A variant of backwards analysis applicable to order-dependent sets

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Recap: Standard backwards analysis

Randomized incremental algorithm

- n elements
- randomization order $\sigma \in S_n$
- T_i: running time of the i-th step
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"Analyze an algorithm as if it were running backwards in time, from output to input. This is based on the observation that often the cost of the last step of an algorithm can be expressed as a function of the complexity of the final product output of the algorithm" [Seidel, 1993].

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- Many other applications [Boissonnat and Yvinec, 1998]:
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  provides a "bad example", where backwards analysis is not applicable
- Key requirement:
  algorithm's output is independent of the randomization order

Variant of backwards analysis: groups of permutations

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$$G_\sigma = \left\{ \begin{array}{l} \sigma = (\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n) = \sigma^{(n)} \\ \hline \sigma^{(1)} = (\sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n, \sigma_1) \\ \sigma^{(2)} = (\sigma_1, \sigma_3, ..., \sigma_{n-1}, \sigma_n, \sigma_2) \\ ... \\ \sigma^{(i)} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1} ..., \sigma_n, \sigma_i) \\ ... \\ \sigma^{(n-1)} = (\sigma_1, \sigma_2, ..., \sigma_{n-2}, \sigma_n, \sigma_{n-1}) \end{array} \right\}$$

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Definition

A set $R_n \subset S_n$, which satisfies $\dot{\bigcup}_{\sigma \in R_n} G_\sigma = S_n$ is called **set of representatives**.

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$$R_4 = \{(1,2,3,4),(2,1,4,3),(3,1,4,2),(3,2,4,1),(4,1,3,2),(4,2,3,1)\}$$

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Why choose this grouping?

- each element appears exactly once as last element

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- minimizes the number of inversions between the base permutation σ and the permutations within its group G_σ
  $\Rightarrow$ similarity between $O_i(\sigma)$ and $O_i(\sigma^{(j)})$ for all $j \leqslant i$

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Example: Triangulation algorithm [Seidel, 1993]

Input : Set of n points $P = \{p_1, p_2, ..., p_n\}$ within simplex $\Delta \subset \mathbb{R}^d$
Output : Triangulation T of P and Δ

```
T ← Δ;
for i = 1 → n do
    Remove simplex τ, which contains p_i, from T;
    Partition τ into d + 1 simplices, each having p_i as corner;
    Add the new d + 1 simplices to T;
return T;
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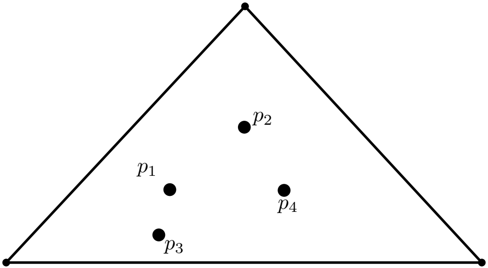

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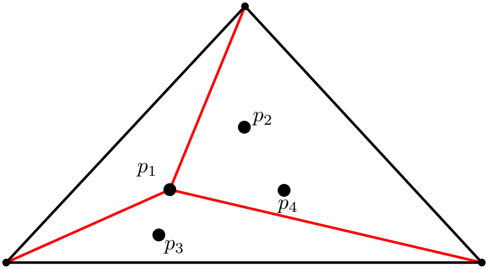

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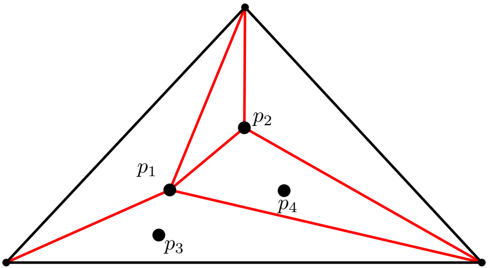

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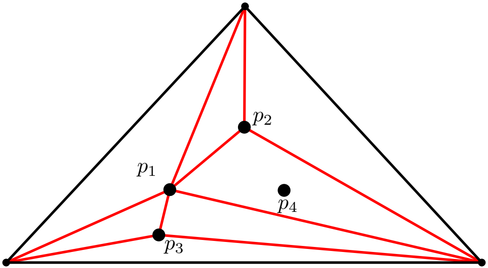

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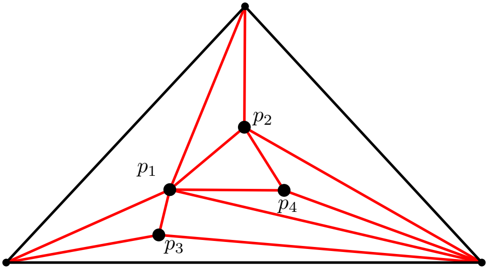

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For any fixed dimension, the triangulation algorithm has $O(n \log n)$ *expected time complexity.*

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1. for each triangle we know the points of P, which are contained in it
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$\Rightarrow$ Count how many points fall into the deleted triangle in expectation!

Connection: shifted permutations ↔ group representative

For $0 \leqslant j \leqslant i \leqslant n$, permutation $\sigma \in S_n$ denote by

T_σ^i	partial triangulation after processing i many points specified by σ	
${}^jA_\sigma^i$	union of triangles of T_σ^i, which are incident to point $p_{\sigma(j)}$	
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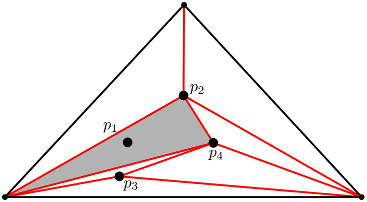
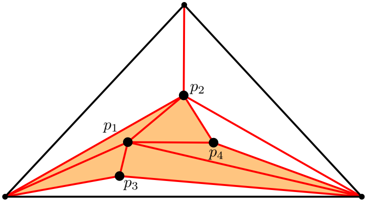
Lemma

For all $1 \leqslant j \leqslant i \leqslant n$ and any permutation $\sigma \in S_n$ it holds: $\Delta_{\sigma(j)}^{i-1} \subset {}^jA_\sigma^i$.

For insertion order $\sigma = (2, 1, 4, 3)$:



$T_{\sigma^{(2)}}^3$ with highlighted $\Delta_{\sigma^{(2)}}^3$



T_σ^4 with highlighted ${}^2A_\sigma^4$

Corollary

In the i-th insertion step, each of the $n - i$ remaining points has to be rebucketed for at most 3 permutations within a group G_σ for any $\sigma \in S_i$.

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Triangulation expected time complexity:

$$E(T(n)) = \sum_{i=1}^{n} \frac{\sum_{\sigma \in R_i} T_i(G_\sigma)}{i!} = \sum_{i=1}^{n} O\left(\frac{\sum_{\sigma \in R_i} 3(n-i)}{i!}\right)$$

$$= O\left(\sum_{i=1}^{n} \frac{n}{i}\right) = O(n \log n)$$

Another example: Voronoi-like diagrams

Voronoi-like diagrams [Junginger and Papadopoulou, 2023]

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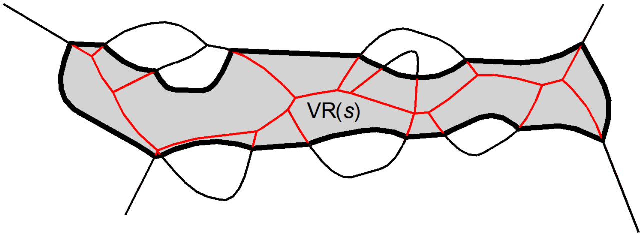

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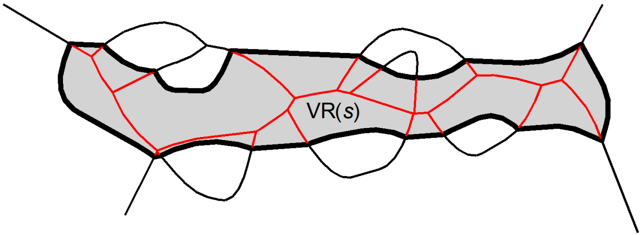

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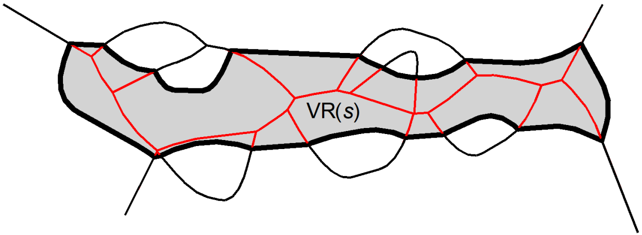
- intermediate structure in the incremental construction of Voronoi diagrams
- perform site-deletion in an abstract Voronoi diagram in linear time
- intermediate Voronoi-like diagrams are order dependent
- final Voronoi-like diagram is oder independent!
  (it corresponds to a real Voronoi diagram)



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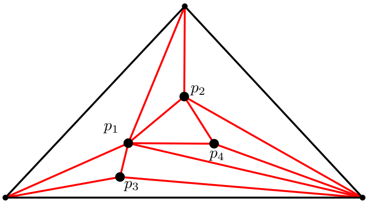

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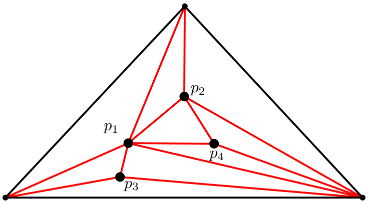

Looking for other applications!

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Literature —— Thank you for your attention!

J. Boissonnat and M. Yvinec
Algorithmic Geometry.
Cambridge University Press, New York, NY, USA, 1998.

L. Chew
Building Voronoi diagrams for convex polygons in linear expected time.
Tech. Report, Dartmouth College, Hanover, 1985.

K. Junginger and E. Papadopoulou
Deletion in abstract Voronoi diagrams in expected linear time and related problems.
Discrete & Computational Geometry, 2023.

V. Levenshtein
On perfect codes in deletion and insertion metric.
Discrete Mathematics and Applications, 1992.

R. Seidel
Backwards analysis of randomized geometric algorithms.
New trends in discrete and computational geometry, 1993.