Reconfiguration of plane trees in convex geometric graphs

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March 13, 2024

Two **geometric objects** A and B.

A B

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Can we transform A into B?

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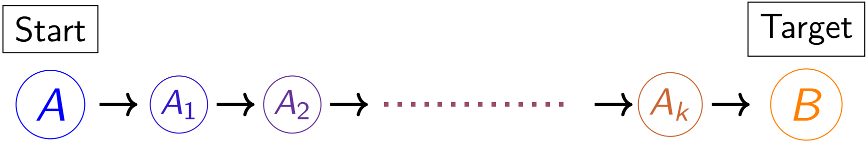
Can we transform A into B via a sequence of **flips** (keeping the same kind of geomertic objects all along) ?



Reconfiguration / Flip graphs

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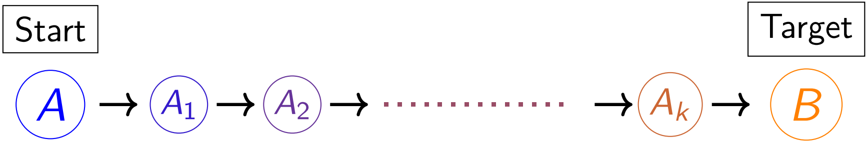

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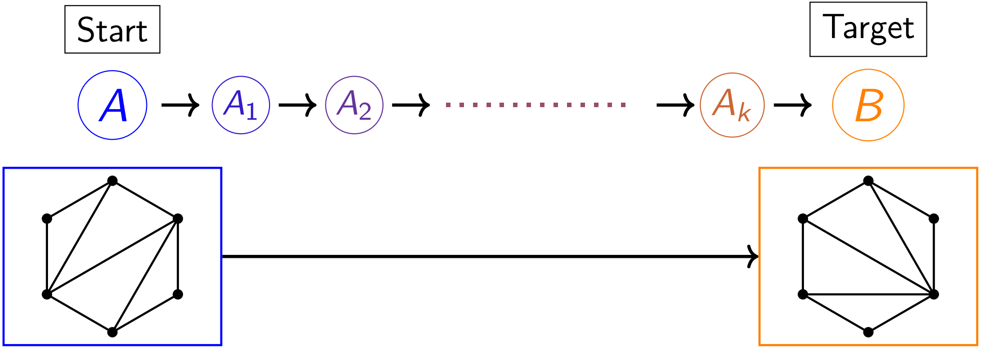

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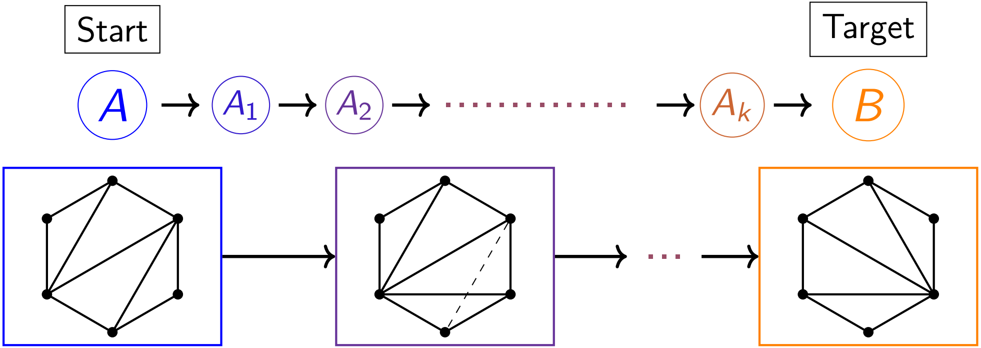
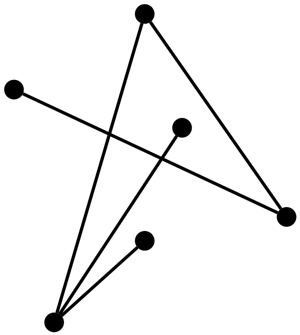


Spanning trees

- **Geo. objects:** Two spanning trees on a set of n points



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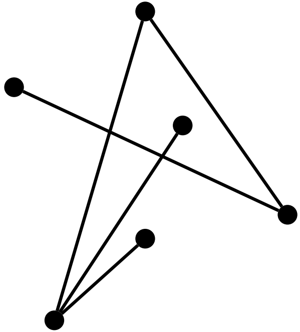

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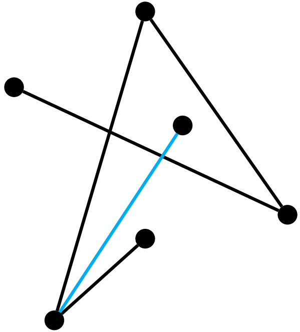

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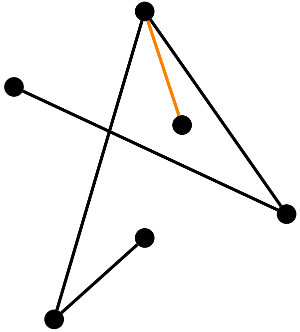

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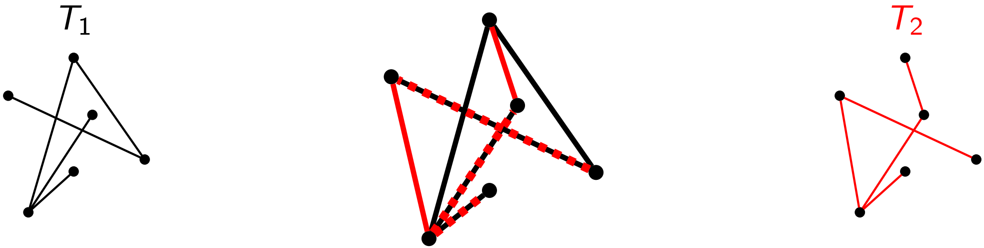

bicolored $= T_1 \cup T_2$, black $= T_1 \setminus T_2$, red $= T_2 \setminus T_1$

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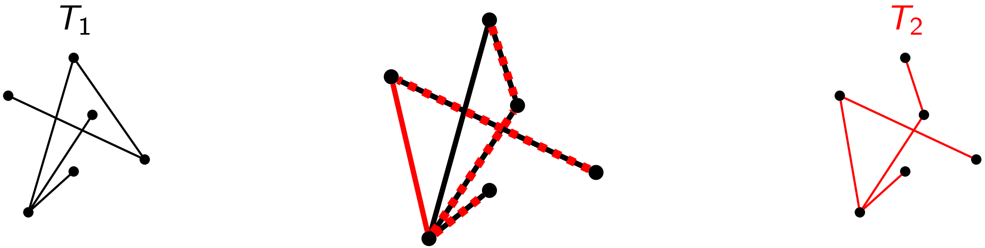

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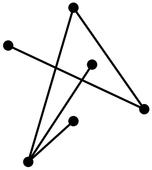
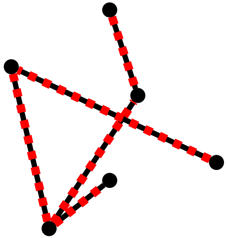
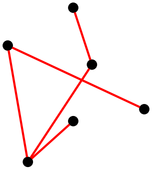
T_1





T_2



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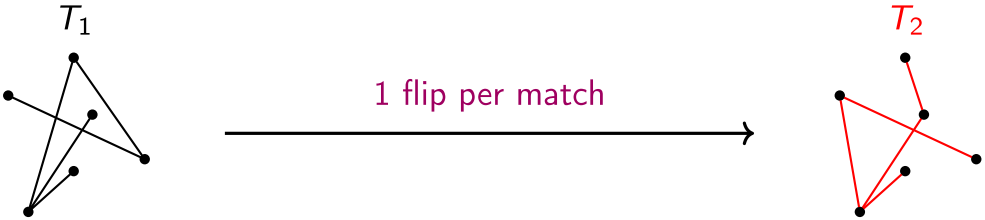

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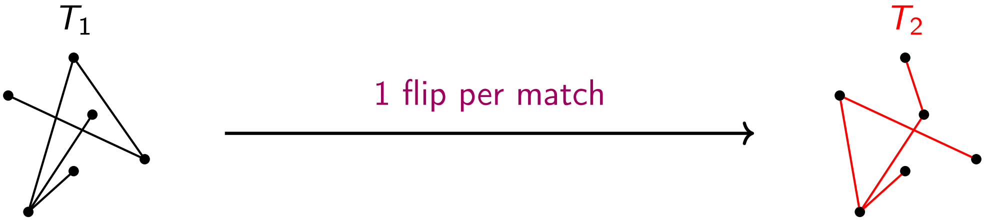
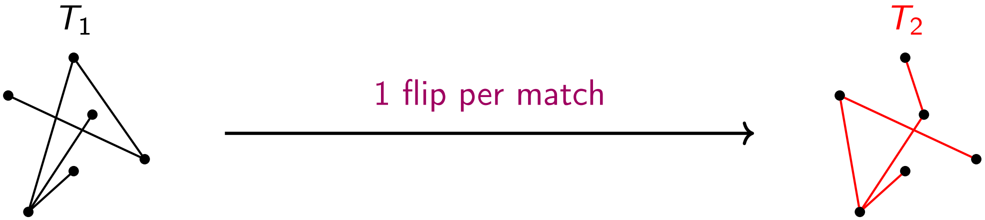


Theorem (folklore)

A minimal transformation from a spanning tree T_1 to another spanning tree T_2 uses exactly $d(T_1, T_2)$ flips.

$$d(T_1, T_2) = |T_1 \setminus T_2| = |T_2 \setminus T_1|$$

Reconfiguration of spanning trees



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Reconfiguration of spanning trees

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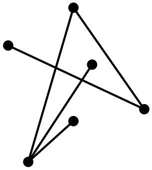
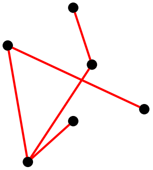


1 flip per match →

T_2



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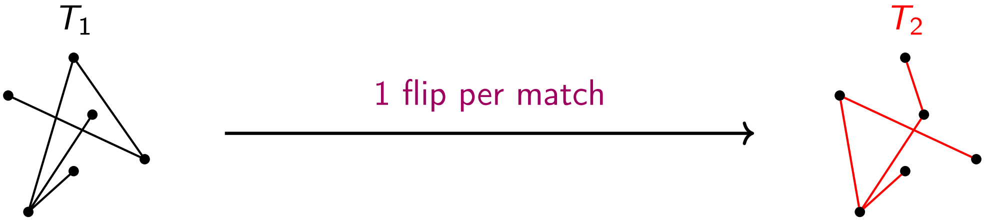
Reconfiguration of spanning trees



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Non-crossing spanning tree on a convex set

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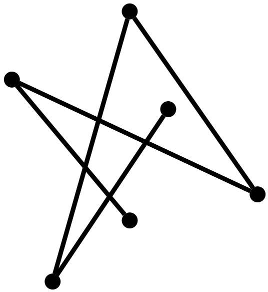

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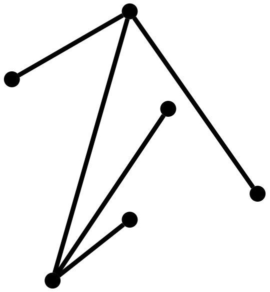
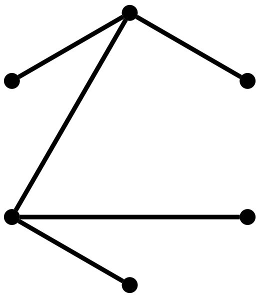


Non-crossing spanning tree on a convex set

- **Geo. objects:** Two non-crossing spanning trees on a set of n points in convex position



Non-crossing spanning tree on a convex set

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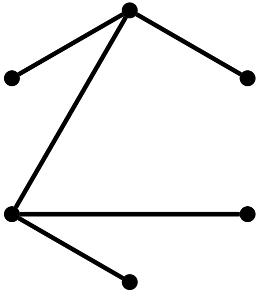

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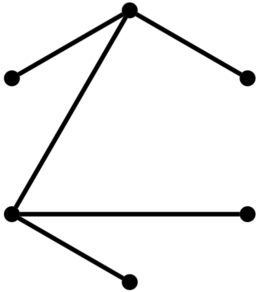
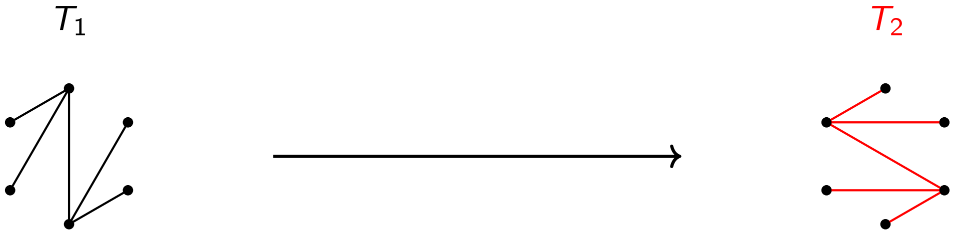


Tree = non-crossing spanning tree on a convex set.

Reconfiguration of n.-c. spanning trees on convex set



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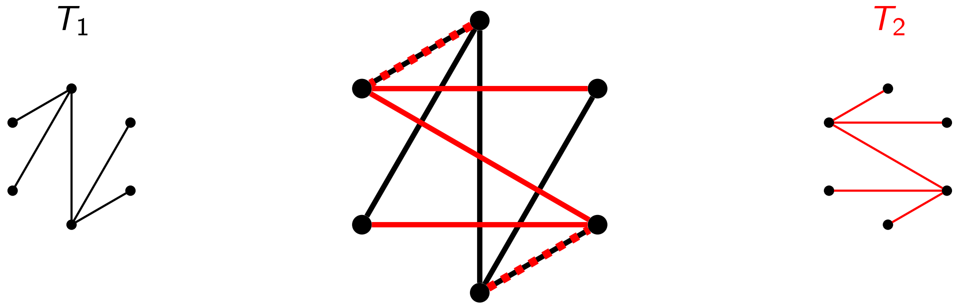
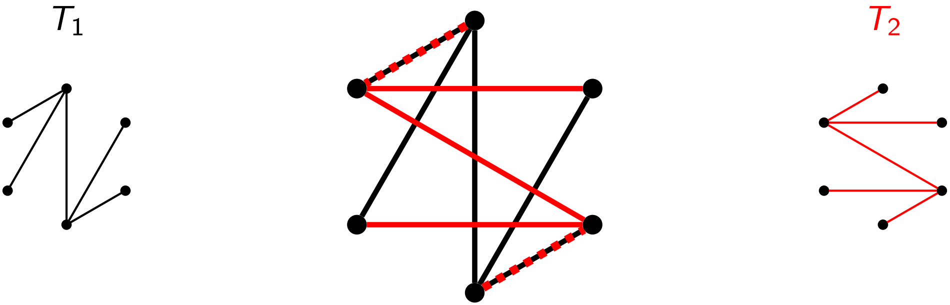


Avis and Fukuda ('96)

For every pair of non-crossing spanning trees T_1 and T_2, there exists a transformation from T_1 to T_2 using flips.



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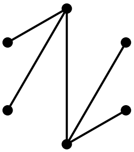
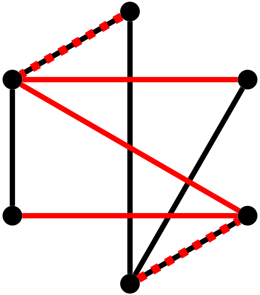
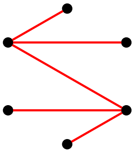




T_2



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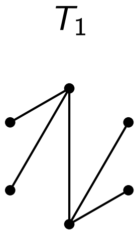

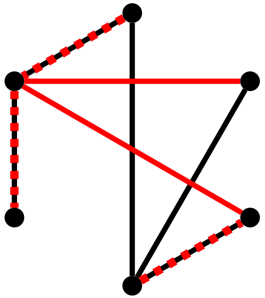

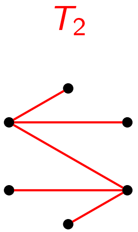

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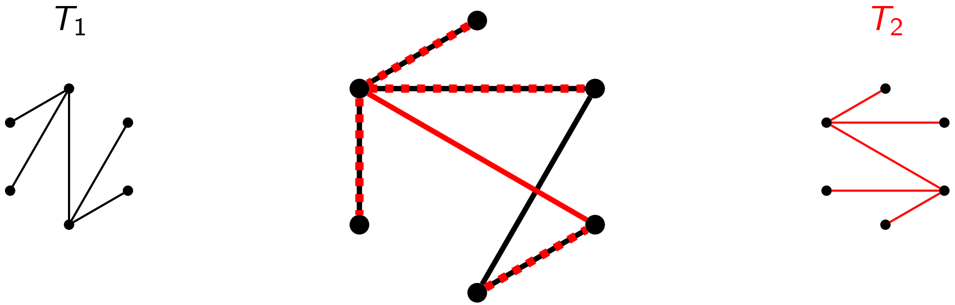

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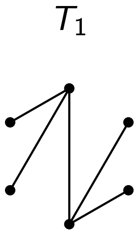

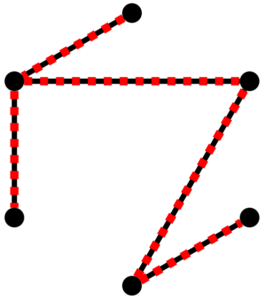

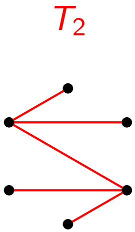

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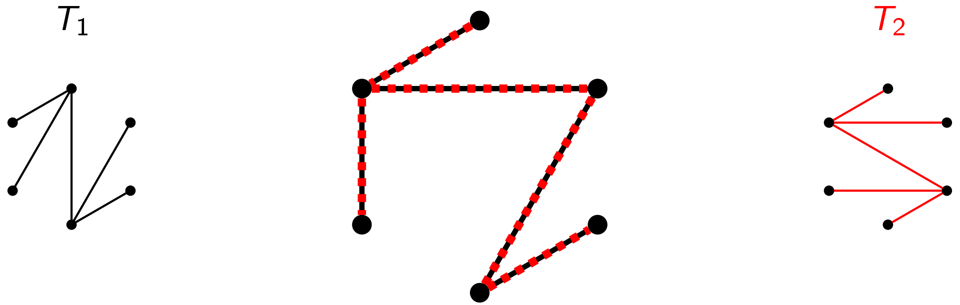
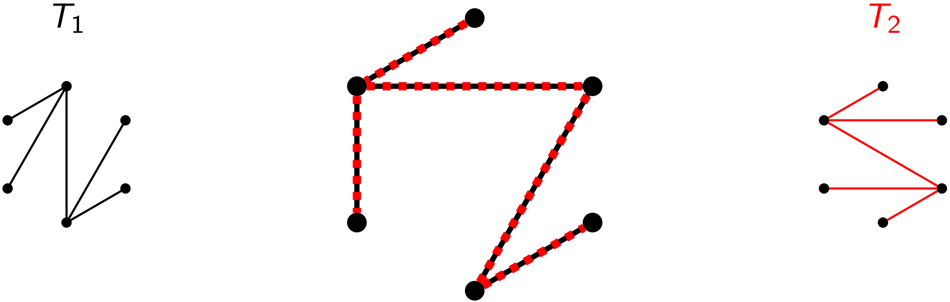


How many flips are needed in the worst case?

Avis and Fukuda ('96)

For every pair of non-crossing spanning trees T_1 and T_2, there exists a transformation from T_1 to T_2 using at most $2n - 4$ flips.



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Existing results

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  - $\frac{3}{2}n - 5$ flips (Hernando et al., 1999).

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Conjecture

For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $\frac{3}{2}n$ flips.

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  - $c \cdot d \approx 1.95d$ flips.
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  - $\frac{5}{3}d$ flips.

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For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $\frac{3}{2}n$ flips.

Result on upper bound

Theorem (Bousquet, DM, Pierron, Wesolek)

For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $c \cdot d$ flips with:

$$c = \frac{1}{12}(22 + \sqrt{2}) \approx 1.95$$

Result on upper bound

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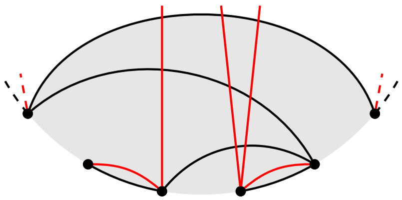
$$c = \frac{1}{12}(22 + \sqrt{2}) \approx 1.95$$

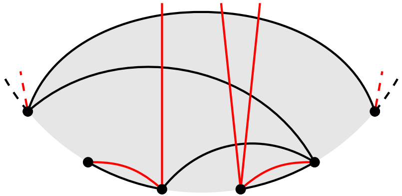
$\implies$ there is always a transformation using at most $c \cdot n$ flips.

Proof sketch



Proof sketch

Very good side

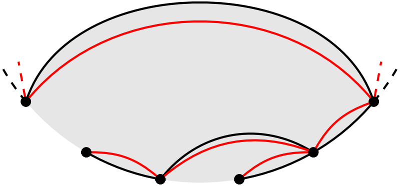


Proof sketch

Very good side

$\leq \frac{5}{3}$ flips per match

Matched side



Proof sketch

Existing side with weaker properties

Very good side

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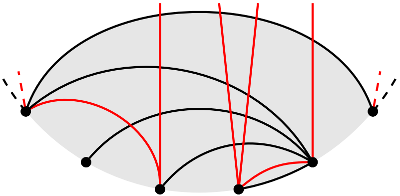
Matched side



Proof sketch

Existing side with weaker properties

≤ 2 flips per match

Very good side

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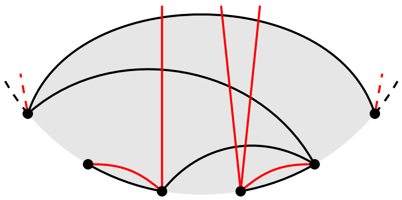
Matched side



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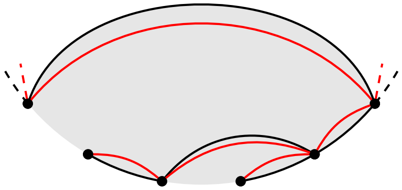
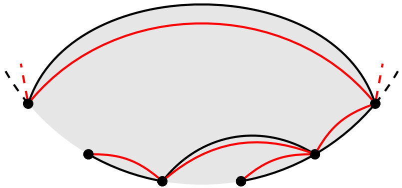


End of proof

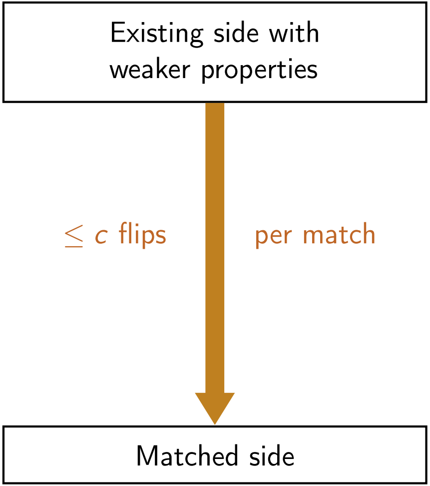
Existing side with weaker properties

$\leq c$ flips per match

Matched side



End of proof



Existing side with weaker properties
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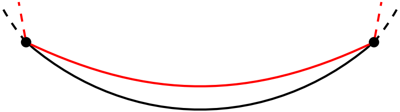
per match
Matched side



Conclusion

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number of flips $\leq c \cdot d \approx 1.95d$

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Thanks for your attention

END