EuroCG 2024
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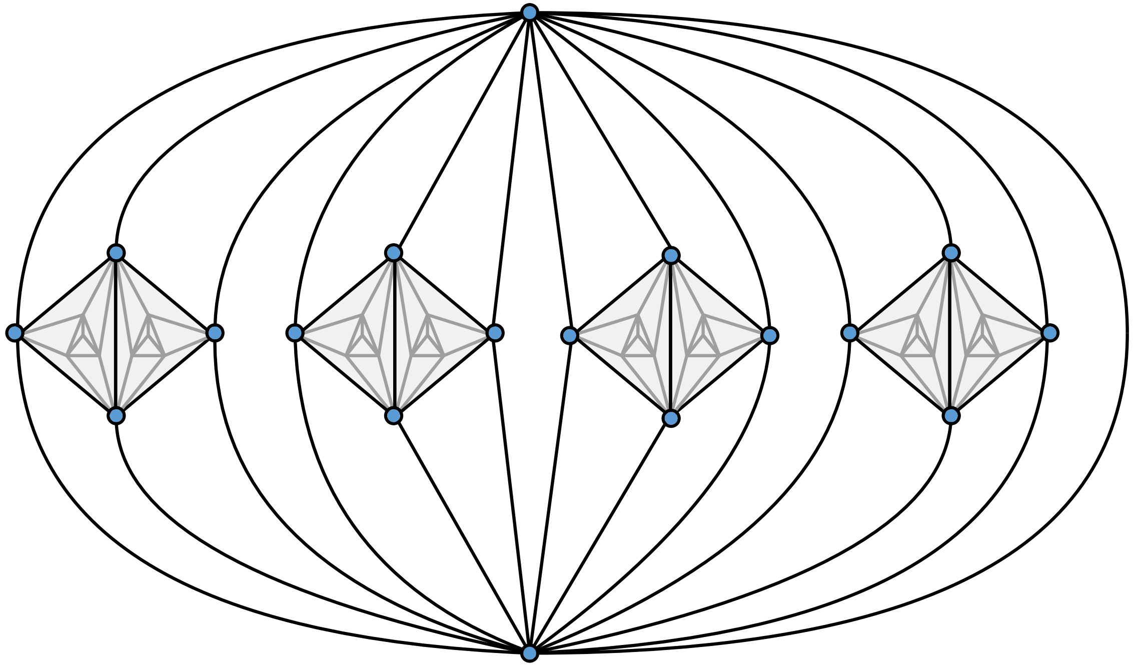
Ioannina, Greece

A Note on Mixed Linear Layouts of Planar Graphs



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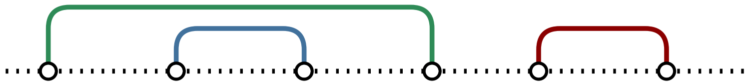
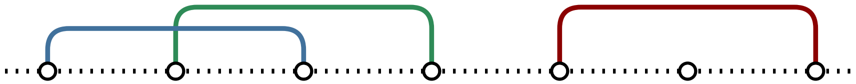
Mixed Linear Layouts

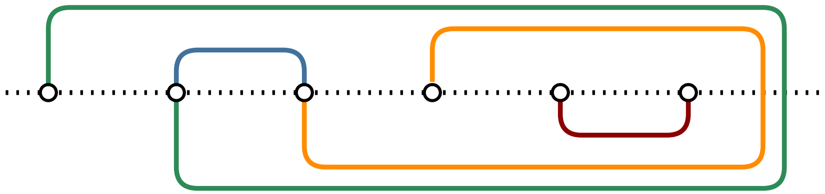
- In a mixed linear layout with k pages, the task is to:
  - arrange the vertices of the graph on a line.
  - partition the edges into k pages s.t. each page satisfies a specific property:



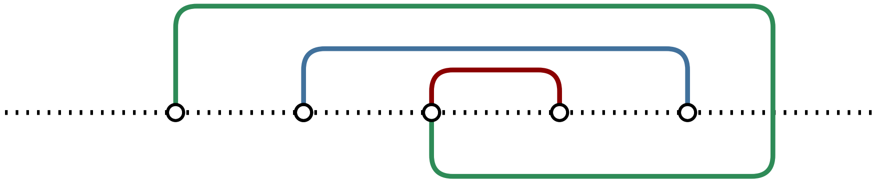
stack



queue



deque

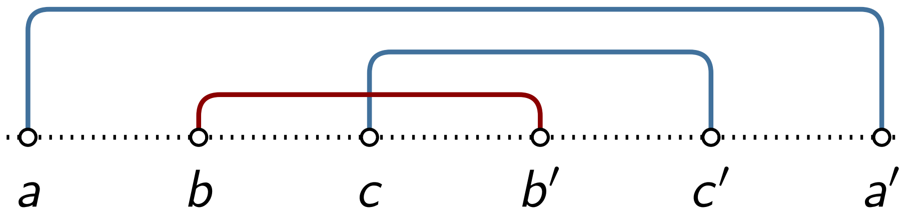


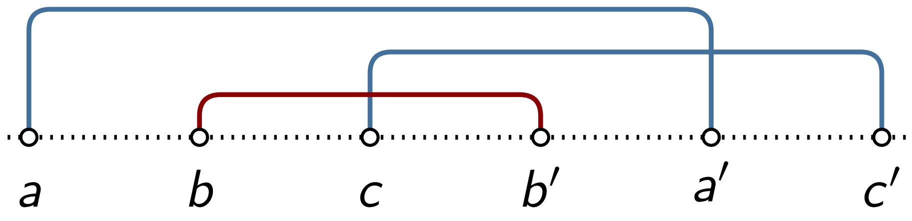
rique

- Our goal: Determine whether all planar graphs admit mixed linear layouts with one rique, and either one stack or one queue.

Known characterization

- A graph admits a linear layout with
  - one stack ⇔ it is outerplanar — Kainen, Bernhart (1979)
  - one queue ⇔ it is level planar — Heath et al. (1992)
  - one deque ⇔ it is subgraph of a planar graph with Hamiltonian path — Auer et al. (2010)
  - one rique ⇔ it admits a vertex order avoiding the following: — Bekos et al. (2022)





Known results on planar graphs

- A general planar graph admits a linear layout with:
  - ≤ 4 stacks (which sometimes are required) — Yannakakis (1986), Bekos at el. (2020)
  - ≤ 42 queues (4 is the lower bound) — Bekos et al. (2018, 2020)
  - ≤ 2 deques (which sometimes are required) — Auer et al. (2010)
  - ≤ 4 riques (2 is the lower bound). — Bekos et al. (2022)
- Note 1: Any two stack pages form a deque
- Note 2: A rique page can be split into a stack and a queue
  the reverse is not true

Motivation and our result

- Theorem: There exist planar graphs that do not admit mixed linear layouts with one rique, and either one stack or one queue.

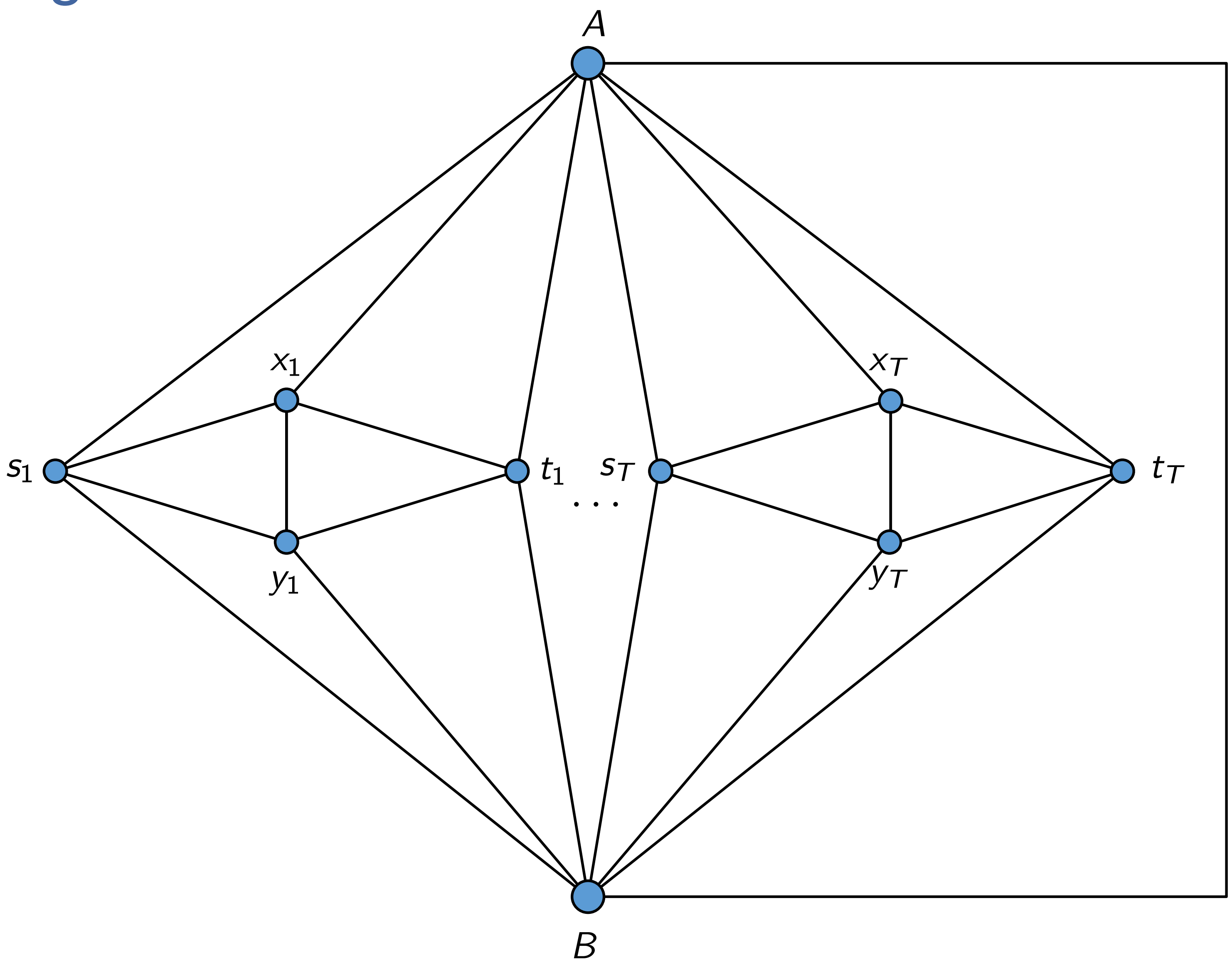
- Motivation: Stems from a conjecture by Heath and Rosenberg (1992):

  Every planar graph admits a mixed linear layout with one stack and one queue

  - Pupyrev (2017) disproved the conjecture.
  - Angelini et al. (2022) showed that the conjecture does not hold for 2-trees.
  - Note 1: Our result strengthens Pupyrev's result.
  - Note 2: 2-trees admit 2-stack layouts. Rengarajan, Madhavan (1995)

    thus also mixed with one rique and one stack

The graph G_T supporting the theorem



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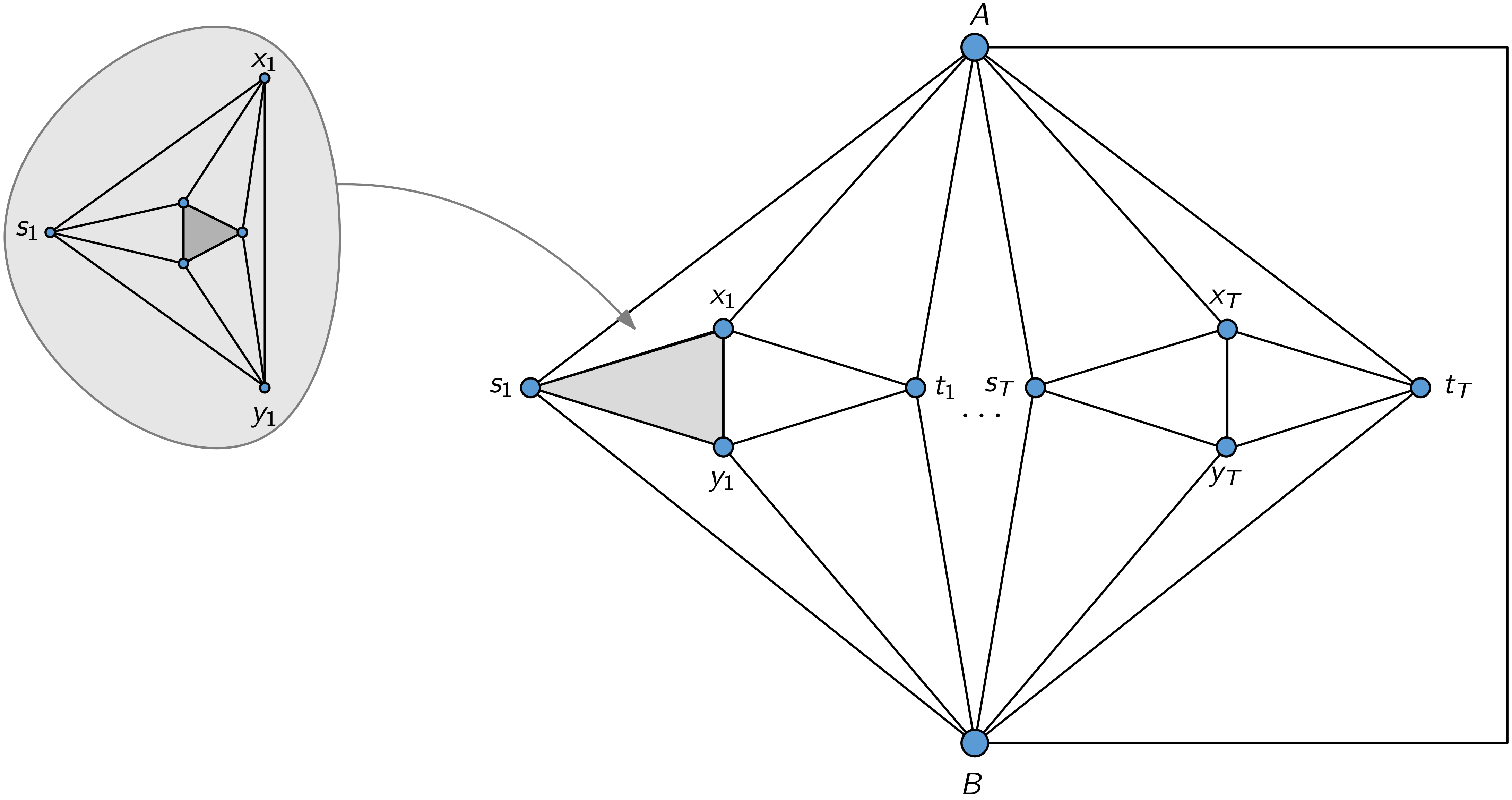

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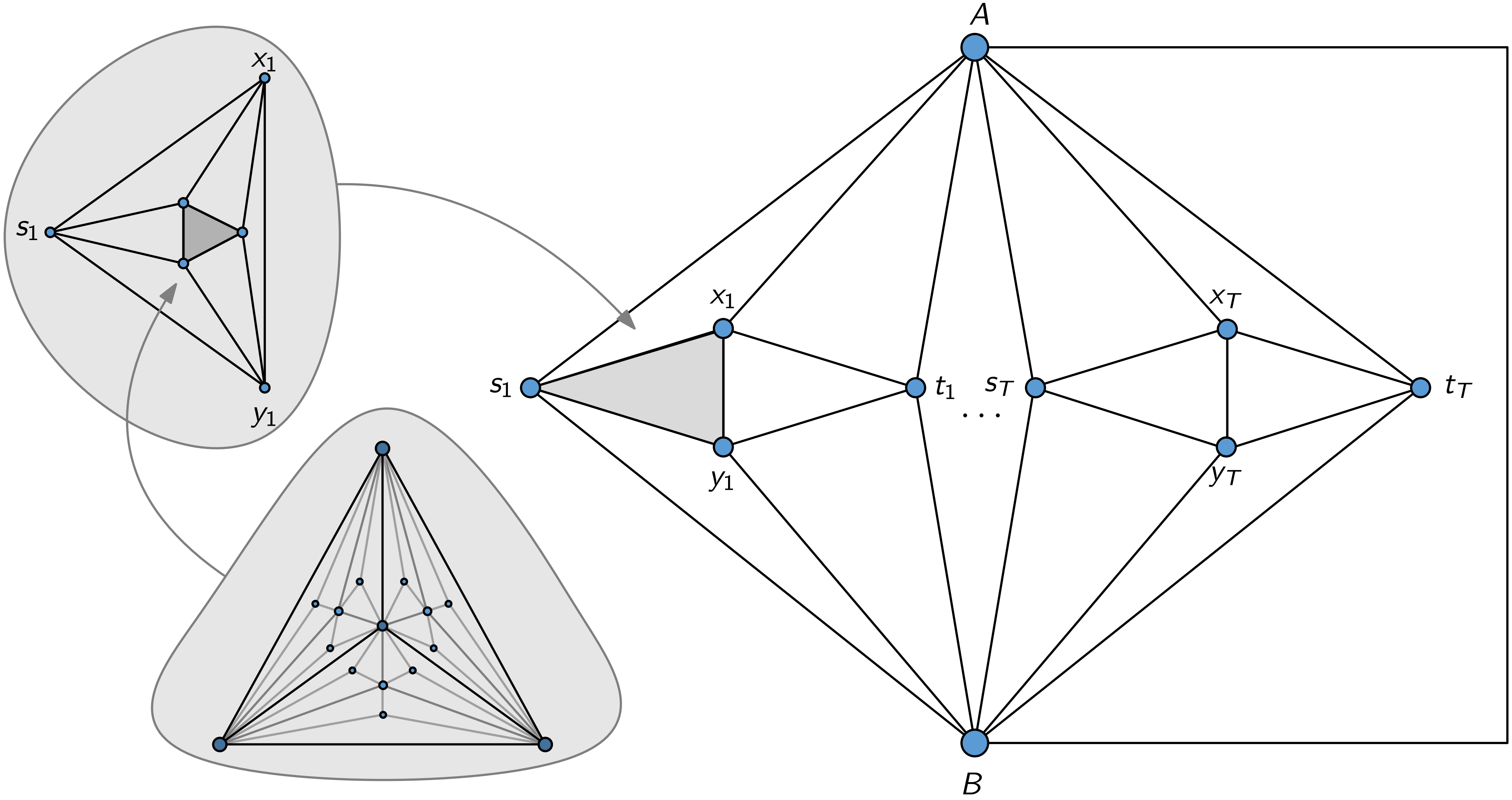

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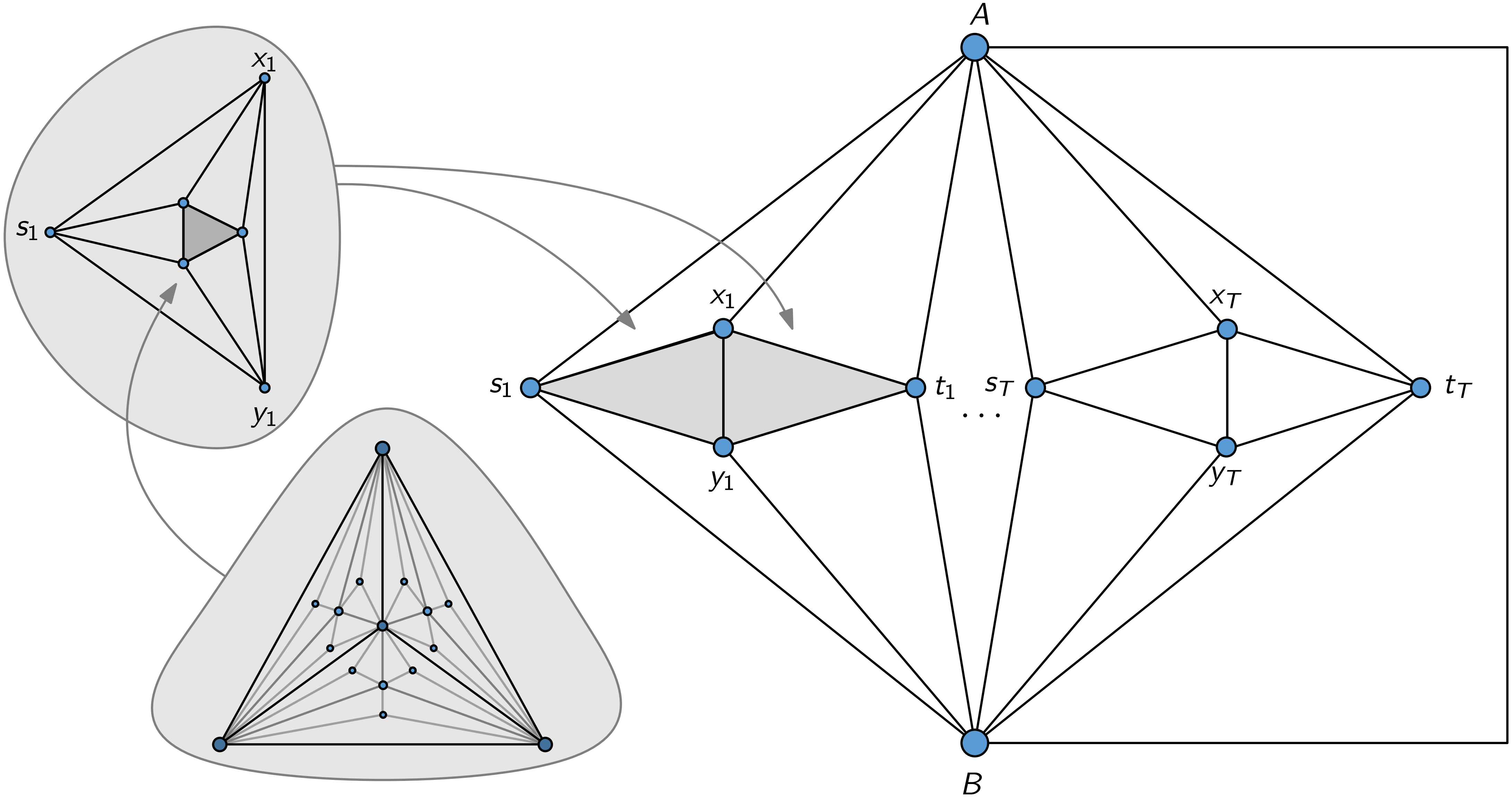

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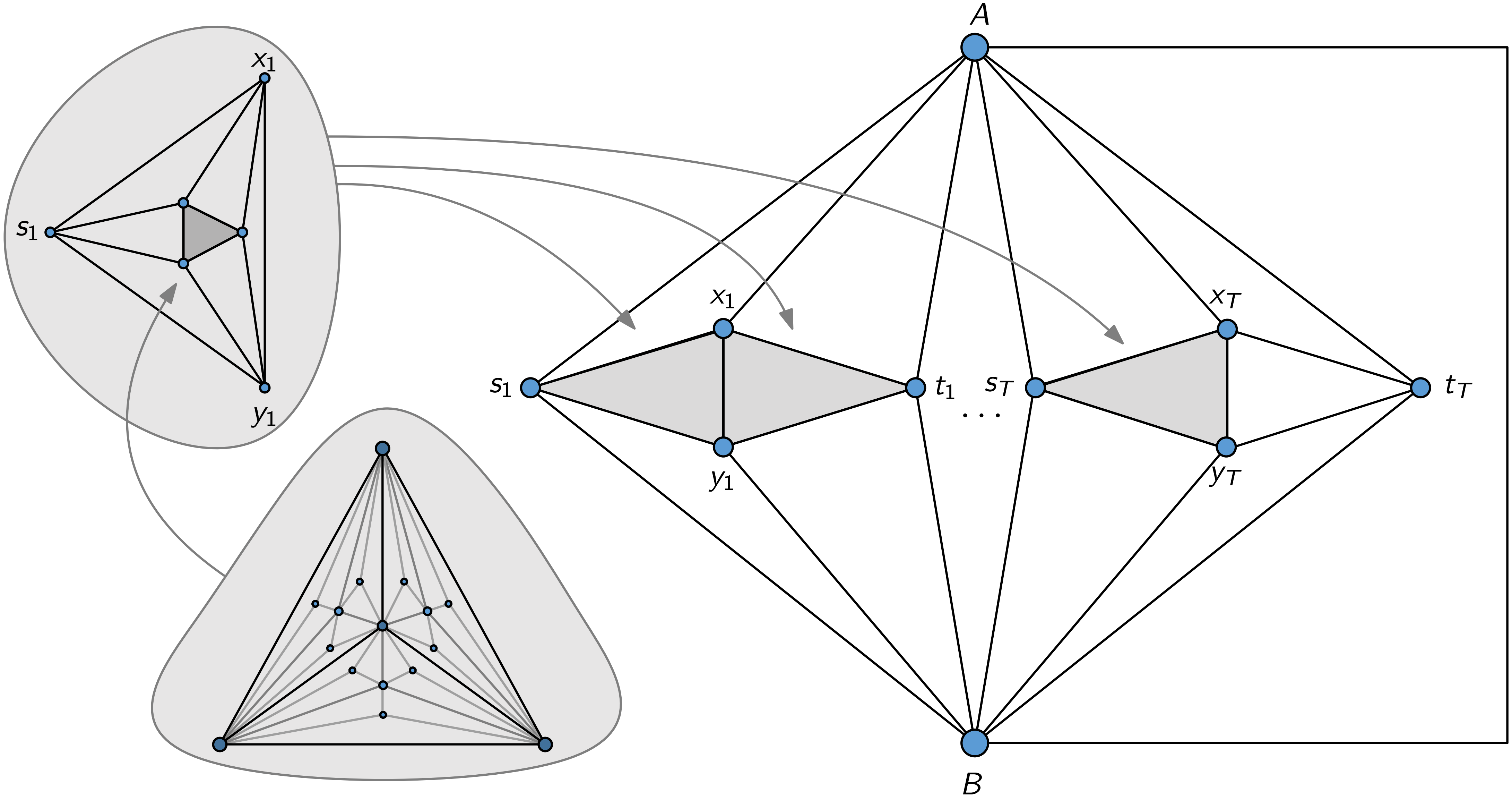

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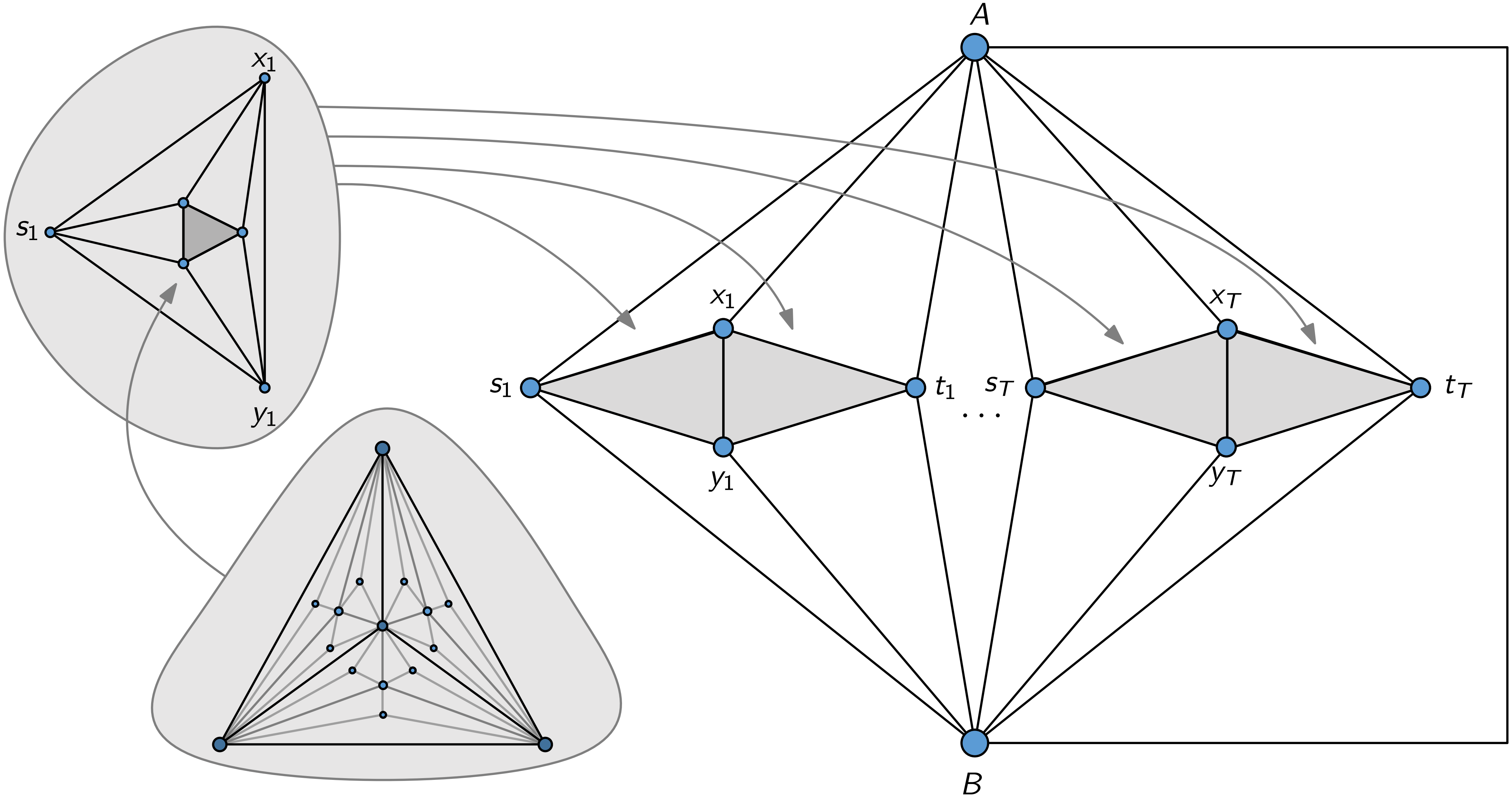

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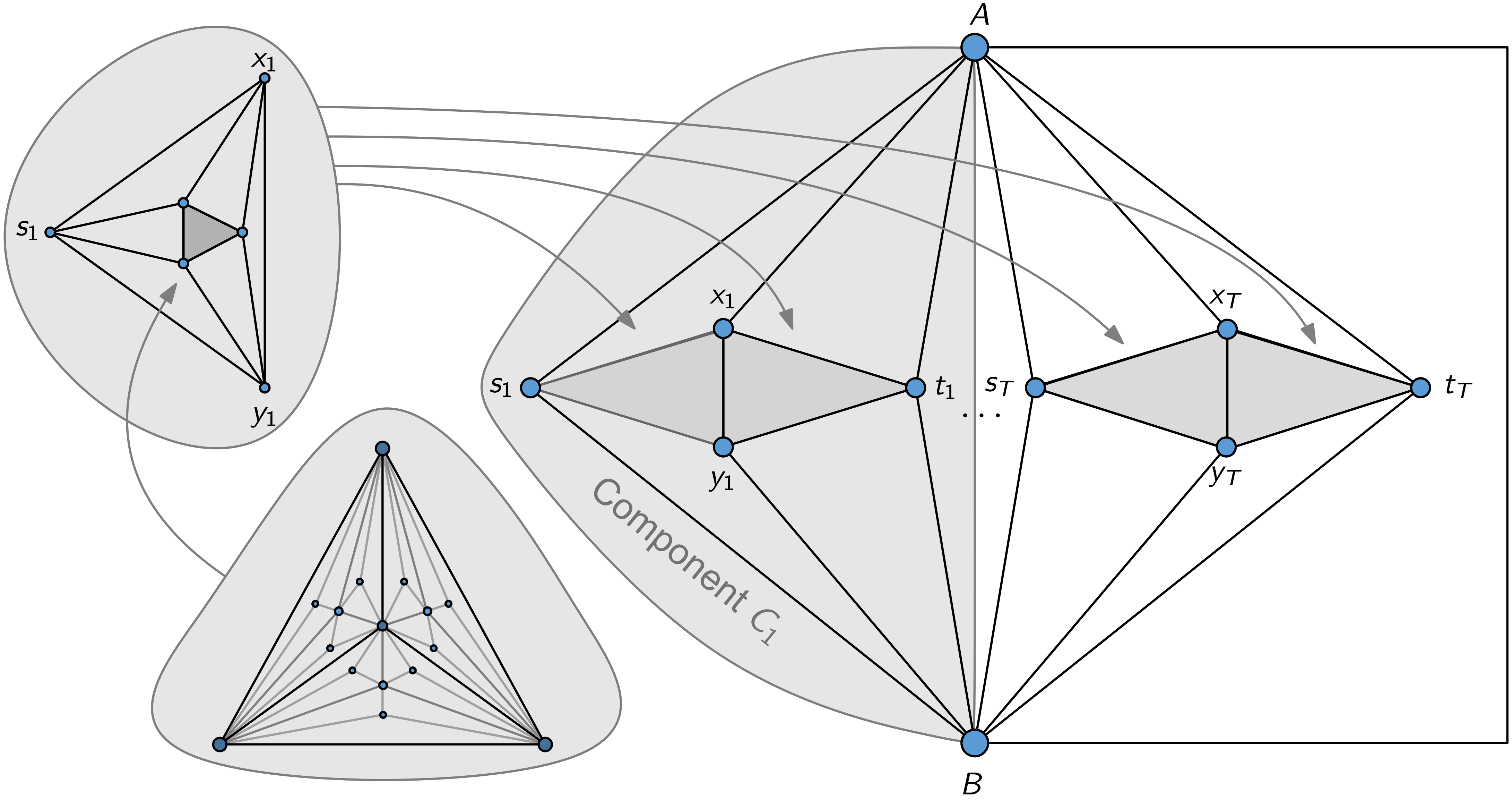

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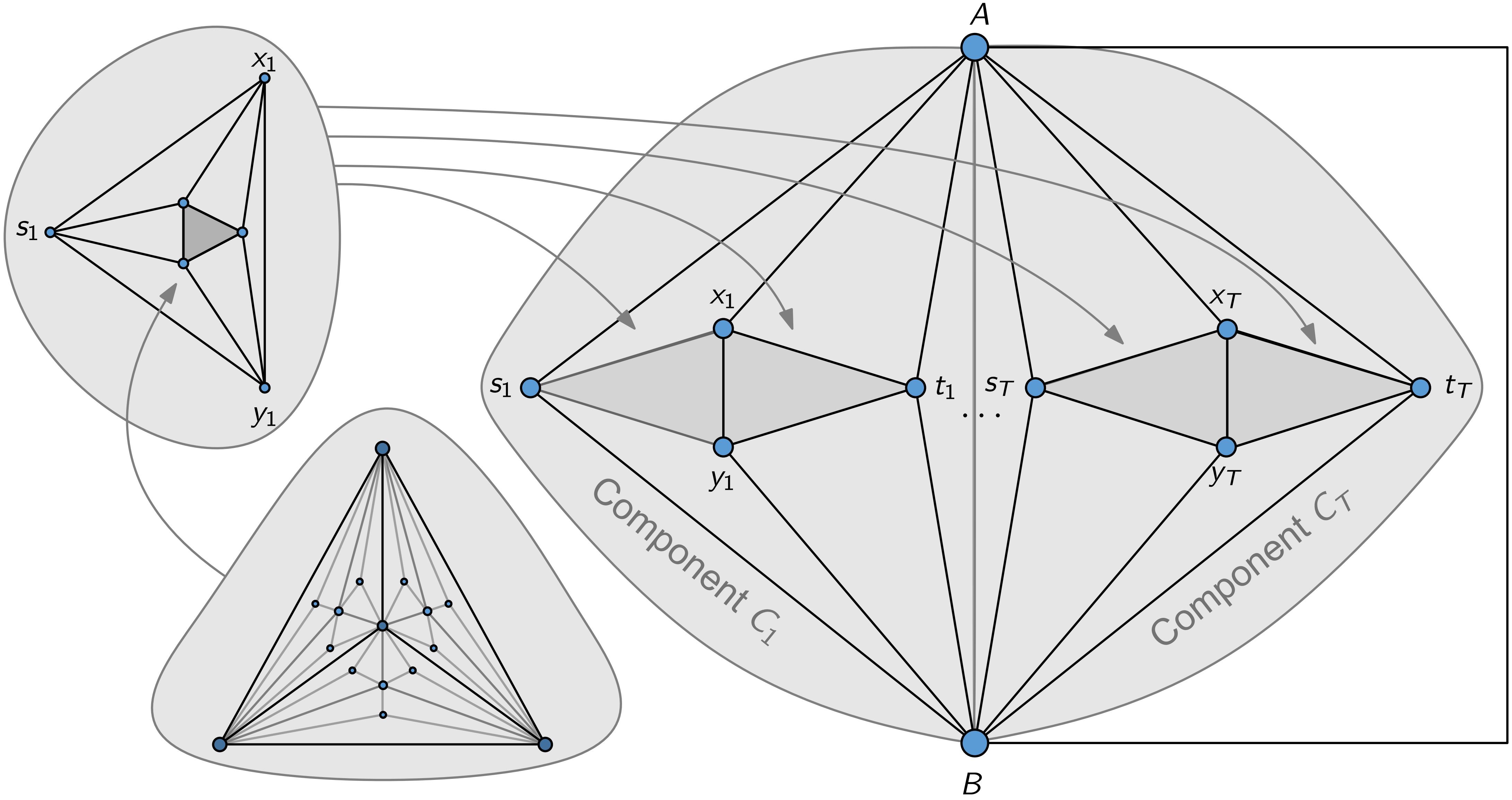
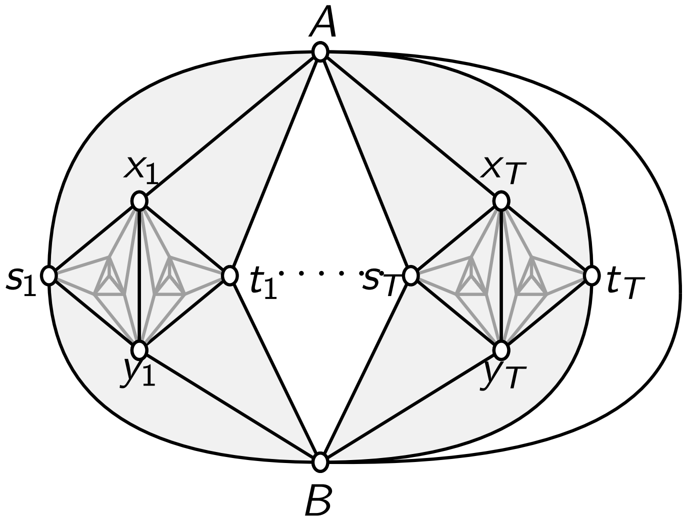


The combinatorial part of the proof



- By contradiction: Assume that G_T has a mixed linear layout with one rique, and either one stack or one queue.
- $T \gg 1 \Rightarrow \exists k$ components $C_1, \ldots, C_k$ of G_T with the same layout:
  - For C_i, C_j, $1 \leq i, j \leq k$, the order that the vertices u, v of C_i appear in the mixed linear layout, is the same with the order of their twin vertices u', v' of C_j (e.g. $s_i \prec t_i \Rightarrow s_j \prec t_j$).
  - Twin edges of C_i and C_j:
    - are in the same page (e.g. $(A, x_i) \in P \Rightarrow (A, x_j) \in P$).
    - are pairwise nesting or crossing or separated.
    - are of the same type, if they are in the rique (head-head or head-tail).

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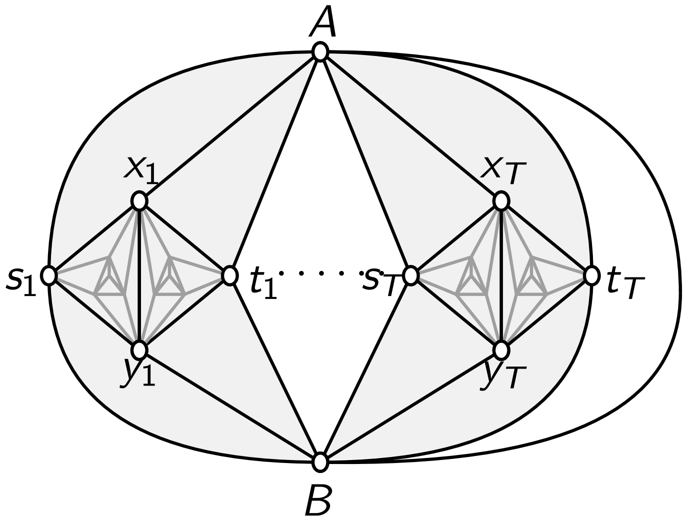

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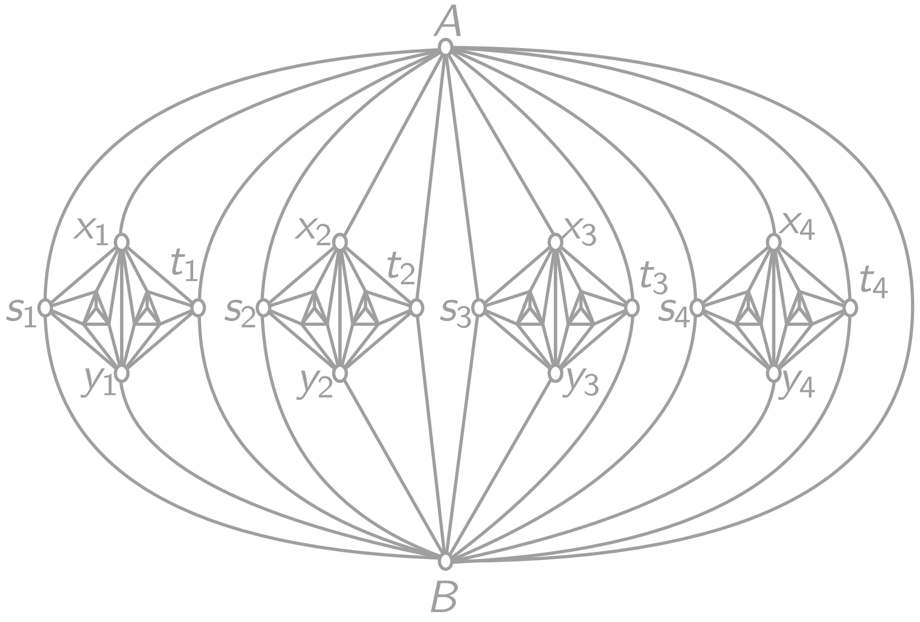
↓ monotonically ordered

- Let T be s.t. $k = 4$ and w.l.o.g. $s_1 \prec s_2 \prec s_3 \prec s_4 \Rightarrow w_1 \prec w_2 \prec w_3 \prec w_4$ or $w_4 \prec w_3 \prec w_2 \prec w_1$ $\forall$ quadruple of twin vertices w_1, w_2, w_3, w_4

The computer aided part of the proof



- We tested with SAT if the subgraph induced by the four components C_1, C_2, C_3, C_4 has a mixed linear layout s.t.:
  - $A \prec B$
  - $s_i \prec t_i, \quad \forall i = 1, 2, 3, 4$
  - $s_1 \prec s_2 \prec s_3 \prec s_4$
  - Every quadruple of twin edges is assigned to the same page.
  - Every quadruple of twin vertices:
    - is monotically ordered
    - either precedes or follows pole A
    - either precedes or follows pole B

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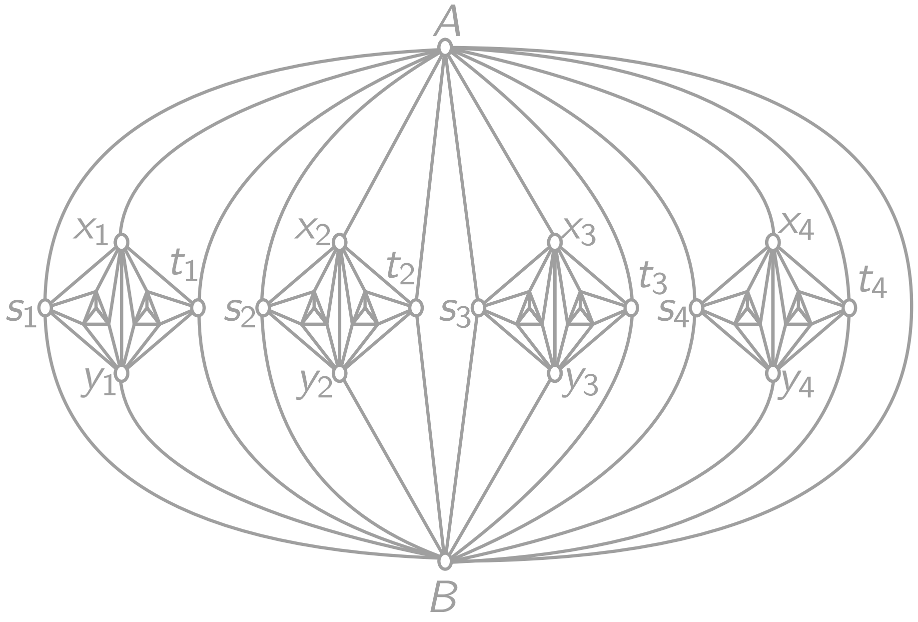

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    - either precedes or follows pole B
- SAT solver concluded $\nexists$ such a layout $\Rightarrow$ G_T does not admit the claimed mixed linear layout

Conclusions

- We proved that there exist planar graphs that do not admit mixed linear layouts with one rique, and either one stack or one queue.

- Open Problems:
  - Does there exist a planar graph that admits no linear layout with two riques?
  - Do all planar graphs admit mixed linear layouts with one deque and one stack or one queue?

Conclusions

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  - Does there exist a planar graph that admits no linear layout with two riques?
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Thanks for your attention