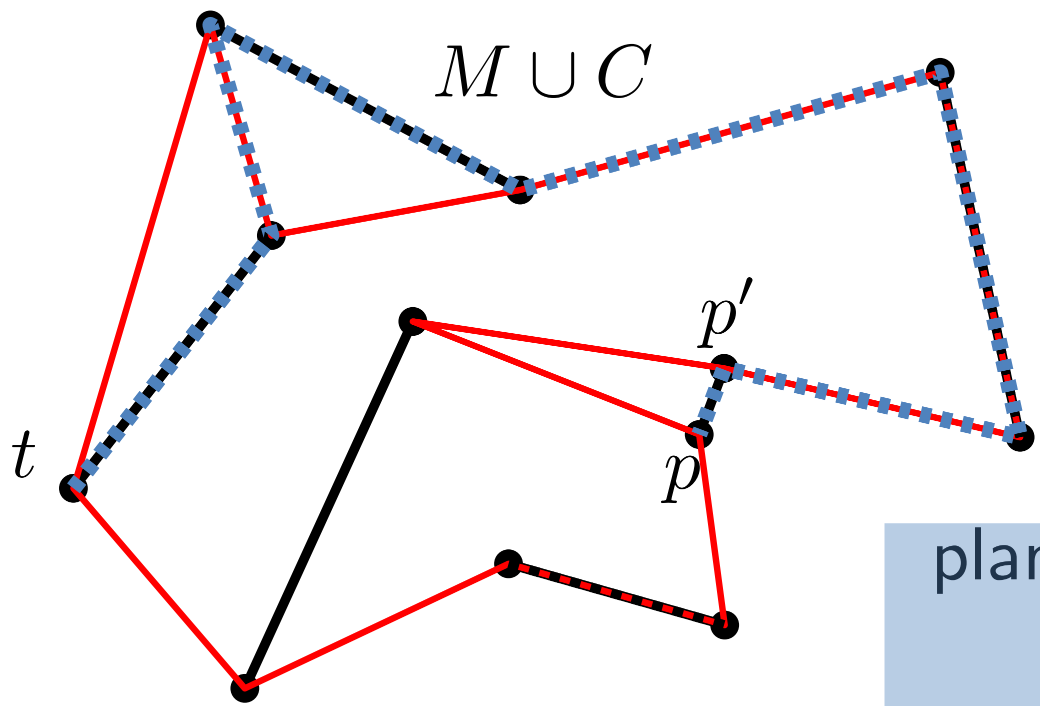

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plane alternating path
$\widehat{=}$
sequence of flips

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Thank you!

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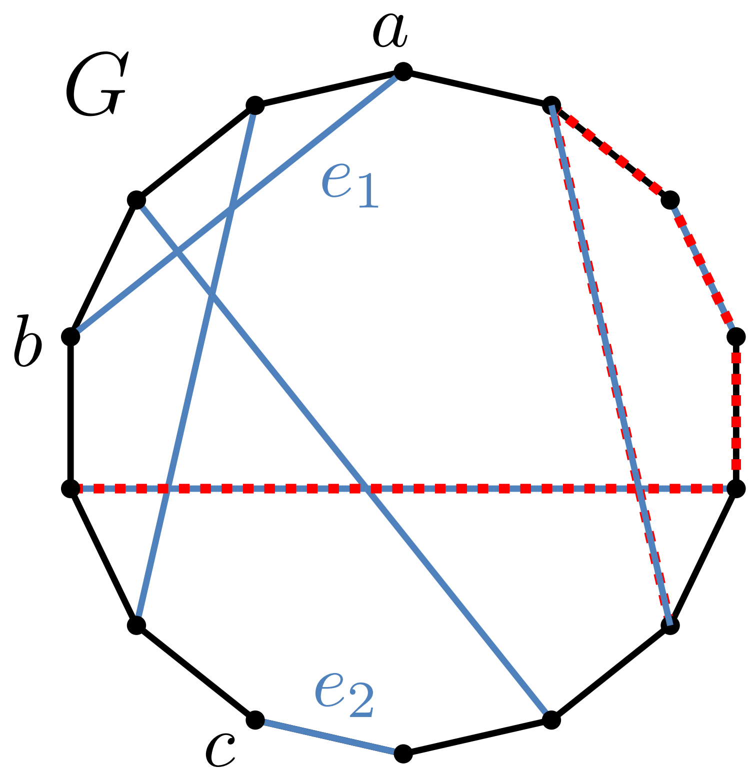

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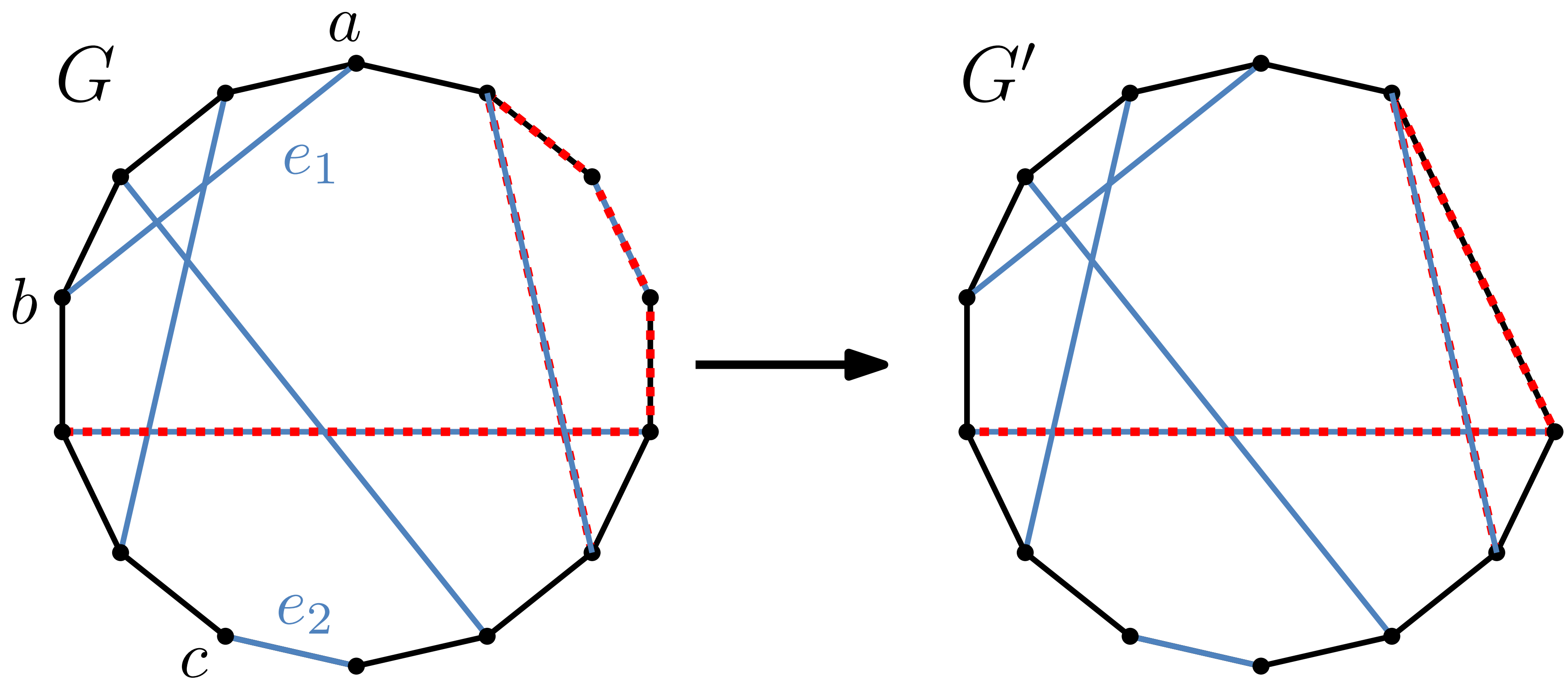

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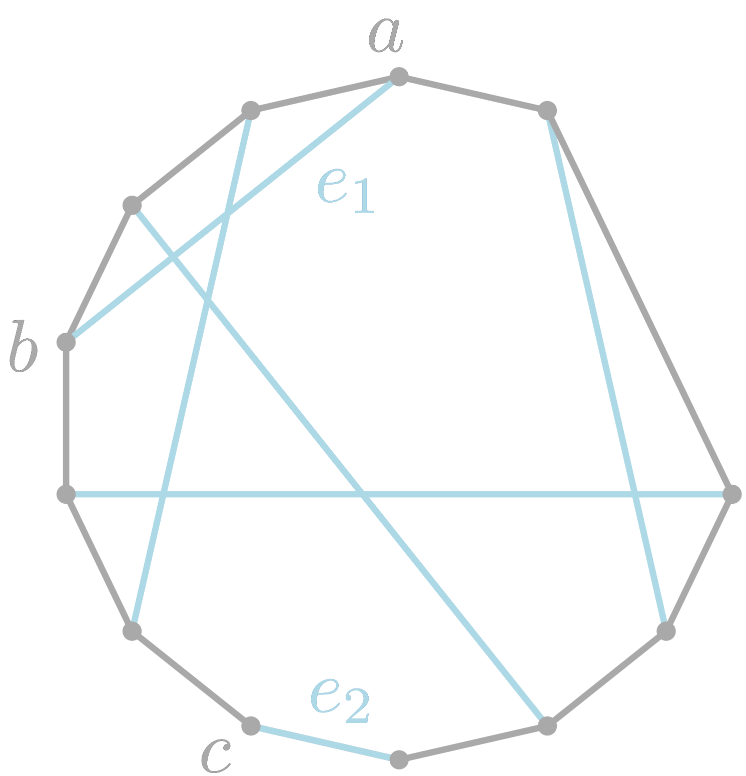

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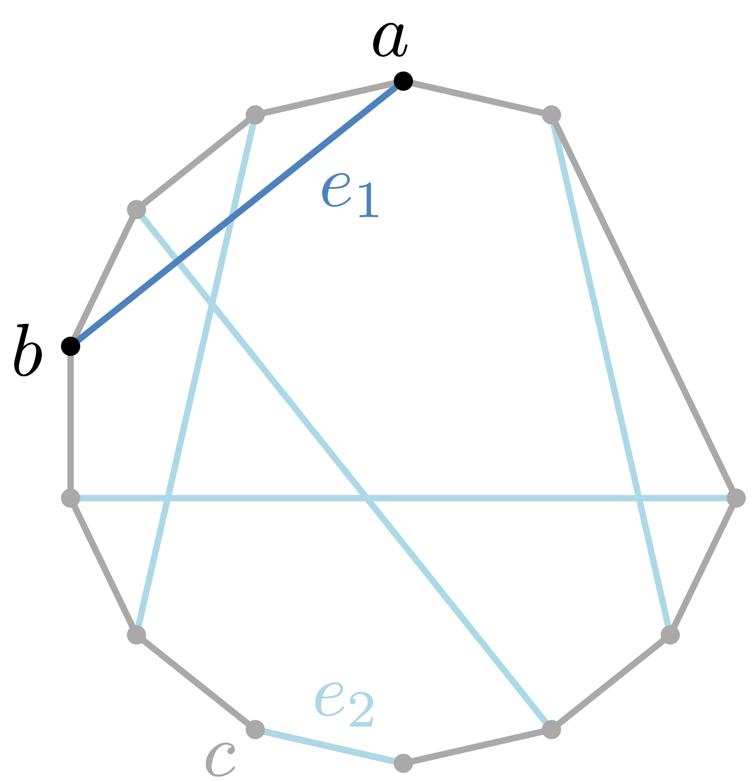

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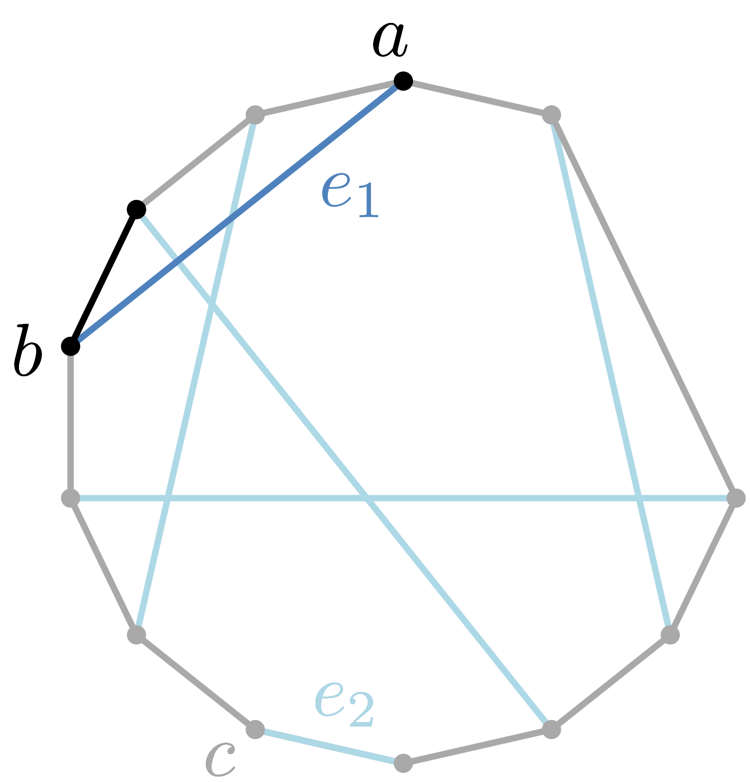

$G_2 = \{e_1\}$, G_3

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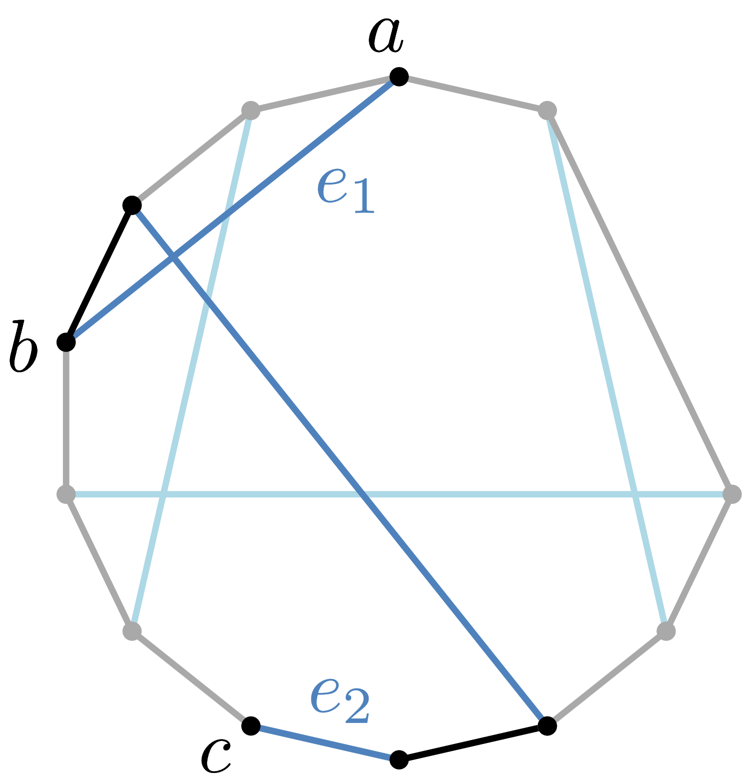

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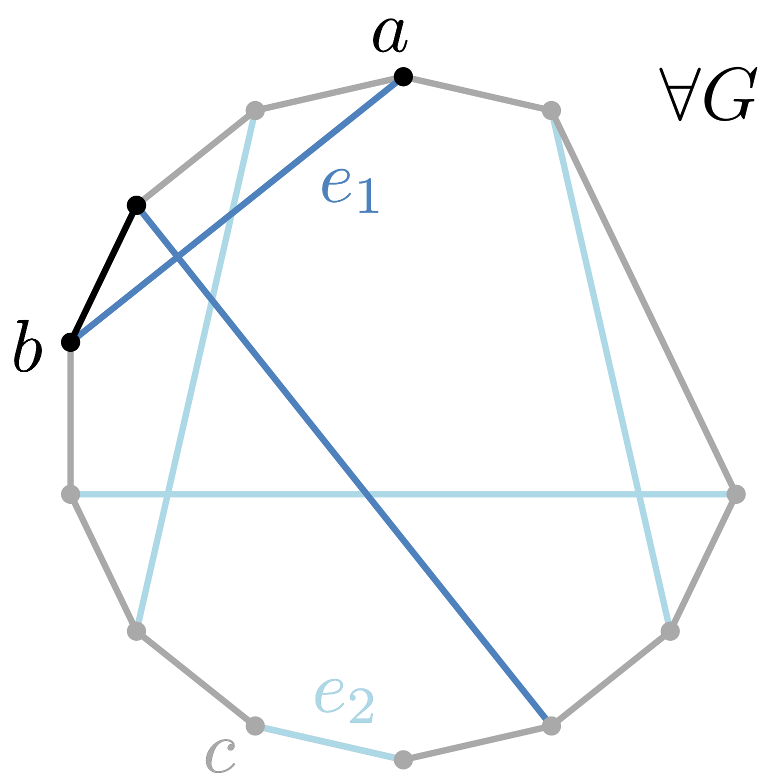

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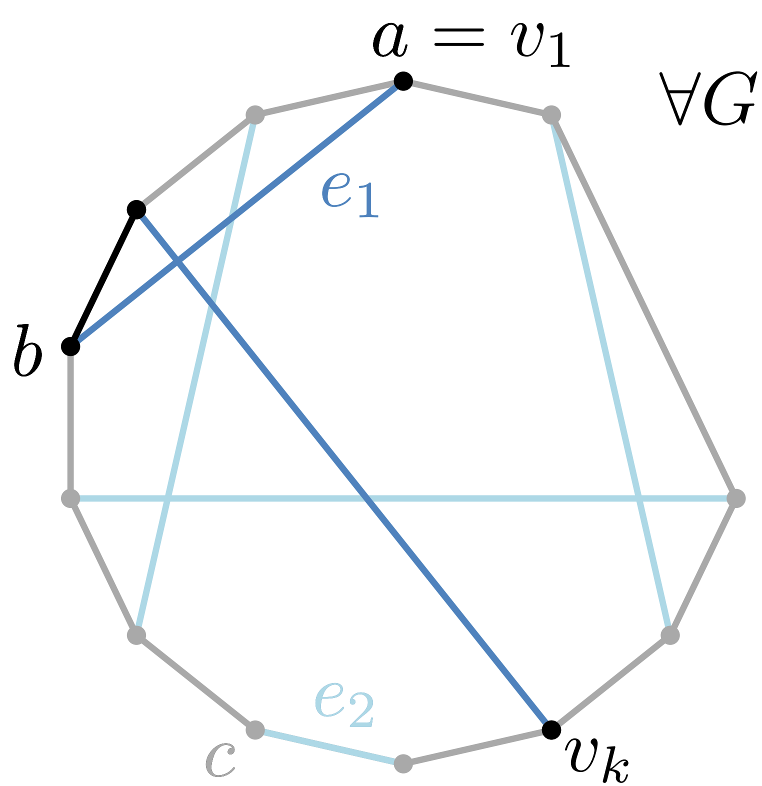

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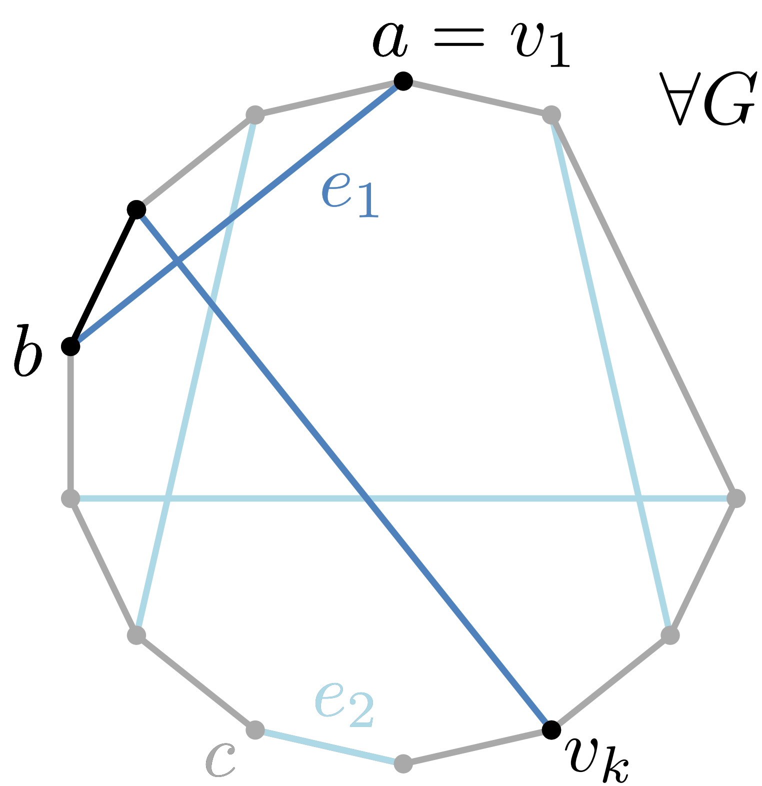

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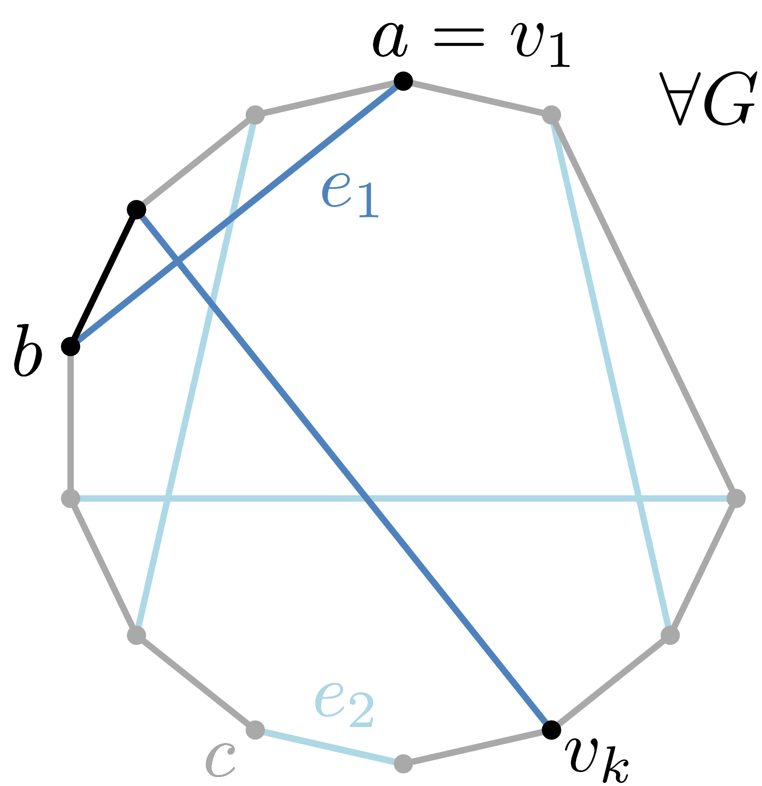

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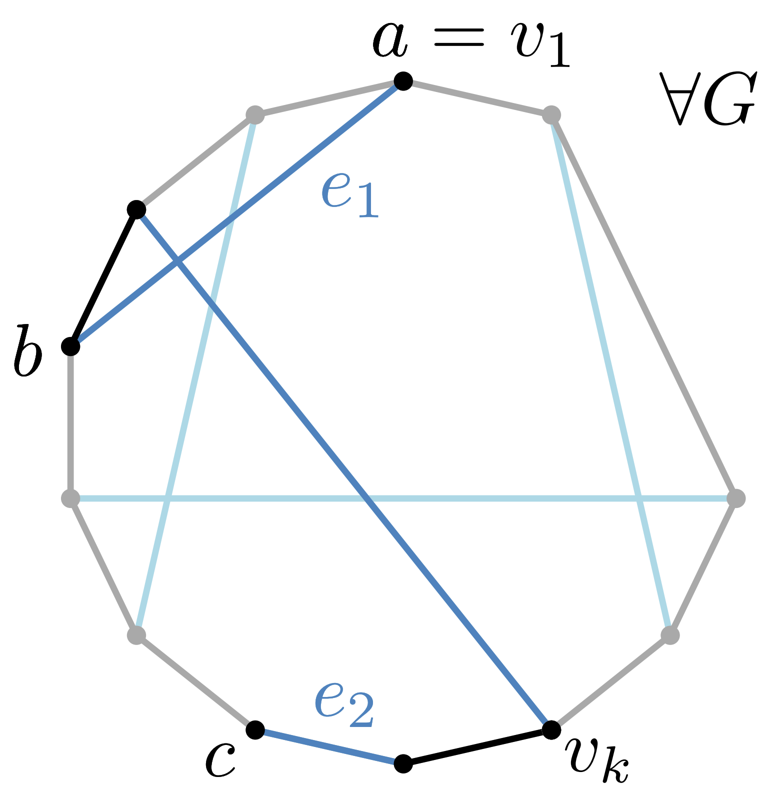

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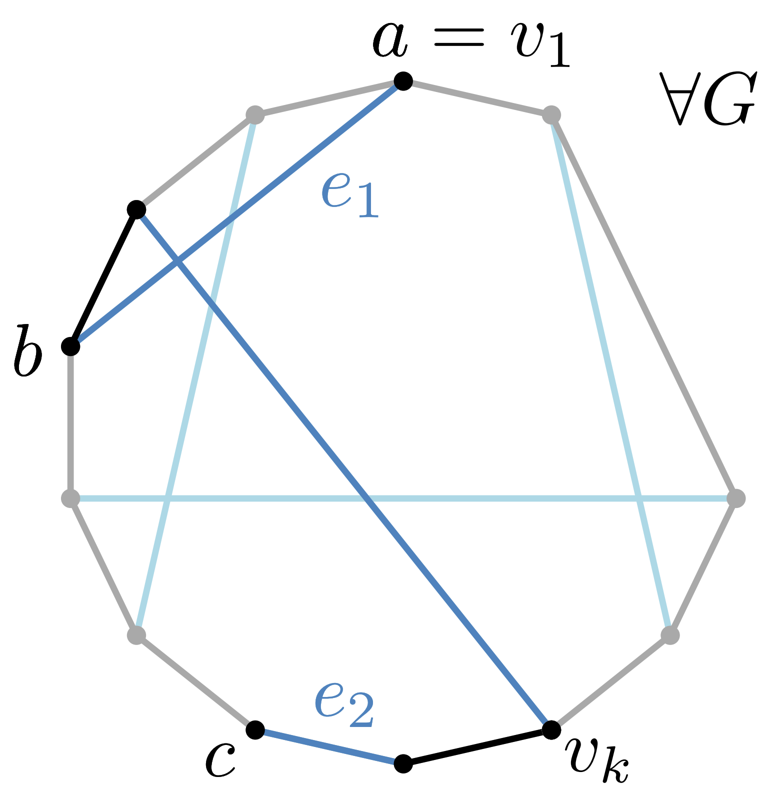

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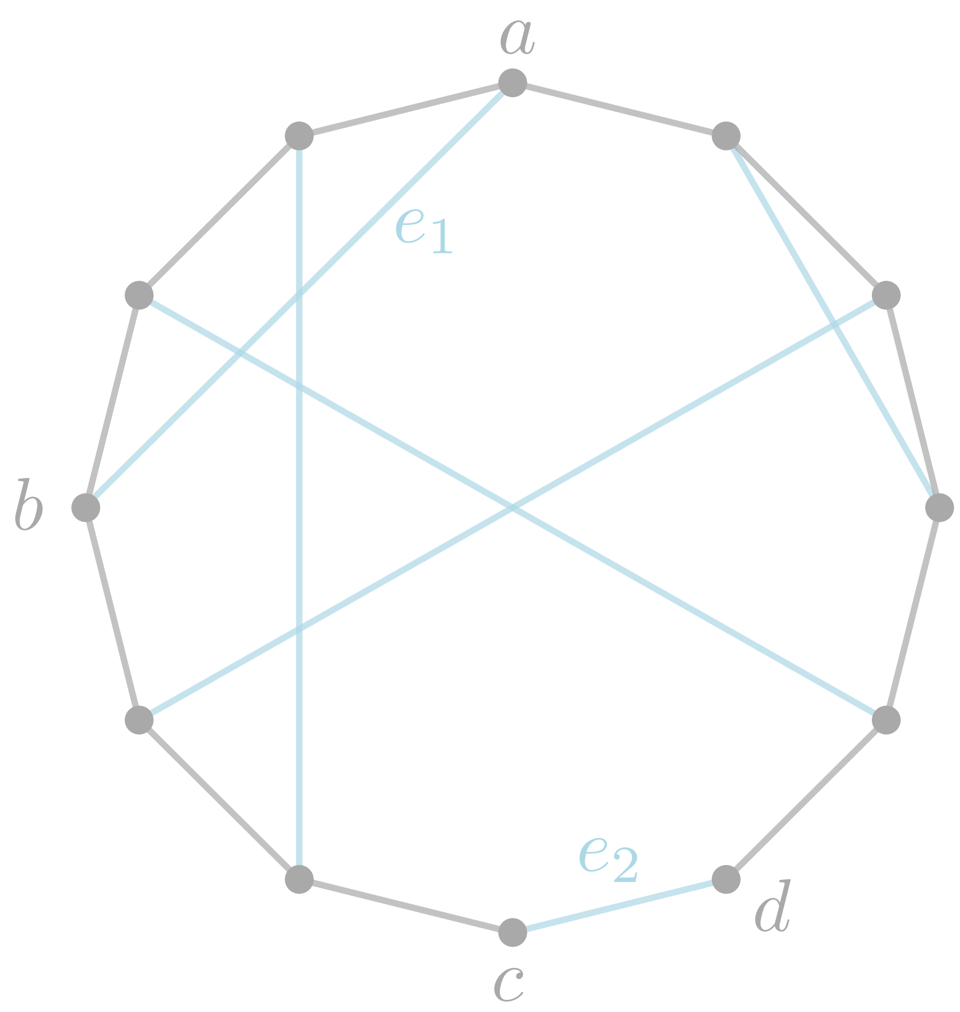
$$G_p = \dot{\bigcup} \{\text{cycles}\} \,\dot{\cup}\, P$$

Construction of G_{k+1}

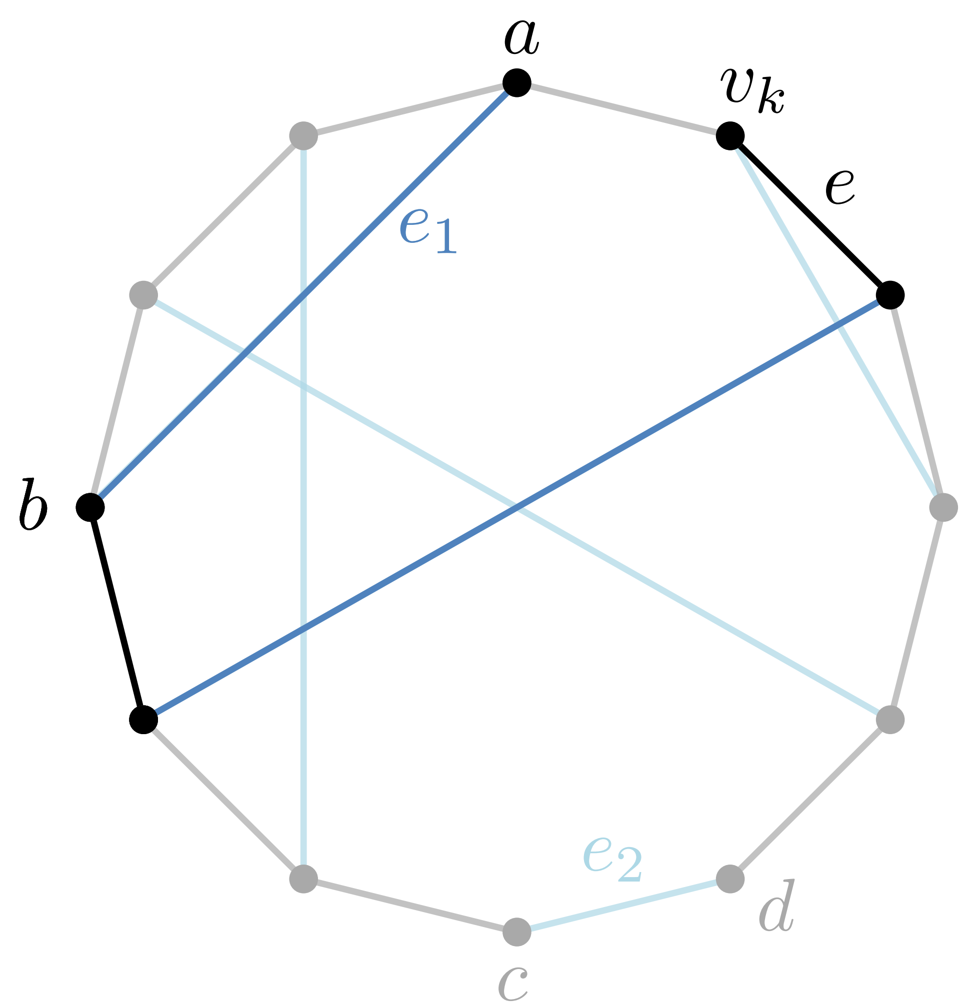
Let e be incident to v_k.



Construction of G_{k+1}

Let e be incident to v_k.

- **Case 1:** $e \in C \setminus M$



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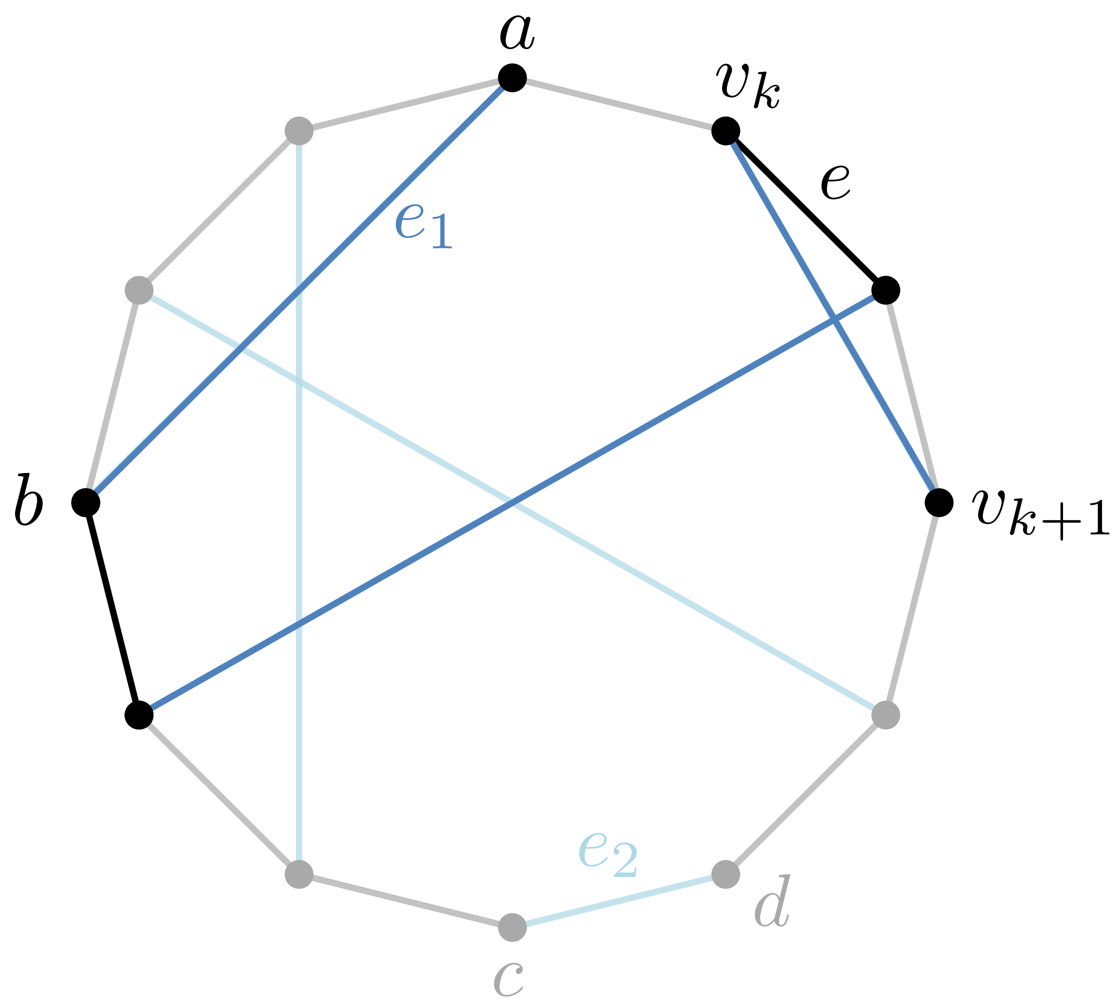

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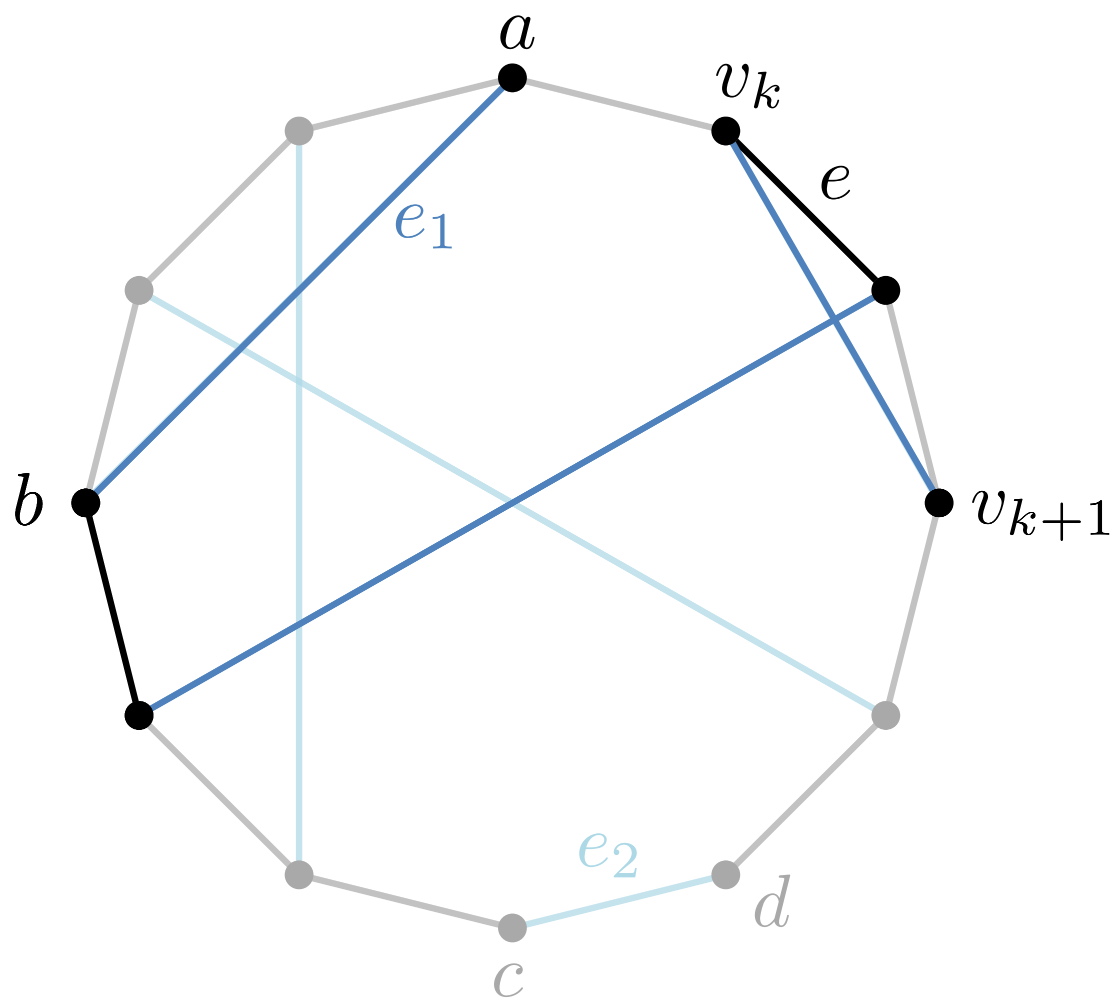

(1) $|V(G_{k+1})| = k + 1$

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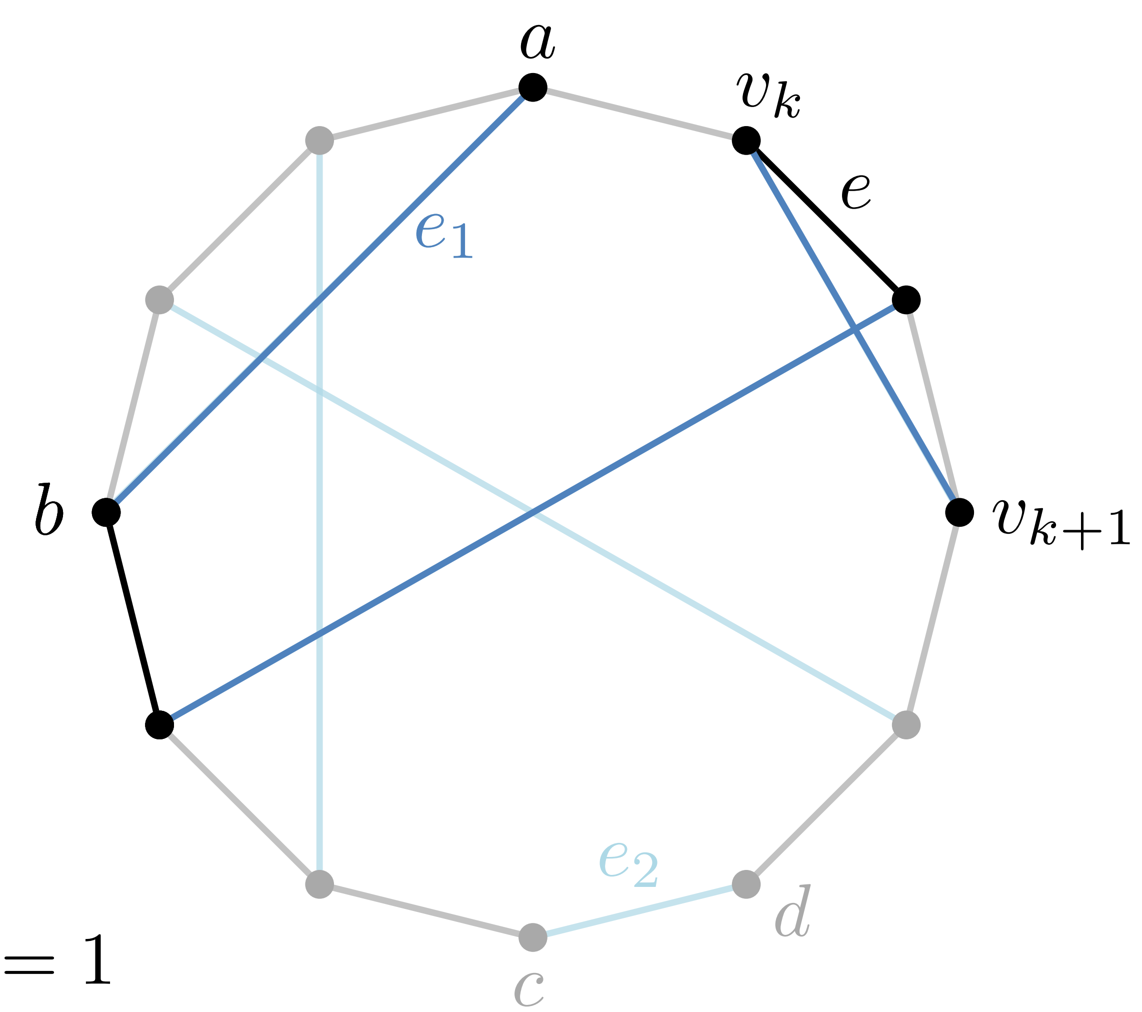

(1) $|V(G_{k+1})| = k + 1$

(2) $\deg(v_1) = \deg(v_{k+1}) = 1$

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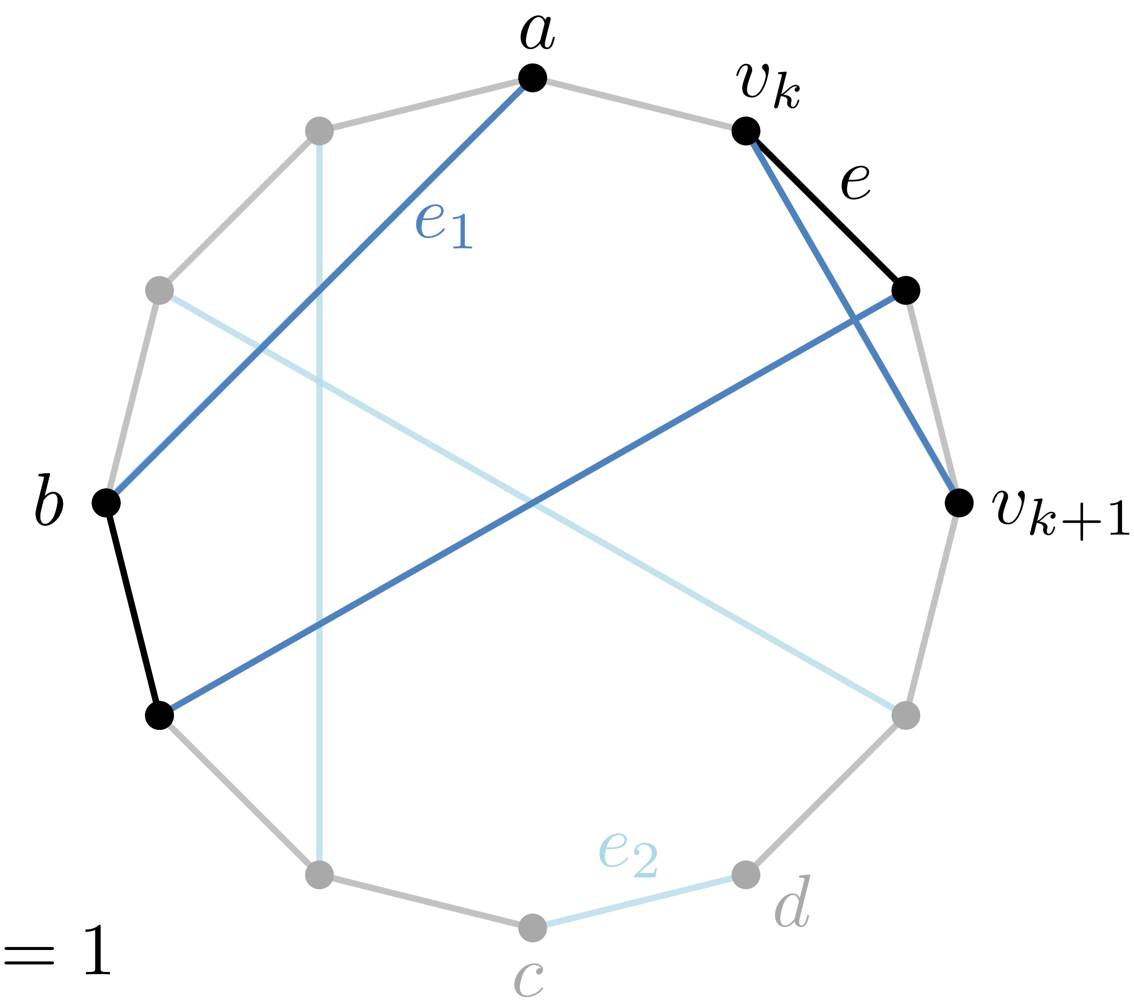


(1) $|V(G_{k+1})| = k + 1$

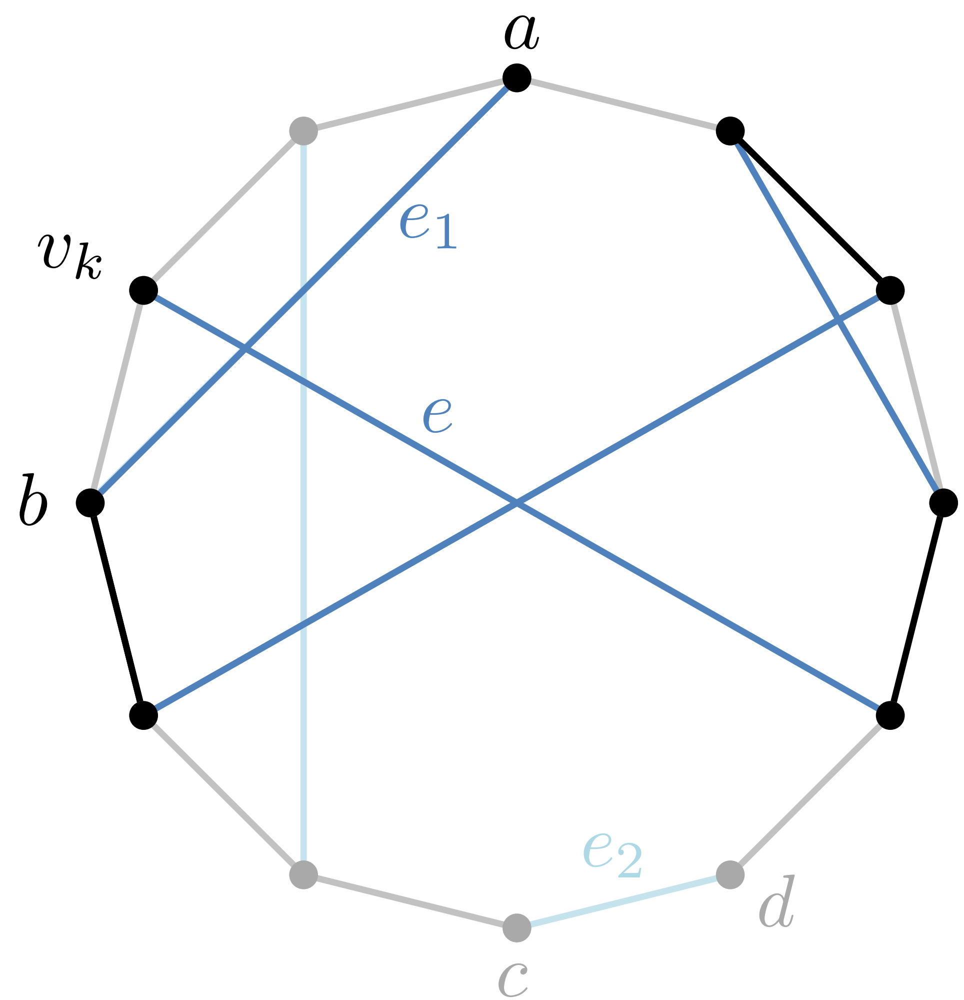
(2) deg$(v_1) =$ deg$(v_{k+1}) = 1$

(3) deg$(v_i) = 2 \quad \forall 1 < i \leq k$, edges are alternating

Construction of G_{k+1}

Let e be incident to v_k.

- **Case 1:** $e \in C \setminus M$
- **Case 2:** $e \in M$



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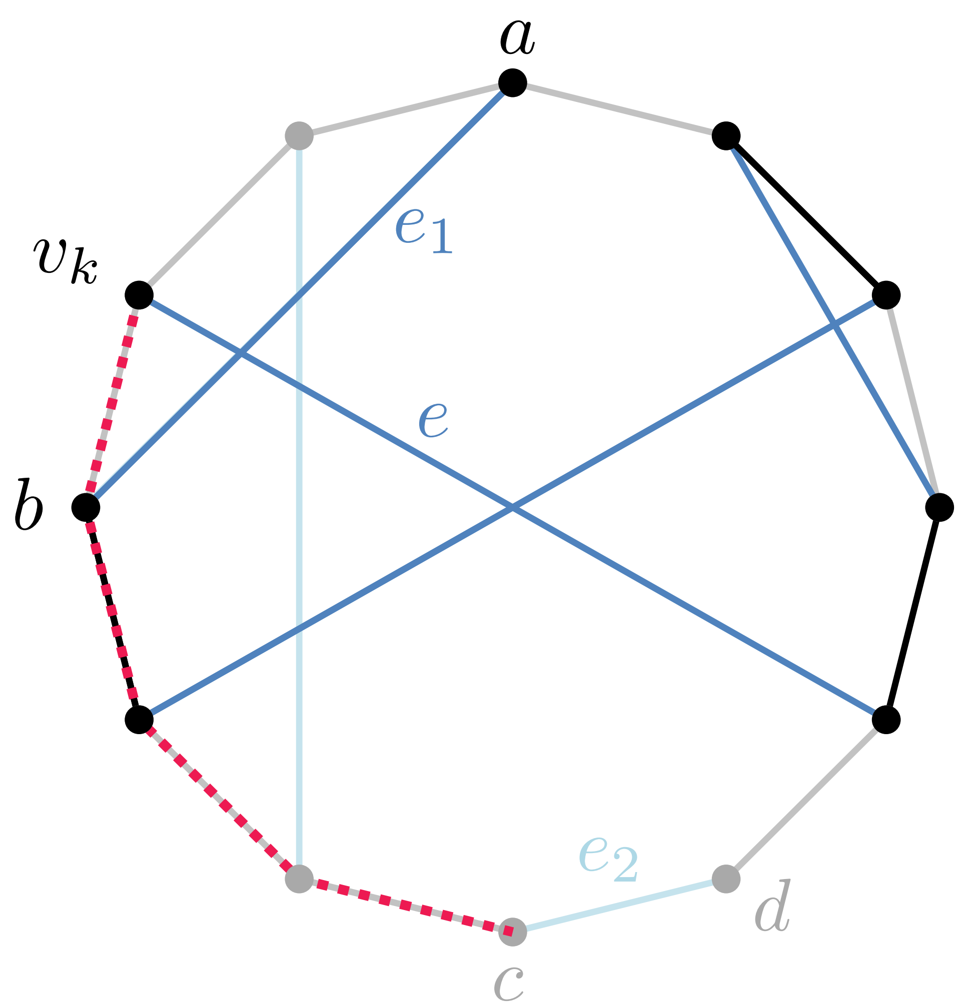

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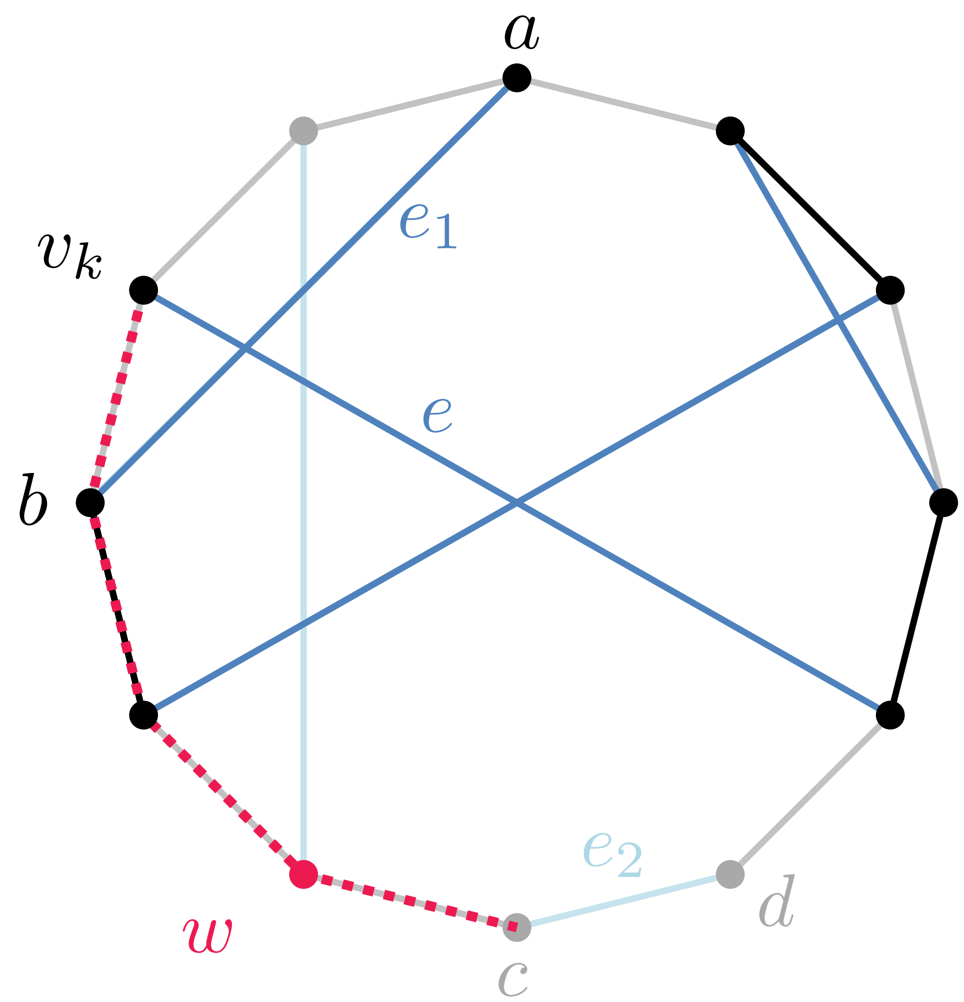

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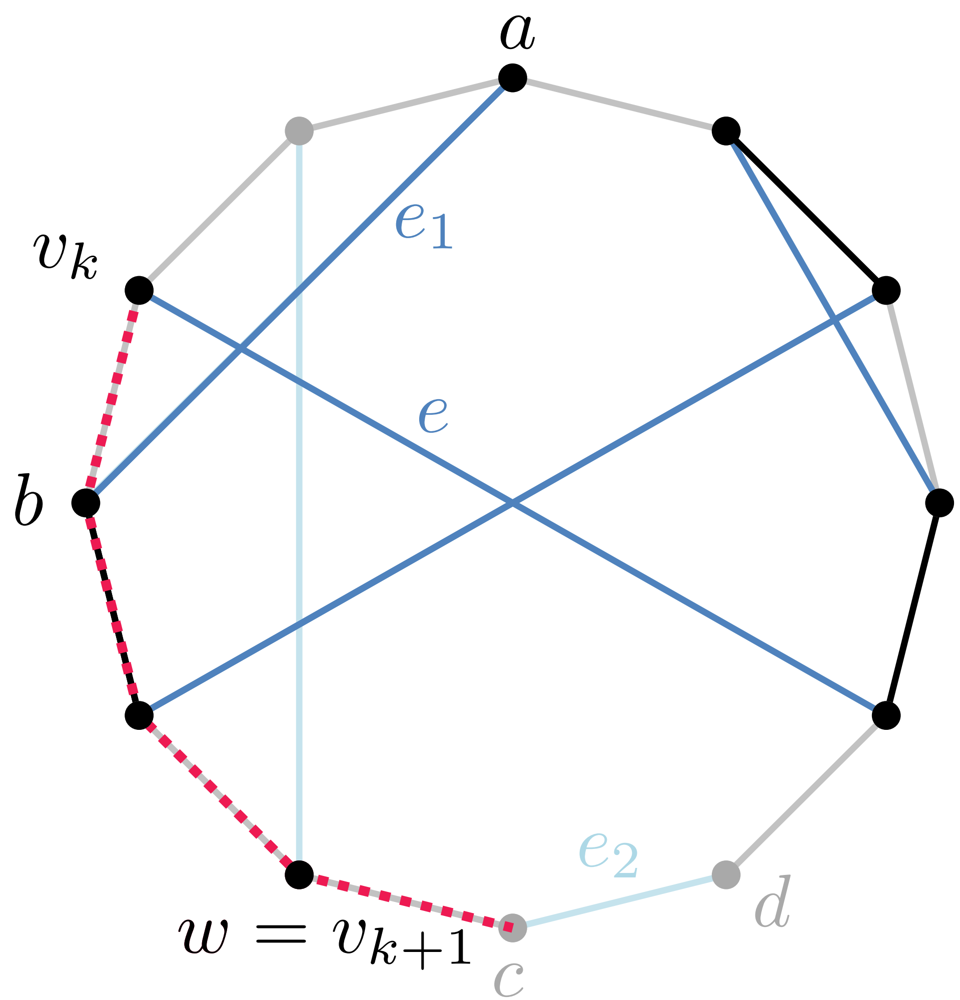
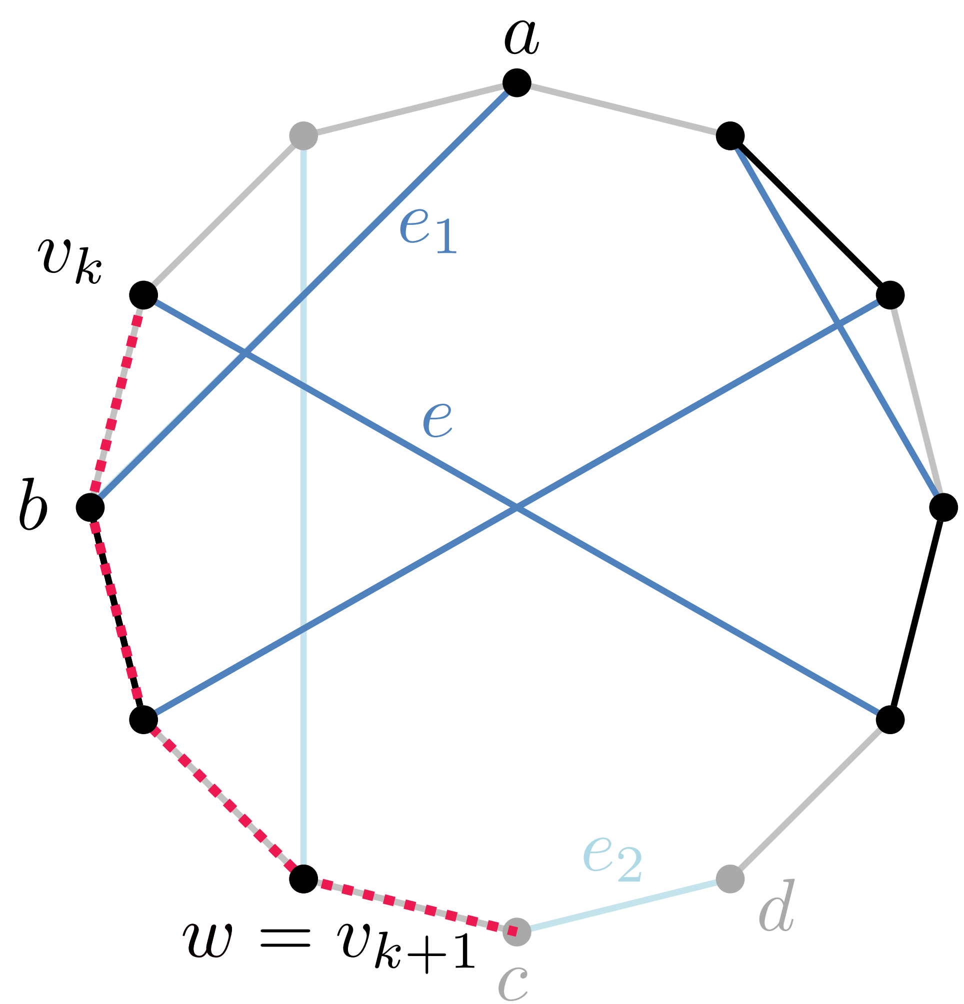


Construction of G_{k+1}

Let e be incident to v_k.

- **Case 1:** $e \in C \setminus M$
- **Case 2:** $e \in M$

alternate edges along the path



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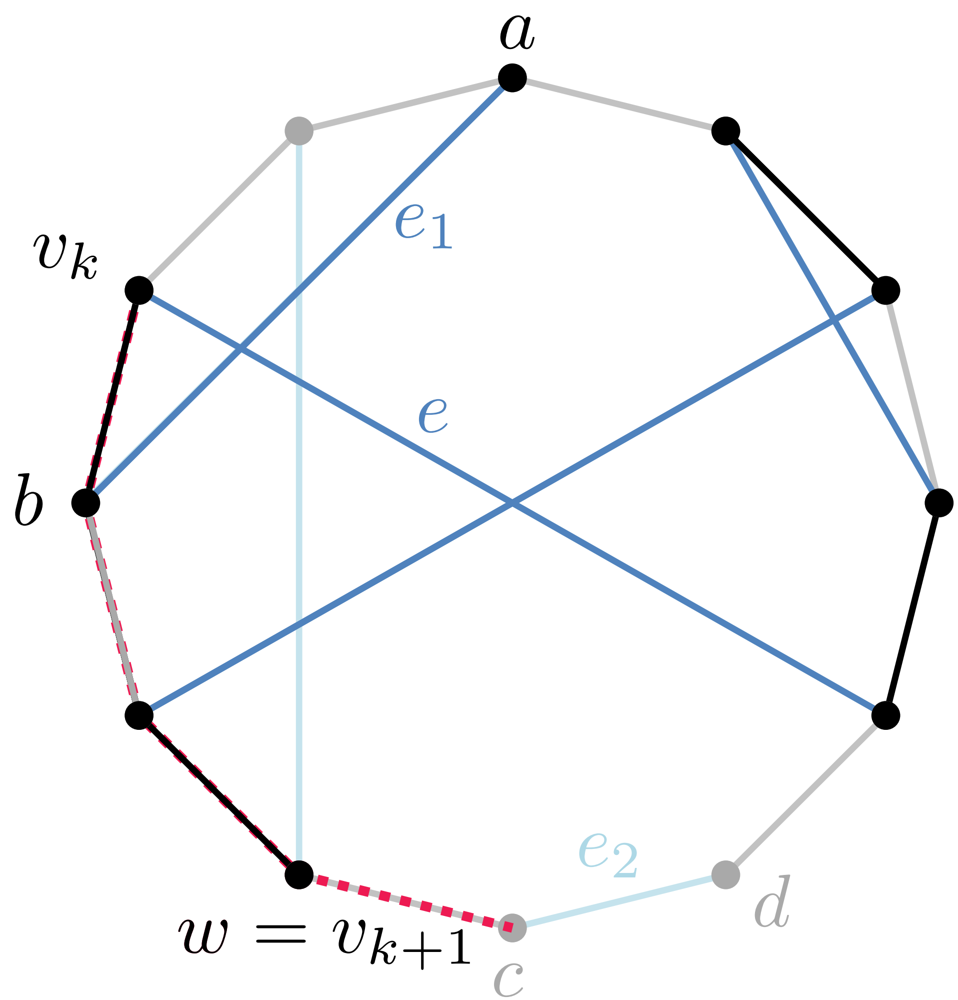

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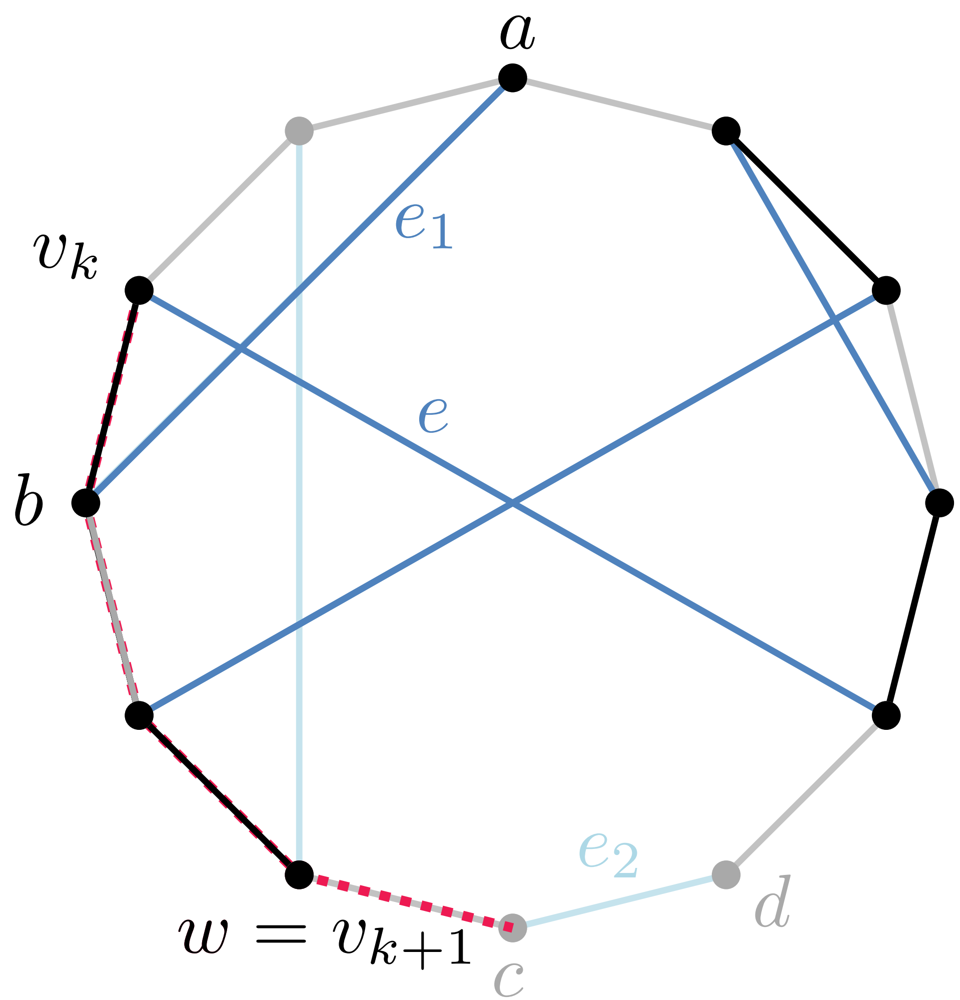

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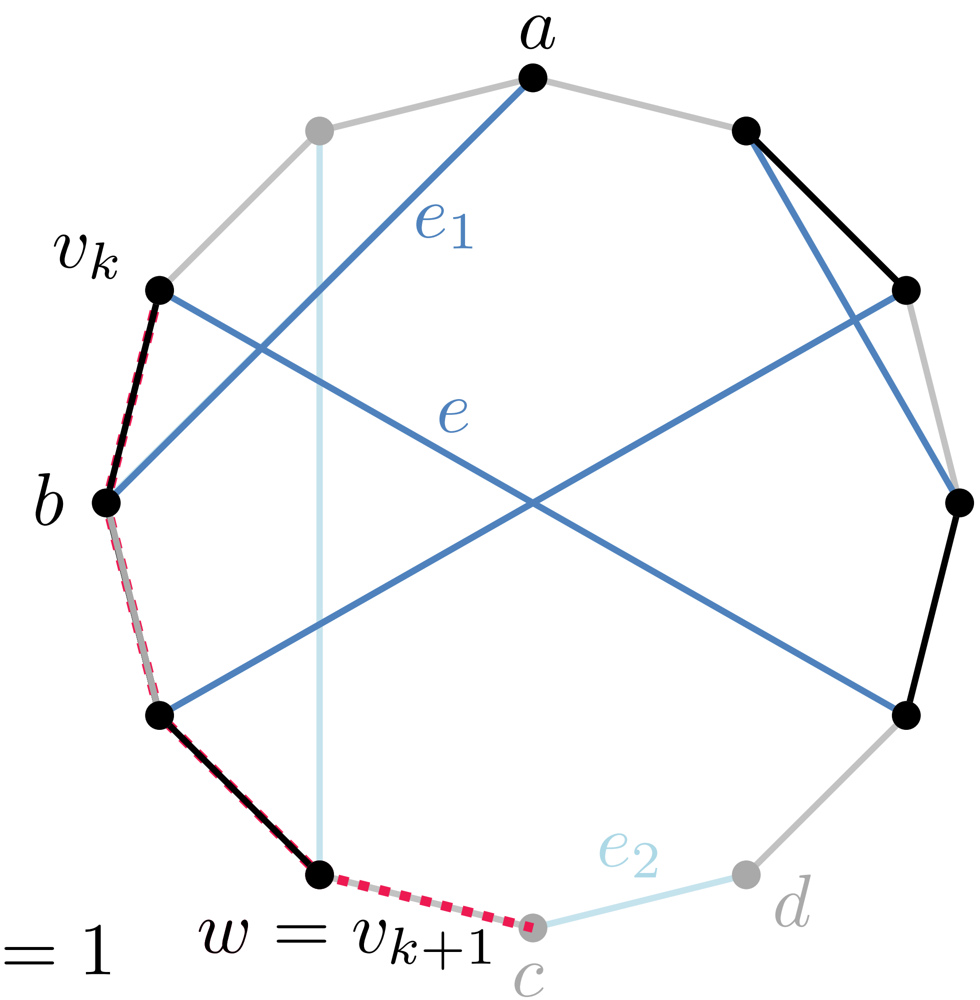

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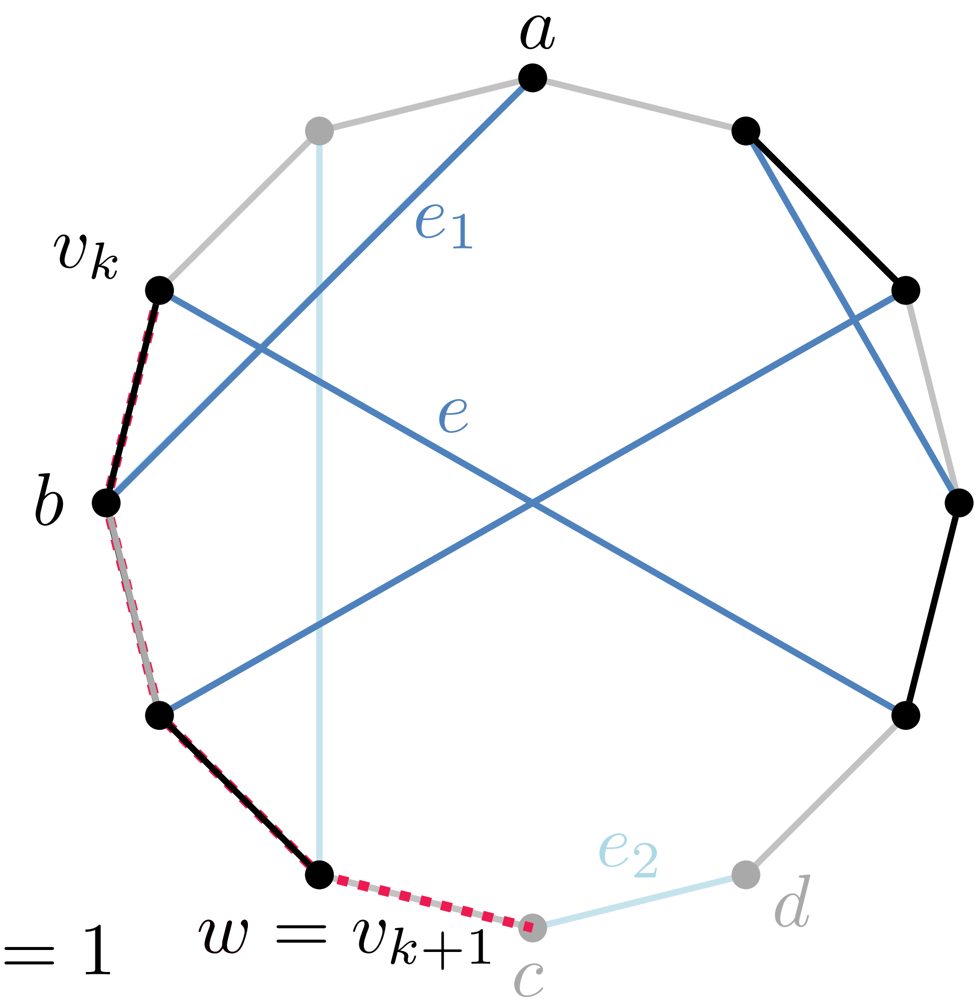

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